VERIFICATION OF POLYTOPES
BY BRIGHTNESS FUNCTIONS

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Abstract. We show that in the class of origin-centered convex bodies in Euclidean space of dimension at least three, a polytope is uniquely determined by its brightness function in a suitably chosen, but very small set of directions.

1. Introduction and result

Aleksandrov’s projection theorem (see, e.g., Gardner [1, Th. 3.3.6]) is one of the classical and central results of geometric tomography. In its simplest version, it can be formulated as follows. In $d$-dimensional Euclidean space $\mathbb{R}^d$ (we assume $d \geq 3$), let $K$ be a convex body (a compact convex set with interior points, in this paper). For $u \in S^{d-1}$ (the unit sphere), we denote the hyperplane through 0 orthogonal to $u$ by $u^\perp$ and the orthogonal projection to $u^\perp$ by \( \cdot | u^\perp \). The function $u \mapsto V_{d-1}(K|u^\perp)$, where $V_{d-1}$ denotes the $(d-1)$-dimensional volume, is known as the brightness function of $K$. The body $K$ is 0-symmetric (or origin-centered) if $K = -K$. Aleksandrov’s projection theorem says that two 0-symmetric convex bodies with the same brightness function are identical.

It is well known that in this theorem the assumption of central symmetry cannot be deleted; in the following, $K$ and $L$ are always 0-symmetric. It is also known that the equality of the brightness functions in all directions cannot be essentially relaxed. For a precise formulation, we define a direction set as a 0-symmetric closed subset of $S^{d-1}$. If the direction set $A$ is a proper subset of $S^{d-1}$, then for any sufficiently smooth 0-symmetric convex body $K$ there exists a 0-symmetric convex body $L$ with

$$V_{d-1}(K|u^\perp) = V_{d-1}(L|u^\perp) \quad \text{for all } u \in A,$$

but $L \neq K$. Examples were constructed in [5]. There have been several attempts to find additional assumptions on $K$ and $L$ under which smaller sets $A$ in (1.1) still allow the conclusion that $K = L$. For example, this holds if $K$ is a polytope and $A$ is a neighborhood of an equator subsphere (Schneider [5]). As shown by Schneider and Weil [5], it also holds if the dimension $d$ is odd, $A$ is a neighborhood of an equator subsphere with pole $e$, and the supporting hyperplanes of $K$ and $L$ with normal vector $e$ touch each of $K$ and $L$ at a vertex. Results of a different kind were
obtained by Grinberg and Quinto [2], for example the following. Let $K$ and $L$ be of class $C^2_+$. Let $A$ be an open connected subset of $S^{d-1}$ such that $\mathbb{R}^d = \bigcup_{u \in A} u^\perp$. Assume that for some $e \in A$, the products of the principal radii of curvature of $K$ and $L$ agree to infinite order along the equator subsphere $e^\perp \cap S^{d-1}$. If (1.1) holds, then $K = L$.

Note that in each of the previous results, the direction set $A$ has to be of positive (spherical Lebesgue) measure. In contrast to this, we show in the present paper that 0-symmetric polytopes can be verified by their brightness function in suitable direction sets of measure zero. Here, we have adopted the terminology used by Gardner [1] (in the case of X-rays): we say that the convex body $K$ can be verified by the brightness function in a direction set $A$, if any 0-symmetric convex body $L$ satisfying (1.1) is equal to $K$. We prove a result on the verification of general convex bodies. Recall that a vector $u \neq 0$ is an extreme normal vector of $K$ if it cannot be written as $u = u_1 + u_2$, where $u_1, u_2$ are linearly independent normal vectors of $K$ at the same boundary point. Let $E(K)$ denote the set of extreme unit normal vectors of $K$. For $e \in S^{d-1}$, let $S_e := e^\perp \cap S^{d-1}$; this is the equator subsphere with pole $e$.

**Theorem 1.1.** Let $d \geq 3$. Let $K$ and $L$ be 0-symmetric convex bodies in $\mathbb{R}^d$. Let $A$ be a direction set that contains $S_e$ for each $e \in E(K)$ and, together with any $d$-dimensional cone spanned by finitely many vectors of $E(K)$, also a vector in the interior of the dual cone. If $V_{d-1}(K|u^\perp) = V_{d-1}(L|u^\perp)$ for all $u \in A$, then $K = L$.

If $K$ is a polytope, then $E(K)$ is the set of unit normal vectors of its facets; hence the set $A$ in the theorem can be chosen as the union of finitely many great subspheres and a finite set, and thus is of spherical Lebesgue measure zero.

