EVERY GRAPH HAS AN EMBEDDING IN $S^3$ CONTAINING NO NON-HYPERBOLIC KNOT

ERICA FLAPAN AND HUGH HOWARDS

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Abstract. In contrast with knots, whose properties depend only on their extrinsic topology in $S^3$, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in $S^3$. For example, it was shown by Conway and Gordon that every embedding of the complete graph $K_7$ in $S^3$ contains a non-trivial knot. Later it was shown that for every $m \in \mathbb{N}$ there is a complete graph $K_n$ such that every embedding of $K_n$ in $S^3$ contains a knot $Q$ whose minimal crossing number is at least $m$. Thus there are arbitrarily complicated knots in every embedding of a sufficiently large complete graph in $S^3$. We prove the contrasting result that every graph has an embedding in $S^3$ such that every non-trivial knot in that embedding is hyperbolic. Our theorem implies that every graph has an embedding in $S^3$ which contains no composite or satellite knots.

In contrast with knots, whose properties depend only on their extrinsic topology in $S^3$, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in $S^3$. For example, it was shown in [2] that every embedding of the complete graph $K_7$ in $S^3$ contains a non-trivial knot. Later in [3] it was shown that for every $m \in \mathbb{N}$, there is a complete graph $K_n$ such that every embedding of $K_n$ in $S^3$ contains a knot $Q$ (i.e., $Q$ is a subgraph of $K_n$) such that $|a_2(Q)| \geq m$, where $a_2$ is the second coefficient of the Conway polynomial of $Q$. More recently, in [4] it was shown that for every $m \in \mathbb{N}$, there is a complete graph $K_n$ such that every embedding of $K_n$ in $S^3$ contains a knot $Q$ whose minimal crossing number is at least $m$. Thus there are arbitrarily complicated knots (as measured by $a_2$ and the minimal crossing number) in every embedding of a sufficiently large complete graph in $S^3$.

In light of these results, it is natural to ask whether there is a graph such that every embedding of that graph in $S^3$ contains a composite knot. Or more generally, is there a graph such that every embedding of the graph in $S^3$ contains a satellite knot? Certainly, $K_7$ is not an example of such a graph since Conway and Gordon [2] exhibit an embedding of $K_7$ containing only the trefoil knot. In this paper we answer this question in the negative. In particular, we prove that every graph has an embedding in $S^3$ such that every non-trivial knot in that embedding is hyperbolic. Our theorem implies that every graph has an embedding in $S^3$ which contains no composite or satellite knots. By contrast, for any particular embedding of a graph...
we can add local knots within every edge to get an embedding such that every knot in that embedding is composite.

Let $G$ be a graph. There is an odd number $n$ such that $G$ is a minor of $K_n$. We will show that for every odd number $n$, there is an embedding of $K_n$ in $S^3$ such that every non-trivial knot in that embedding of $K_n$ is hyperbolic. It follows that there is an embedding of $G$ in $S^3$ which contains no non-trivial non-hyperbolic knots.

Let $n$ be a fixed odd number. We begin by constructing a preliminary embedding of $K_n$ in $S^3$ as follows. Let $h$ be a rotation of $S^3$ of order $n$ with fixed point set $\alpha \cong S^1$. Let $V$ denote the complement of an open regular neighborhood of the fixed point set $\alpha$. Let $v_1, \ldots, v_n$ be points in $V$ such that for each $i$, $h(v_i) = v_{i+1}$ (throughout the paper we shall consider our subscripts mod $n$). These $v_i$ will be the vertices of the preliminary embedding of $K_n$.

**Definition 1.** By a solid annulus we shall mean a 3-manifold with boundary which can be parametrized as $D \times I$ where $D$ is a disk. We use the term the annulus boundary of a solid annulus $D \times I$ to refer to the annulus $\partial D \times I$. The ends of $D \times I$ are the disks $D \times \{0\}$ and $D \times \{1\}$. If $A$ is an arc in a solid annulus $W$ with one endpoint in each end of $W$ and $A$ co-bounds a disk in $W$ together with an arc in $\partial W$, then we say that $A$ is a longitudinal arc of $W$.

As follows, we embed the edges of $K_n$ as simple closed curves in the quotient space $S^3/h = S^3$. Observe that since $V$ is a solid torus, $V' = V/h$ is also a solid torus. Let $D'$ denote a meridional disk for $V'$ which does not contain the point $v = v_1/h$. Let $W'$ denote the solid annulus $\text{cl}(V' - D')$ with ends $D'_+$ and $D'_-$. Since $n$ is odd, we can choose unknotted simple closed curves $S_1, \ldots, S_{n+1}$ in the solid torus $V'$ such that each $S_i$ contains $v$ and has winding number $n + i$ in $V'$, the $S_i$ are pairwise disjoint except at $v$, and for each $i$, $W' \cap S_i$ is a collection of $n + i$ untangled longitudinal arcs (see Figure 1).

