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ATTRACTIVITY FOR TWO-DIMENSIONAL LINEAR SYSTEMS WHOSE ANTI-DIAGONAL COEFFICIENTS ARE PERIODIC

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ABSTRACT. This paper deals with the linear system $\mathbf{x}' = A(t)\mathbf{x}$ with $A(t)$ being a 2×2 matrix. The anti-diagonal components of $A(t)$ are assumed to be periodic, but the diagonal components are not necessarily periodic. Our concern is to establish sufficient conditions for the zero solution to be attractive. Floquet theory is of no use in solving our problem, because not all components are periodic. Another approach is adopted. Some simple examples are included to illustrate the main result.

1. INTRODUCTION

We consider the linear system

$$\mathbf{x}' = A(t)\mathbf{x} = \begin{pmatrix} -\,r(t) & p(t) \\ -\,p(t) & -\,q(t) \end{pmatrix}\mathbf{x}, \tag{1}$$

where the prime denotes d/dt; the coefficients $p(t)$, $q(t)$ and $r(t)$ are continuous for $t \geq 0$, and $p(t)$ is a periodic function with period $\omega > 0$. The coefficients $q(t)$ and $r(t)$ are not always assumed to be periodic. Since system (1) has such a simple form, it has broad applications to science and engineering.

It is well-known that the zero solution of (1) is asymptotically stable if it is *attractive*; that is, every solution $\mathbf{x}(t)$ of (1) tends to $\mathbf{0} \in \mathbb{R}^2$ as $t \to \infty$. The purpose of this paper is to give sufficient conditions on $p(t)$, $q(t)$ and $r(t)$ which guarantee the attractivity of the zero solution of (1).

Floquet's theorem is available for the special case where $q(t)$ and $r(t)$ are also periodic functions with period ω. Let $\Phi(t)$ be the fundamental matrix of (1) with $\Phi(0) = E$, the 2×2 identity matrix. Then $\Phi(\omega)$ is called the *monodromy matrix* of (1). Let μ_1 and μ_2 be the eigenvalues of the monodromy matrix $\Phi(\omega)$. The eigenvalues μ_1 and μ_2 are often called the Floquet multipliers of (1). By Abel's formula,

$$\det\Phi(\omega) = \det\Phi(0)\exp\biggl(-\int_0^\omega (q(s)+r(s))ds\biggr) = \exp\biggl(-\int_0^\omega (q(s)+r(s))ds\biggr).$$

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Thus, the Floquet multipliers μ_1 and μ_2 are the roots of the equation

$$\mu^2 - \mathrm{tr}\Phi(\omega)\mu + \exp\left(-\int_0^\omega (q(s) + r(s))ds\right) = 0.$$

It follows from Floquet's theorem that the zero solution of (1) with periodic coefficients $q(t)$ and $r(t)$ is attractive if and only if the Floquet multipliers μ_1 and μ_2 have magnitude strictly less than 1. Hence, in this special case, necessary and sufficient conditions for the zero solution of (1) to be attractive are that

$$|\mathrm{tr}\Phi(\omega)| < 1 + \exp\left(-\int_0^\omega (q(s) + r(s))ds\right)$$

and

$$\exp\left(-\int_0^\omega (q(s) + r(s))ds\right) < 1.$$

For example, we can find Floquet's theorem in the books [2, 3, 5, 8, 16].

Although the above conditions are necessary and sufficient for the zero solution of (1) to be attractive, it is difficult to estimate the absolute value of the trace of $\Phi(\omega)$, because it is impossible to find a fundamental matrix of (1) in general. Of course, Floquet's theorem is useless when $q(t)$ or $r(t)$ is not periodic. Then, without knowledge of a fundamental matrix of (1), can we decide whether the zero solution is attractive? What kind of condition on $A(t)$ will guarantee the attractivity of the zero solution of (1)?

We give an answer to our question in Sections 2 and 3. In Section 2, we state the main result and present some preparatory lemmas. In Section 3, we give the proof of the main result. To illustrate our main result, we take some concrete examples and exhibit positive orbits of (1) in Section 4. In addition, we mention the approach via Floquet theory.