**2. Proof of the theorem**

Let the assumptions of the theorem be satisfied. Then $V_{d-1}(K|u^\perp) = V_{d-1}(L|u^\perp)$ if $v \perp u$ and $u \in E(K)$. Let $\overline{E}$ denote the closure of $E(K)$. It follows from the continuity of the brightness functions that $V_{d-1}(K|v^\perp) = V_{d-1}(L|v^\perp)$ holds also if $v \perp u$ and $u \in \overline{E}$. We shall make use of the fact that $\overline{E}$ is the support of the surface area measure $S_{d-1}(K, \cdot)$ of $K$ (see [3, Th. 4.6.3]).

For standard notation from the theory of convex bodies, we refer to [4]. In particular, $h(K, \cdot)$ denotes the support function of $K$, $H(K, u)$ is the supporting hyperplane and $H^-(K, u)$ is the supporting halfspace of $K$, both with outer normal vector $u$. The scalar product of $\mathbb{R}^d$ is $(\cdot, \cdot)$. By $\Pi K$ we denote the projection body of $K$. Its support function is given by

$$h(\Pi K, u) = V_{d-1}(K|u^\perp) = \frac{1}{2} \int_{S^{d-1}} |(u, v)| S_{d-1}(K, dv)$$

for $u \in S^{d-1}$.

Let $u \in \overline{E}$. Then

$$h(\Pi K, v) = h(\Pi L, v) \quad \text{for all } v \in u^\perp.$$  \hfill (2.1)

For any convex body $M$ we have $h(M|u^\perp, v) = h(M, v)$ if $v \in u^\perp$; hence (2.1) gives

$$h(\Pi K|u^\perp, v) = h(\Pi L|u^\perp, v) \quad \text{for all } v \in u^\perp$$

and, therefore,

$$\Pi K|u^\perp = \Pi L|u^\perp.$$  \hfill (2.2)
It follows that the cylinder \( C(u) := \Pi L + \text{lin} \{u\} \) contains \( \Pi K \). The set
\[
D := \bigcap_{u \in \mathcal{E}} C(u)
\]
is a convex body, which satisfies \( \Pi K \subset D \), \( \Pi L \subset D \), and
\[
(2.3) \quad h(\Pi L, v) = h(D, v) \quad \text{for } v \in u^\perp, \text{ if } u \in \mathcal{E}.
\]
Suppose that \( \Pi K \not\subset D \). Then \( \Pi K \) is a proper subset of \( D \); hence the interior of \( D \) contains a nonempty relatively open subset of the boundary of \( \Pi K \) and hence a regular boundary point of \( \Pi K \). Therefore, there is a vector \( w \in S^{d-1} \) such that the supporting hyperplane \( H(\Pi K, w) \) contains a regular boundary point of \( \Pi K \) and is not a supporting hyperplane of \( D \); hence \( h(\Pi K, w) < h(D, w) \). If \( w \in u^\perp \) for some \( u \in \mathcal{E} \), then \( (2.1) \) and \( (2.3) \) would imply \( h(\Pi K, w) = h(\Pi L, w) = h(D, w) \), a contradiction. Therefore, \( w \not\in u^\perp \) for all \( u \in \mathcal{E} \) and thus \( w^\perp \cap \mathcal{E} = \emptyset \). Since \( \mathcal{E} \) is closed, a whole neighborhood of the equator subsphere \( S_w \) does not meet \( \mathcal{E} \). Thus, \( S_y \cap \mathcal{E} = \emptyset \) for all \( y \) in a neighborhood of \( w \).

We use a formula for support sets of zonoids. For a convex body \( M \), let \( F(M, y) \) be the support set of \( M \) with outer normal vector \( y \). Then, for the zonoid \( \Pi K \) we have (see [4, Lemma 3.5.5])
\[
(2.4) \quad h(F(\Pi K, y), x) = \langle x, e_y \rangle + \frac{1}{2} \int_{S_y} |\langle x, v \rangle| S_{d-1}(K, dv)
\]
for \( x \in \mathbb{R}^d \), with
\[
(2.5) \quad e_y := \int_{S^{d-1}} 1\{\langle v, y \rangle > 0\} v S_{d-1}(K, dv).
\]
In our case, the integral in \( (2.4) \) vanishes for all \( y \) in a neighborhood of \( w \), since \( \mathcal{E} \) is the support of the measure \( S_{d-1}(K, \cdot) \). This means that \( F(\Pi K, y) = \{e_y\} \) for these \( y \), and from \( (2.5) \) it follows that \( e_y = e_w \) for \( y \) in a neighborhood of \( w \). Hence, \( H(\Pi K, w) \cap \Pi K = \{e_w\} \), and \( e_w \) is a singular point of \( \Pi K \); thus, the supporting hyperplane \( H(\Pi K, w) \) does not contain a regular boundary point of \( \Pi K \). This contradiction shows that \( \Pi K = D \).