![Figure 1](image.png)

**Figure 1.** For each $i$, $W' \cap S_i$ is a collection of $n + i$ untangled longitudinal arcs.

We define as follows two additional simple closed curves $J'$ and $C'$ in $V'$ whose intersections with $W'$ are illustrated in Figure 1. First, choose a simple closed curve $J'$ in $V'$ whose intersection with $W'$ is a longitudinal arc which is disjoint from and untangled with $S_1 \cup \cdots \cup S_{n+1}$. Next we let $C'$ be the unknotted simple closed...
curve in $W' - (S_1 \cup \cdots \cup S_{n-1} \cup J')$ whose projection is illustrated in Figure 1. In particular, $C$ contains one half twist between $J'$ and the set of arcs of $S_1 \cup \cdots \cup S_{n-1}$ which do not contain $v$, another half twist between those arcs of $S_1 \cup \cdots \cup S_{n-1}$ and the set of arcs containing $v$, and $r$ full twists between each of the individual arcs of $S_i$ and $S_{i+1}$ containing $v$. We will determine the value of $r$ later.

Each of the $\frac{n-1}{2}$ simple closed curves $S_1, \ldots, S_{n-1}$ lifts to a simple closed curve consisting of $n$ consecutive edges of $K_n$. The vertices $v_1, \ldots, v_n$ together with these $\frac{n(n-1)}{2}$ edges give us a preliminary embedding $\Gamma_1$ of $K_n$ in $S^3$.

Lift the meridional disk $D'$ of the solid torus $V'$ to $n$ disjoint meridional disks $D_1, \ldots, D_n$ of the solid torus $V$. Lift the simple closed curve $C'$ to $n$ disjoint simple closed curves $C_1, \ldots, C_n$, and lift the simple closed curve $J'$ to $n$ consecutive arcs $J_1, \ldots, J_n$ whose union is a simple closed curve $J$. The closures of the components of $V - (D_1 \cup \cdots \cup D_n)$ are solid annuli, which we denote by $W_1, \ldots, W_n$. The subscripts of all of the lifts are chosen consistently so that for each $i$, $v_i \in W_i$, $C_i \cup J_i \subset W_i$, and $D_i$ and $D_{i+1}$ are the ends of the solid annulus $W_i$. For each $i$, the pair $(W_i - (C_i \cup J_i), (W_i - (C_i \cup J_i)) \cap \Gamma_1)$ is homeomorphic to $(W' - (C' \cup J'), (W' - (C' \cup J')) \cap (S_1 \cup \cdots \cup S_{n-1}))$. For each $i$, the solid annulus $W'$ contains $n + i - 1$ arcs of $S_i$ which are disjoint from $v$. Hence each edge of the embedded graph $\Gamma_1$ meets each solid annulus $W_i$ in at least one arc not containing $v_i$.

Let $\kappa$ be a simple closed curve in $\Gamma_1$. For each $i$, we let $k_i$ denote the set of those arcs of $\kappa \cap W_i$ which do not contain $v_i$, and we let $e_i$ denote either the single arc of $\kappa \cap W_i$ which does contain $v_i$ or the empty set if $v_i$ is not on $\kappa$. Observe that since $\kappa$ is a simple closed curve, it contains at least three edges of $\Gamma_1$; and as we observed above, each edge of $\kappa$ contains at least one arc of $k_i$. Thus for each $i$, $k_i$ contains at least three arcs. Either $e_i$ is empty, the endpoints of $e_i$ are in the same end of the solid annulus $W_i$, or the endpoints of $e_i$ are in different ends of $W_i$. We illustrate these three possibilities for $(W_i, C_i \cup J_i \cup k_i \cup e_i)$ in Figure 2 as forms a), b) and c) respectively. The number of full twists represented by the labels $t, u, x, o, z$ in Figure 2 is some multiple of $r$ depending on the particular simple closed curve $\kappa$.