2. Some lemmas

Let

$$R(t) = \int_0^t r(s)ds \quad \text{and} \quad \psi(t) = 2(q(t) - r(t))$$

for $t \geq 0$. For the sake of convenience, we write

$$\psi_+(t) = \max\{0, \psi(t)\} \quad \text{and} \quad \psi_-(t) = \max\{0, -\psi(t)\}.$$

Note that $\psi(t) = \psi_+(t) - \psi_-(t)$ and $|\psi(t)| = \psi_+(t) + \psi_-(t)$. If $r(t) \equiv 0$ and $p(t) \equiv k > 0$, then $\psi(t) = 2q(t)$ and system (1) is equivalent to the damped linear oscillator of one degree of freedom,

$$x'' + q(t)x' + k^2x = 0. \tag{2}$$

It is clear that the equilibrium $(x, x') = (0, 0)$ of (2) corresponds to the zero solution of (1). It is well-known that the divergence of an indefinite integral of $q(t)$ is not sufficient to guarantee that the equilibrium of (2) will be attractive. For this reason, it is natural to make a stronger assumption on $\psi(t)$.

We introduce an important concept here. A nonnegative function $\phi(t)$ is said to be *weakly integrally positive* if

$$\int_I \phi(t)dt = \infty$$

for every set $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$ such that $\tau_n + \delta < \sigma_n < \tau_{n+1} < \sigma_n + \Delta$ for some $\delta > 0$ and $\Delta > 0$. For example, $1/(1+t)$ and $\sin^2 t/(1+t)$ are weakly integrally positive functions (see [6, 7, 13, 14, 15]).

Our main result is as follows:

Theorem 1. *Suppose that $q(t)$ and $R(t)$ are bounded for $t \geq 0$. Suppose also that*

(i) *$\psi_+(t)$ is weakly integrally positive;*

(ii) $\displaystyle\int_0^\infty \psi_-(t)dt < \infty.$

Then the zero solution of (1) *is attractive.*

Before proving our result, we present some lemmas.

Lemma 2. *Suppose that assumption* (ii) *in Theorem* 1 *holds. Let $v(t)$ be nonnegative and continuously differentiable on $[t_0, \infty)$ for some $t_0 > 0$. If*

$$v'(t) \leq \psi_-(t)v(t) \quad \textit{for } t \geq t_0, \tag{3}$$

then $v'(t)$ is absolutely integrable, and therefore $v(t)$ has a nonnegative limiting value.

Proof. By (3), we have

$$\left(v(t)\exp\left(-\int_{t_0}^t \psi_-(s)ds\right)\right)' = (v'(t) - \psi_-(t)v(t))\exp\left(-\int_{t_0}^t \psi_-(s)ds\right) \leq 0$$

for $t \geq t_0$. Integrating this inequality from t_0 to t, we obtain

$$v(t) \leq v(t_0)\exp\left(\int_{t_0}^t \psi_-(s)ds\right) \quad \text{for } t \geq t_0.$$

Hence, using (3) again, we get

$$v'(t) \leq v(t_0)\exp\left(\int_{t_0}^t \psi_-(s)ds\right)\psi_-(t) \quad \text{for } t \geq t_0.$$

It follows from assumption (ii) that

$$v'(t) \leq v(t_0)\exp\left(\int_{t_0}^\infty \psi_-(s)ds\right)\psi_-(t) \quad \text{for } t \geq t_0.$$

Since the right-hand side of the above inequality is positive for $t \geq t_0$, we see that

$$(v')_+(t) \leq v(t_0)\exp\left(\int_{t_0}^\infty \psi_-(s)ds\right)\psi_-(t).$$

Consequently,

$$\int_{t_0}^\infty (v')_+(s)ds \leq v(t_0)\exp\left(\int_{t_0}^\infty \psi_-(s)ds\right)\int_{t_0}^\infty \psi_-(s)ds < \infty.$$