From \( \Pi K = D \) we get \( \Pi L \subset \Pi K \), and by the monotonicity of mixed volumes this implies that
\[
(2.6) \quad V(\Pi K, \Pi L, \ldots, \Pi L) \leq V(\Pi K, \Pi K, \Pi L, \ldots, \Pi L) \leq \cdots \leq V(\Pi K, \ldots, \Pi K),
\]
where \( V(\cdot, \ldots, \cdot) \) is the mixed volume.

If \( M \) is a convex body, then, using a well-known representation of mixed volumes together with Fubini's theorem, we get (all integrals are over \( S^{d-1} \))
\[
V(\Pi K, M, \ldots, M) = \frac{1}{d} \int h(\Pi K, v) S_{d-1}(M, dv)
\]
\[
= \frac{1}{d} \int \frac{1}{2} \int |\langle u, v \rangle| S_{d-1}(K, du) S_{d-1}(M, dv)
\]
\[
= \frac{1}{d} \int \frac{1}{2} \int |\langle u, v \rangle| S_{d-1}(M, dv) S_{d-1}(K, du)
\]
\[
= \frac{1}{d} \int V_{d-1}(M | u^\perp) S_{d-1}(K, du).
\]
If \( u \in E \), then \( V_{d-1}(\Pi K|u^+) = V_{d-1}(\Pi L|u^+) \) by (2.2). Since this holds for all \( u \) in the support of the measure \( S_{d-1}(K, \cdot) \), we get
\[
V(\Pi K, \Pi L, \ldots, \Pi L) = V(\Pi K, \ldots, \Pi K).
\]

By (2.6), this implies, in particular, that
\[
V(\Pi K, \Pi L, \ldots, \Pi L) = V(\Pi K, \Pi K, \Pi L, \ldots, \Pi L).
\]

By [4] Th. 6.6.16, this is only possible if \( \Pi K \) is a 1-tangential body of \( \Pi L \). A 1-tangential body is a cap body (see [4], p. 76); hence \( \Pi K \) is the convex hull of \( \Pi L \) and a (possibly empty) set \( X \) of points not in \( \Pi L \) such that any segment joining two of these points meets \( \Pi L \). If \( X = \emptyset \), then \( \Pi K = \Pi L \). Since \( K \) and \( L \) are centrally symmetric with respect to 0, Aleksandrov's projection theorem yields \( K = L \). Therefore, it remains to consider the case where \( X \neq \emptyset \). (Note that a zonoid may well be a cap body of another zonoid. For example, a rhombic dodecahedron is a cap body of a cube. Therefore, we do not immediately get a contradiction. It would be interesting to classify all pairs of zonoids where one is a cap body of the other.)

Let \( p \in X \). Let \( C_p \) denote the cone with apex \( p \) spanned by \( \Pi K \). Since \( p \notin \Pi L \), there is a hyperplane \( H \) that strictly separates \( p \) and \( \Pi L \). It intersects the cone \( C_p \) in a \((d-1)\)-dimensional convex body \( Q \). Let \( x \) be an exposed point of \( Q \). The halfline with endpoint \( p \) through \( x \) is an exposed ray of \( C_p \); hence there is a supporting hyperplane of \( \Pi K \) through \( p \) that intersects \( \Pi K \) in a nondegenerate line segment \( S_x \); thus \( F(\Pi K, w) = S_x \) for a suitable unit vector \( w \). Let \( u \) be a unit vector parallel to \( S_x \). Since \( F(\Pi K, w) \) is a segment of direction \( u \), it follows from (2.3) (together with the uniqueness theorem [4] Th. 3.5.3) that the measure \( S_{d-1}(K, \cdot) \) has point masses at \( \pm u \). Therefore, the support sets \( F(K, \pm u) \) of \( K \) are of dimension \( d - 1 \), which implies that \( u \in E(K) \). To each exposed point \( x \) of \( Q \) there corresponds such a segment \( S_x \). It is a summand of \( \Pi K \) (by [4] Cor. 3.5.6), every support set of a zonoid is a summand of the zonoid). Since all the segments \( S_x \) have different directions and their lengths are bounded from below by a positive constant, there can only be finitely many such segments, since otherwise their sum would be unbounded. Thus, the cone \( C_p - p \) is the positive hull of finitely many vectors from \( E(K) \). By assumption, the interior of its dual cone contains a vector \( v \in A \), and we have \( F(\Pi K, v) = \{p\} \) and, therefore, \( h(\Pi K, v) > h(\Pi L, v) \). On the other hand, the assumptions of the theorem give \( V_{d-1}(K|v^+) = V_{d-1}(L|v^+) \) and thus \( h(\Pi K, v) = h(\Pi L, v) \). This contradiction shows that the case \( X \neq \emptyset \) cannot occur, which completes the proof.

References


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