![Figure 2](image_url)  

**Figure 2.** The forms of $(W_i, C_i \cup J_i \cup k_i \cup e_i)$.

With each of the forms of $(W_i, C_i \cup J_i \cup k_i \cup e_i)$ illustrated in Figure 2 we will associate an additional arc and an additional collection of simple closed curves as follows (illustrated in Figure 3). Let the arc $B_i$ be the core of a solid annulus neighborhood of the union of the arcs $k_i$ in $W_i$ such that $B_i$ is disjoint from $J_i, C_i$, ...
and $e_i$. Let the simple closed curve $Q$ be obtained from $C_i$ by removing the full twists $z, x, t,$ and $u$. Let $Z, X, T,$ and $U$ be unknotted simple closed curves which wrap around $Q$ in place of $z, x, t,$ and $u$ as illustrated in Figure 3.

For each $i$, let $M_i$ denote an unknotted solid torus in $S^3$ obtained by gluing together two identical copies of $W_i$ along $D_i$ and $D_{i+1}$, making sure that the endpoints of the arcs of $J_i, B_i,$ and $e_i$ match up with their counterparts in the second copy to give simple closed curves $j, b,$ and $E$, respectively, in $M_i$. Thus $M_i$ has a $180^\circ$ rotational symmetry around a horizontal line which goes through the center of the figure and the endpoints of both copies of $J_i, B_i,$ and $e_i$. Recall that in form a), $e_i$ is the empty set, and hence so is $E$. Let $Q_1$ and $Q_2, X_1$ and $X_2, Z_1$ and $Z_2, T_1$ and $T_2,$ and $U_1$ and $U_2$ denote the doubles of the unknotted simple closed curves $Q, X, Z, T,$ and $U$ respectively.

Let $Y$ denote the core of the solid torus $c(S^3 - M_i)$. We associate to Form a) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y$. We associate to Form b) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup X_1 \cup X_2 \cup Z_1 \cup Z_2$. We associate to Form c) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup T_1 \cup T_2 \cup U_1 \cup U_2$. Figure 4 illustrates the three forms of the link $L$.

The software program SnapPea can be used to determine whether or not a given knot or link in $S^3$ is hyperbolic, and if it is, SnapPea estimates the hyperbolic volume of the complement. We used SnapPea to verify that each of the three forms of the link $L$ illustrated in Figure 4 is hyperbolic.

A 3-manifold is unchanged by doing Dehn surgery on an unknot if the boundary slope of the surgery is the reciprocal of an integer (though such surgery may change a knot or link in the manifold). According to Thurston’s Hyperbolic Dehn Surgery Theorem [1, 5], all but finitely many Dehn fillings of a hyperbolic link complement yield a hyperbolic manifold. Thus there is some $r \in \mathbb{N}$ such that for any $m \geq r$, if we do Dehn filling with slope $\frac{1}{m}$ along the components $X_1, X_2, Z_1, Z_2$ of the link $L$ in form b) or along the components $T_1, T_2, U_1, U_2$ of the link $L$ in form c), then we obtain a hyperbolic link $\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup j \cup b \cup Y \cup E$, where the simple closed curves $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are obtained by adding $m$ full twists to $Q_1$ and $Q_2$ in place of each of the surgered curves.

\[1\text{Available at http://www.geometrygames.org/SnapPea/index.html.}\]
Figure 4. The possible forms of the link $L$.

We fix the value of $r$ according to the above paragraph, and this is the value of $r$ that we use in Figure 1. Recall that the number of twists $x$, $z$, $u$, and $t$ in the simple closed curves $C_i$ in Figure 2 are each a multiple of $r$. Thus the particular simple closed curves $C_i$ are determined by our choice of $r$ together with our choice of the simple closed curve $\kappa$. Now we do Dehn fillings along $X_1$ and $X_2$ with slope $\frac{1}{x}$, along $Z_1$ and $Z_2$ with slope $\frac{1}{z}$, and along $T_1$ and $T_2$ with slope $\frac{1}{u}$. Since $x$, $z$, $u$, and $t$ are each greater than or equal to $r$, the link $Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E$ that we obtain will be hyperbolic. In Form a), $E$ is the empty set, and the link $Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E$ was already seen to be hyperbolic from using SnapPea. In this case, we do no surgery and let $Q_1 = Q_1$ and $Q_2 = Q_2$. It follows that each form of $M_i - (Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E)$ is a hyperbolic 3-manifold. Observe that $M_i - (Q_1 \cup Q_2 \cup j \cup b \cup E)$ is the double of $W_i - (C_i \cup J_i \cup B_i \cup e_i)$.