On the other hand, since $v(t) \geq 0$ for $t \geq t_0$, we get

$$\int_{t_0}^\infty (v')_-(s)ds = \int_{t_0}^\infty (v')_+(s)ds - \int_{t_0}^\infty v'(s)ds \leq \int_{t_0}^\infty (v')_+(s)ds + v(t_0) < \infty.$$

Hence, we obtain

$$\int_{t_0}^{\infty} |v'(s)| ds = \int_{t_0}^{\infty} \big((v')_+(s) + (v')_-(s)\big) ds < \infty.$$

Since $v(t)$ is nonnegative for $t \geq t_0$ and $v'(t)$ is absolutely integrable, it turns out that $v(t)$ has a limiting value $v_0 \geq 0$. This completes the proof of Lemma 2. □

Using a classical Lyapunov's direct method, we can prove that all solutions of (1) are *uniformly bounded*; that is, for any $\alpha > 0$, there exists a $\beta(\alpha) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \alpha$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \beta$ for all $t \geq t_0$. For details about the direct method of Lyapunov, see the books [1, 2, 4, 5, 9, 10, 11, 12, 17, 18, 19], for example.

Lemma 3. *Suppose that* $R(t)$ *is bounded for* $t \geq 0$. *If assumption* (ii) *in Theorem* 1 *holds, then all solutions of* (1) *are uniformly bounded.*

Proof. Let $\mathbf{x} = (x, y)$ and define two Lyapunov functions

$$V(t, \mathbf{x}) = \frac{1}{2} e^{2R(t)} \big(x^2 + y^2\big)$$

and

$$U(t, \mathbf{x}) = V(t, \mathbf{x}) \exp\left(-\int_0^t \psi_-(s) ds\right)$$

on $[0, \infty) \times \mathbb{R}^2$. From the boundedness of $R(t)$, we can choose an $L > 0$ such that $|R(t)| < L$ for $t \geq 0$. Let

$$M = \int_0^{\infty} \psi_-(s) ds$$

(because of assumption (ii), such an M exists). Then, we have

$$\frac{1}{2} e^{-(2L+M)} \big(x^2 + y^2\big) \leq V(t, \mathbf{x}) e^{-M} \leq U(t, \mathbf{x}) \leq V(t, \mathbf{x}) \leq \frac{1}{2} e^{2L} \big(x^2 + y^2\big).$$

Thus, $U(t, \mathbf{x})$ tends to ∞ as $\|\mathbf{x}\| \to \infty$ uniformly for $t \geq 0$ (i.e. it is radially unbounded), and it is decrescent. Differentiate $V(t, \mathbf{x})$ along any solution of (1) to obtain

$$\dot{V}_{(1)}(t, \mathbf{x}) = -((q(t) - r(t)) e^{2R(t)} y^2 \leq \psi_-(t) V(t, \mathbf{x})$$

on $[0, \infty) \times \mathbb{R}^2$. Hence, we have

$$\dot{U}_{(1)}(t, \mathbf{x}) = \Big\{\dot{V}_{(1)}(t, \mathbf{x}) - \psi_-(t) V(t, \mathbf{x})\Big\} \exp\left(-\int_0^t \psi_-(s) ds\right) \leq 0.$$

We therefore conclude that all solutions of (1) are uniformly bounded by using a Lyapunov-type theorem due to Yoshizawa [17, 18, 19]. □

Remark 1. Using the same Lyapunov function $U(t, \mathbf{x})$, we can prove that the zero solution of (1) is uniformly stable.

Recall that $p(t)$ is a periodic function with period $\omega > 0$. Let

$$\overline{p} = \max_{t \in [0, \omega]} p(t) \quad \text{and} \quad \underline{p} = \min_{t \in [0, \omega]} p(t).$$

Taking $\overline{p} \geq \underline{p}$ into account, we see that if $\overline{p} + \underline{p} \geq 0$, then $\overline{p} > 0$; if $\overline{p} + \underline{p} < 0$, then $\underline{p} < 0$. Since $p(t)$ is continuous for $t \geq 0$, we see that $p(t)$ has the following property (we omit the proof).