Now that we have fixed $C_i$, we let $N(C_i)$, $N(J_i)$, $N(B_i)$, and $N(e_i)$ be pairwise disjoint regular neighborhoods of $C_i$, $J_i$, $B_i$, and $e_i$ respectively in the interior of each of the forms of the solid annulus $W_i$ (illustrated in Figure 2). We choose $N(B_i)$ such that it contains the union of the arcs $k_i$. Note that in Form a) $e_i$ is the empty set and hence so is $N(e_i)$. Let $N(k_i)$ denote a collection of pairwise disjoint regular neighborhoods, each containing an arc $k_i$, such that $N(k_i) \subseteq N(B_i)$. Let
$V_i = \text{cl}(W_i - (N(C_i) \cup N(J_i) \cup N(B_i) \cup N(\epsilon_i)))$, let $\Delta = \text{cl}(N(B_1) - N(k_i))$, and let $V'_i = V_i \cup \Delta$. Since $N(B_1)$ is a solid annulus, it has a product structure $D^2 \times I$. Without loss of generality, we assume that each of the components of $N(k_i)$ respects the product structure of $N(B_1)$. Thus $\Delta = F \times I$ where $F$ is a disk with holes.

**Definition 2.** Let $X$ be a 3-manifold. A sphere in $X$ is said to be **essential** if it does not bound a ball in $X$. A properly embedded disk $D$ in $X$ is said to be **essential** if $\partial D$ does not bound a disk in $\partial X$. A properly embedded annulus is said to be **essential** if it is incompressible and not boundary parallel. A torus in $X$ is said to be **essential** if it is incompressible and not boundary parallel.

**Lemma 1.** For each $i$, $V'_i$ contains no essential torus, sphere, or disk whose boundary is in $D_i \cup D_{i+1}$. Also, any incompressible annulus in $V'_i$ whose boundary is in $D_i \cup D_{i+1}$ either is boundary parallel or can be expressed as $\sigma \times I$ (possibly after a change in parameterization of $\Delta$), where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$.

**Proof.** Since $k_i$ contains at least three disjoint arcs, $F$ is a disk with at least three holes. Let $\beta$ denote the double of $\Delta$ along $\Delta \cap (D_i \cup D_{i+1})$. Then $\beta = F \times S^1$. Now it follows from Waldhausen [7] that $\beta$ contains no essential sphere or properly embedded disk and that any incompressible torus in $\beta$ can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of $\beta$) where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$.

Let $\nu$ denote the double of $V_i$ along $V_i \cap (D_i \cup D_{i+1})$. Observe that $\nu \cup \beta$ is the double of $V'_i$ along $V'_i \cap (D_i \cup D_{i+1})$. Now the interior of $\nu$ is homeomorphic to $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$. Since we saw above that $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$ is hyperbolic, it follows from Thurston [5, 6] that $\nu$ contains no essential sphere or torus and no properly embedded disk or annulus.

We see as follows that $\nu \cup \beta$ contains no essential sphere and that any essential torus in $\nu \cup \beta$ can be expressed (after a possible change in parameterization of $\beta$) as $\sigma \times S^1$, where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$. Let $\tau$ be an essential sphere or torus in $\nu \cup \beta$, and let $\gamma$ denote the torus $\nu \cap \beta$. By doing an isotopy as necessary, we can assume that $\tau$ intersects $\gamma$ in a minimal number of disjoint simple closed curves. Suppose there is a curve of intersection which bounds a disk in the essential surface $\tau$. Let $c$ be an innermost curve of intersection on $\tau$ which bounds a disk $\delta$ in $\tau$. Then $\delta$ is a properly embedded disk in either $\gamma$ or $\beta$. Since neither $\nu$ nor $\beta$ contains a properly embedded essential disk or an essential sphere, there is an isotopy of $\tau$ which removes $c$ from the collection of curves of intersection. Thus by the minimality of the number of curves in $\tau \cap \gamma$, we can assume that none of the curves in $\tau \cap \gamma$ bounds a disk in $\tau$.

Suppose that $\tau$ is an essential sphere in $\nu \cup \beta$. Since none of the curves in $\tau \cap \gamma$ bounds a disk in $\tau$, $\tau$ must be contained entirely in either $\nu$ or $\beta$. However, we saw above that neither $\nu$ nor $\beta$ contains any essential sphere. Thus $\tau$ cannot be an essential sphere and hence must be an essential torus. Since $\tau \cap \gamma$ is minimal, if $\tau \cap \nu$ is non-empty, then the components of $\tau$ in $\nu$ are all incompressible annuli. However, we saw above that $\nu$ contains no essential annuli. Thus $\tau \cap \nu$ is empty. Since $\nu$ contains no essential torus, the essential torus $\tau$ must be contained in $\beta$. Hence $\tau$ can be expressed (after a possible change in parameterization of $\beta$) as $\sigma \times S^1$, where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$. 

Now we consider essential surfaces in $V'_i$. Suppose that $V'_i$ contains an essential sphere $S$. Since $\nu \cap \beta$ contains no essential sphere, $S$ bounds a ball $B$ in $\nu \cap \beta$. Now the ball $B$ cannot contain any of the boundary components of $\nu \cap \beta$. Thus $B$ cannot contain either $D_i$ or $D_{i+1}$. Since $S$ is disjoint from $D_i \cup D_{i+1}$, it follows that $B$ must be disjoint from $D_i \cup D_{i+1}$. Thus $B$ is contained in $V'_i$. Hence $V'_i$ cannot contain an essential sphere.

We see as follows that $V'_i$ cannot contain an essential disk whose boundary is in $D_i \cup D_{i+1}$. Let $\epsilon$ be a disk in $V'_i$ whose boundary is in $D_i \cup D_{i+1}$. Let $\epsilon'$ denote the double of $\epsilon$ in $\nu \cap \beta$. Then $\epsilon'$ is a sphere which meets $D_i \cup D_{i+1}$ in the simple closed curve $\partial \epsilon$. Since $\nu \cap \beta$ contains no essential sphere, $\epsilon'$ bounds a ball $B$ in $\nu \cap \beta$. It follows that $B$ cannot contain any of the boundary components of $\nu \cap \beta$. Thus $B$ cannot contain any of the boundary components of $D_i \cup D_{i+1}$. Therefore, $D_i \cup D_{i+1}$ intersects the ball $B$ in a disk bounded by $\partial \epsilon$. Hence the simple closed curve $\partial \epsilon$ bounds a disk in $(D_i \cup D_{i+1}) \cap V'_i$, and therefore the disk $\epsilon$ was not essential in $V'_i$. Thus, $V'_i$ contains no essential disk whose boundary is in $D_i \cup D_{i+1}$.

Now suppose that $V'_i$ contains an essential torus $T$. Suppose that $T$ is not essential in $\nu \cap \beta$. Then either $T$ is boundary parallel or $T$ is compressible in $\nu \cap \beta$. However, $T$ cannot be boundary parallel in $\nu \cap \beta$ since $T \subseteq V'_i$. Thus $T$ must be compressible in $\nu \cap \beta$. Let $\delta$ be a compression disk for $T$ in $\nu \cap \beta$. Since $V'_i$ contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$, we can use an innermost disk argument to push $\delta$ off of $D_i \cup D_{i+1}$. Hence $T$ is compressible in $V'_i$, contrary to our initial assumption. Thus $T$ must be essential in $\nu \cap \beta$. It follows that $T$ has the form $\sigma \times S^1$, where $\sigma \subset D_i \cap \Delta$. However, since $\nu \cap \beta$ is the double of $V'_i$, the intersection of $\sigma \times S^1$ with $V'_i$ is an annulus $\sigma \times I$. In particular, $V'_i$ cannot contain $\sigma \times S^1$. Therefore, $V'_i$ cannot contain an essential torus.

Suppose that $V'_i$ contains an incompressible annulus $\alpha$ whose boundary is in $D_i \cup D_{i+1}$. Let $\tau$ denote the double of $\alpha$ in $\nu \cap \beta$. Then $\tau$ is a torus. If $\tau$ is essential in $\nu \cap \beta$, then we saw above that $\tau$ can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of $\beta$) where $\sigma$ is a non-trivial simple closed curve in $D_i \cap \Delta$. In this case, $\alpha$ can be expressed as $\sigma \times I$.

On the other hand, if $\tau$ is inessential in $\nu \cap \beta$, then either $\tau$ is parallel to a component of $\partial (\nu \cap \beta)$, or $\tau$ is compressible in $\nu \cap \beta$. If $\tau$ is parallel to a boundary component of $\nu \cap \beta$, then $\alpha$ is parallel to the annulus boundary component of $W_i$, $N(J_i)$, $N(e_i)$, or one of the boundary components of $N(k_i)$.

Thus we suppose that the torus $\tau$ is compressible in $\nu \cap \beta$. In this case, it follows from an innermost loop–outermost arc argument that either the annulus $\alpha$ is compressible in $V'_i$ or $\alpha$ is $\partial$-compressible in $V'_i$. Since we assumed $\alpha$ was incompressible in $V'_i$, $\alpha$ must be $\partial$-compressible in $V'_i$. Now according to a lemma of Waldhausen [7], if a 3-manifold contains no essential sphere or properly embedded essential disk, then any annulus which is incompressible but boundary compressive must be boundary parallel. We saw above that $V''_i$ contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$. Since the boundary of the incompressible annulus $\alpha$ is contained in $D_i \cup D_{i+1}$, it follows from Waldhausen’s lemma that $\alpha$ is boundary parallel in $V''_i$.

It follows from Lemma[8] that for any $i$, any incompressible annulus in $V''_i$ whose boundary is in $D_i \cup D_{i+1}$ either is parallel to an annulus in $D_i$ or $D_{i+1}$ or co-bounds a solid annulus in the solid annulus $W_i$ with ends in $D_i \cup D_{i+1}$. Recall that $\kappa$ is a simple closed curve in $\Gamma_1$ such that $\kappa \cap W_i = k_i \cup e_i$. Also $J = J_1 \cup \cdots \cup J_n$. Let $N(\kappa)$
and $N(J)$ be regular neighborhoods of the simple closed curves $\kappa$ and $J$ respectively, such that for each $i$, $N(\kappa) \cap W_i = N(k_i) \cup N(e_i)$ and $N(J) \cap W_i = N(J_i)$. Recall that $V = W_1 \cup \cdots \cup W_n$. Thus $\text{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J) \cup N(\kappa))) = V'_1 \cup \cdots \cup V'_n$.

**Proposition 1.** $H = \text{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J) \cup N(\kappa)))$ contains no essential sphere or torus.

**Proof.** Suppose that $S$ is an essential sphere in $H$. Without loss of generality, $S$ intersects the collection of disks $D_1 \cup \cdots \cup D_n$ transversely in a minimal number of simple closed curves. By Lemma 1 for each $i$, $V'_i$ contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$. Thus the sphere $S$ cannot be entirely contained in one $V'_i$. Let $c$ be an innermost curve of intersection on $S$. Then $c$ bounds a disk $\delta$ in some $V'_i$. However, since the number of curves of intersection is minimal, $\delta$ must be essential, contrary to Lemma 1. Hence $H$ contains no essential sphere.

Suppose $T$ is an incompressible torus in $H$. We show as follows that $T$ is parallel to some boundary component of $H$. Without loss of generality, the torus $T$ intersects the collection of disks $D_1 \cup \cdots \cup D_n$ transversely in a minimal number of simple closed curves. By Lemma 1 for each $i$, $V'_i$ contains no essential torus, essential sphere, or essential disk whose boundary is in $D_i \cup D_{i+1}$. Thus the torus $T$ cannot be entirely contained in one $V'_i$. Also, by the minimality of the number of curves of intersection, we can assume that if $V'_i \cap T$ is non-empty, then it consists of a collection of incompressible annuli in $V'_i$ whose boundary components are in $D_i \cup D_{i+1}$. Furthermore, by Lemma 1, each such annulus either is boundary parallel or is contained in $N(B_i)$ and can be expressed (after a possible change in parameterization of $N(B_i)$) as $\sigma_i \times I$ for some non-trivial simple closed curve $\sigma_i$ in $D_i \cap \Delta$. If some annulus component of $V'_i \cap T$ is parallel to an annulus in $D_i \cup D_{i+1}$, then we could remove that component by an isotopy of $T$. Thus we can assume that each annulus in $V'_i \cap T$ is parallel to the annulus boundary component of one of the solid annuli $W_i$, $N(J_i)$, or $N(e_i)$, or can be expressed as $\sigma_i \times I$. In any of these cases the annulus co-bounds a solid annulus in $W_i$ with ends in $D_i \cup D_{i+1}$.

Consider some $i$ such that $V'_i \cap T$ is non-empty. Hence it contains an incompressible annulus $A_i$ which has one of the above forms. By the connectivity of the torus $T$, either there is an incompressible annulus $A_{i+1} \subseteq V'_{i+1} \cap T$ such that $A_i$ and $A_{i+1}$ share a boundary component, or there is an incompressible annulus $A_{i-1} \subseteq V'_{i-1} \cap T$ such that $A_i$ and $A_{i-1}$ share a boundary component, or both. We will assume, without loss of generality, that there is an incompressible annulus $A_{i+1} \subseteq V'_{i+1} \cap T$ such that $A_i$ and $A_{i+1}$ share a boundary component. Now it follows that $A_i$ co-bounds a solid annulus $F_i$ in $W_i$ with ends in $D_i \cup D_{i+1}$ and that $A_{i+1}$ co-bounds a solid annulus $F_{i+1}$ in $W_{i+1}$ together with two disks in $D_{i+1} \cup D_{i+2}$. Hence the solid annuli $F_i$ and $F_{i+1}$ meet in one or two disks in $D_{i+1}$.

We consider several cases where $A_i$ is parallel to some boundary component of $V'_i$. Suppose that $A_i$ is parallel to the annulus boundary component of the solid annulus $N(J_i)$. Then the solid annulus $F_i$ contains $N(J_i)$ and is disjoint from the arcs $k_i$ and $e_i$. Now the arcs $J_i$ and $J_{i+1}$ share an endpoint contained in $F_i \cap F_{i+1}$, and there is no endpoint of any arc of $k_i$ or $e_i$ in $F_i \cap F_{i+1}$. It follows that the solid annulus $F_{i+1}$ contains the arc $J_{i+1}$ and contains no arcs of $k_{i+1}$. Hence, by Lemma 1 the incompressible annulus $A_{i+1}$ must be parallel to $\partial N(J_{i+1})$. Continuing from one $V'_i$ to the next, we see that in this case $T$ is parallel to $\partial N(J)$.
Suppose that $A_i$ is parallel to the annulus boundary component of the solid annulus $\partial N(e_i)$ or one of the solid annuli in $\partial N(k_i)$. Using an argument similar to that in the above paragraph, we see that $A_{i+1}$ is parallel to the annulus boundary component of the solid annulus $\partial N(e_{i+1})$ or one of the solid annuli in $\partial N(k_{i+1})$. Continuing as above, we see that in this case $T$ is parallel to $\partial N(\kappa)$.

Suppose that the annulus $A_i$ is parallel to the annulus boundary component of the solid annulus $W_i$. Then the solid annulus $F_i$ contains all of the arcs of $J_i$, $k_i$, and $e_i$. It follows as above that the solid annulus $F_{i+1}$ contains the arc $J_{i+1}$ and some arcs of $k_{i+1} \cup e_{i+1}$. Thus by Lemma 1, $A_{i+1}$ must be parallel to the annulus boundary component of the solid annulus $W_{i+1}$. Continuing in this way, we see that in this case $T$ is parallel to $\partial V$.

Thus we now assume that no component of any $V_i \cap T$ is parallel to an annulus boundary component of $V_i$. Hence if any $V_i \cap T$ is non-empty, then by Lemma 1 it consists of disjoint incompressible annuli in $N(B_i)$ which can each be expressed (after a possible re-parametrization of $N(B_i)$) as $\sigma_i \times I$ for some non-trivial simple closed curve $\sigma_i \subseteq D_i \cap \Delta$. Choose $i$ such that $V_i \cap T$ is non-empty. Since $N(B_i)$ is a solid annulus, there is an innermost incompressible annulus $A_i$ of $N(B_i)$ \cap $T$. Now $A_i$ bounds a solid annulus $F_i$ in $N(B_i)$, and $F_i$ contains more than one arc of $k_i$. Since $A_i$ is innermost in $N(B_i)$, $\text{int}(F_i)$ is disjoint from $T$. Now there is an incompressible annulus $A_{i+1}$ in $V_i^{i+1} \cap T$ such that $A_i$ and $A_{i+1}$ meet in a circle in $D_i^{i+1}$. Furthermore, this circle bounds a disk in $D_i^{i+1}$ which is disjoint from $T$ and, by our assumption, is contained in $N(B_i)$. Thus by Lemma 1 the incompressible annulus $A_{i+1}$ has the form $\sigma_{i+1} \times I$ for some non-trivial simple closed curve $\sigma_{i+1} \subseteq D_i^{i+1} \cap \Delta$. Thus $A_i^{i+1}$ bounds a solid annulus $F_{i+1}$ in $N(B_{i+1})$, and $\text{int}(F_{i+1})$ is also disjoint from $T$. We continue in this way considering consecutive annuli to conclude that for every $j$, every component $A_j$ of $T \cap V_j$ is an incompressible annulus which bounds a solid annulus $F_j$ whose interior is disjoint from $T$.

Recall that $V = W_1 \cup \cdots \cup W_n$ is a solid torus. Let $Q$ denote the component of $V - T$ which is disjoint from $\partial V$. Then $Q$ is the union of the solid annuli $F_j$. Since some $F_i$ contains some arcs of $k_i$, the simple closed curve $\kappa$ must be contained in $Q$.

Recall that the simple closed curve $\kappa$ contains at least three vertices of the embedded graph $\Gamma_1$. Also each vertex of $\kappa$ is contained in some arc $e_j$. Since each such $e_j$ satisfies $e_j \subset \kappa \subset Q$, some component $F_j$ of $Q \cap W_j$ contains the arc $e_j$. By our assumption, for any $V_j' \cap T$ which is non-empty, $V_j' \cap T$ consists of disjoint incompressible annuli in $N(B_i)$. In particular, $V_j \cap T \subset N(B_i)$. Now the annulus boundary of $F_j$ is contained in $N(B_j)$ and hence $F_j \subset N(B_j)$. But this is impossible since $e_j \subset F_j$ and $e_j$ is disjoint from $N(B_j)$. Hence our assumption that no component of any $V_i' \cap T$ is parallel to an annulus boundary component of $V_i'$ is wrong. Thus, as we saw in the previous cases, $T$ must be parallel to a boundary component of $H$. Therefore $H$ contains no essential annulus.

Recall that the value of $r$, the simple closed curves, and the manifold $H$ all depend on the particular choice of simple closed curve $\kappa$. In the following theorem we do not fix a particular $\kappa$, so none of the above are fixed.

**Theorem 1.** Every graph can be embedded in $S^3$ in such a way that every non-trivial knot in the embedded graph is hyperbolic.
Proof. Let $G$ be a graph, and let $n \geq 3$ be an odd number such that $G$ is a minor of the complete graph on $n$ vertices, $K_n$. Let $\Gamma_1$ be the embedding of $K_n$ given in our preliminary construction. Then $\Gamma_1$ contains at most finitely many simple closed curves, $\kappa_1, \ldots, \kappa_m$. For each $\kappa_j$, we use Thurston’s Hyperbolic Dehn Surgery Theorem \[1\] to choose an $r_j$ in the same manner that we chose $r$ after we fixed a particular simple closed curve $\kappa$. Now let $R = \max\{r_1, \ldots, r_m\}$, and let $R$ be the value of $r$ in Figure \[2\]. This determines the simple closed curves $C_1, \ldots, C_n$.

Let $P = \text{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J)))$ where $V$ and $J$ are given in our preliminary construction. Then the embedded graph is such that $\Gamma_1 \subseteq P$. For each $j = 1, \ldots, m$, let $H_j = \text{cl}(P - N(\kappa_j))$. It follows from Proposition \[1\] that each $H_j$ contains no essential sphere or torus. Since each $H_j$ has more than three boundary components, no $H_j$ can be Seifert fibered. Hence by Thurston’s Hyperbolization Theorem \[4\], every $H_j$ is a hyperbolic manifold.

We will glue solid tori $Y_1, \ldots, Y_{n+2}$ to $P$ along its $n+2$ boundary components $\partial V$, $\partial N(C_1)$, $\ldots$, $\partial N(C_n)$, and $\partial N(J)$ to obtain a closed manifold $\overline{P}$ as follows. For each $j$, any gluing of solid tori along the boundary components of $P$ defines a Dehn filling of $H_j = \text{cl}(P - N(\kappa_j))$ along all of its boundary components except $\partial N(\kappa_j)$. Since each $H_j$ is hyperbolic, by Thurston’s Hyperbolic Dehn Surgery Theorem \[1\], \[3\], all but finitely many such Dehn fillings of $H_j$ result in a hyperbolic 3-manifold. Furthermore, since $P$ is obtained by removing solid tori from $S^3$, for any integer $q$, if we attach the solid tori $Y_1, \ldots, Y_{n+2}$ to $P$ with slope $\frac{1}{q}$, then $\overline{P} = S^3$. In this case each $H_j \cup Y_1 \cup \cdots \cup Y_{n+2}$ is the complement of a knot in $S^3$. There are only finitely many $H_j$’s, and for each $j$, only finitely many slopes $\frac{1}{q}$ are excluded by Thurston’s Hyperbolic Dehn Surgery Theorem. Thus there is some integer $q$ such that if we glue the solid tori $Y_1, \ldots, Y_{n+2}$ to any of the $H_j$ along $\partial N(C_1)$, $\ldots$, $\partial N(C_n)$, $\partial N(J)$, and $\partial V$ with slope $\frac{1}{q}$, then we obtain the complement of a hyperbolic knot in $S^3$.

Let $\Gamma_2$ denote the re-embedding of $\Gamma_1$ obtained as a result of gluing the solid tori $Y_1, \ldots, Y_{n+2}$ to the boundary components of $P$ with slope $\frac{1}{q}$. Now $\Gamma_2$ is an embedding of $K_n$ in $S^3$ such that every non-trivial knot in $\Gamma_2$ is hyperbolic. Now there is a minor $G'$ of the embedded graph $\Gamma_2$, which is an embedding of our original graph $G$, such that every non-trivial knot in $G'$ is hyperbolic. \[\square\]

References


Department of Mathematics, Pomona College, 610 North College Avenue, Claremont, California 91711-6348

Department of Mathematics, Wake Forest University, P.O. Box 7388, Winston-Salem, North Carolina 27109-7388