Lemma 4. *Suppose that $p(t)$ is a nontrivial periodic function with period $\omega > 0$. If $\overline{p} + \underline{p} \geq 0$, then there exist numbers a and b with $0 \leq a < b \leq \omega$ such that*

$$p(t) \geq \frac{1}{2}\overline{p} > 0 \quad \textit{for } a \leq t \leq b.$$

If $\overline{p} + \underline{p} < 0$, then there exist numbers a and b with $0 \leq a < b \leq \omega$ such that

$$p(t) \leq \frac{1}{2}\underline{p} < 0 \quad \textit{for } a \leq t \leq b.$$

Remark 2. Let m be any integer. Since $p(t)$ is a periodic function with period $\omega > 0$, it turns out that if $\overline{p} + \underline{p} \geq 0$, then

$$p(t) \geq \frac{1}{2}\overline{p} > 0 \quad \text{for } a + m\omega \leq t \leq b + m\omega;$$

if $\overline{p} + \underline{p} < 0$, then

$$p(t) \leq \frac{1}{2}\overline{p} < 0 \quad \text{for } a + m\omega \leq t \leq b + m\omega.$$

3. Proof of the main result

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $\mathbf{x}(t; t_0, \mathbf{x}_0)$ be a solution of (1) passing through $(t_0, \mathbf{x}_0) \in [0, \infty) \times \mathbb{R}^2$. It follows from Lemma 3 that for any $\alpha > 0$, there exists a $\beta(\alpha) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \alpha$ imply that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \beta \quad \text{for } t \geq t_0. \tag{4}$$

For the sake of brevity, we write $(x(t), y(t)) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and

$$v(t) = V(t, x(t), y(t)).$$

Then, we have

$$v(t) = \frac{1}{2}e^{2R(t)}\big(x^2(t) + y^2(t)\big) \tag{5}$$

and

$$v'(t) = -((q(t) - r(t))e^{2R(t)}y^2 \leq \psi_-(t)v(t) \tag{6}$$

for $t \geq t_0$ (see the calculation of $\dot{V}_{(1)}(t, \mathbf{x})$ in the proof of Lemma 2). Hence, from Lemma 2, we see that $v(t)$ has a limiting value $v_0 \geq 0$. If $v_0 = 0$, then by (5) the solution $(x(t), y(t))$ tends to $\mathbf{0}$ as $t \to \infty$. This completes the proof. Thus, we need consider only the case in which $v_0 > 0$. We will show that this case does not occur.

Because of (4), we see that $|y(t)|$ is bounded for $t \geq t_0$. Hence, $|y(t)|$ has an inferior limit and a superior limit. First, we will show that the inferior limit of $|y(t)|$ is zero, and we will then show that the superior limit of $|y(t)|$ is also zero.

Suppose that $\liminf_{t\to\infty} |y(t)| > 0$. Then, there exist a $\gamma > 0$ and a $T_1 \geq t_0$ such that $y(t) > \gamma$ for $t \geq T_1$. It follows from (6) and Lemma 2 that

$$\begin{aligned}\infty > \int_{t_0}^{\infty} |v'(s)| ds &= \frac{1}{2}\int_{t_0}^{\infty} |\psi(s)| e^{2R(s)} y^2(s) ds \\ &\geq \frac{1}{2}\gamma^2 \int_{T_1}^{\infty} \psi_+(s) e^{2R(s)} ds \geq \frac{1}{2}\gamma^2 e^{-2L} \int_{T_1}^{\infty} \psi_+(s) ds,\end{aligned}$$

where L is the number given in the proof of Lemma 3. This contradicts assumption (i). Thus, we see that $\liminf_{t\to\infty} |y(t)| = 0$.

Suppose that $\limsup_{t\to\infty}|y(t)| > 0$. Let $\nu = \limsup_{t\to\infty}|y(t)|$. Since $q(t)$ is bounded, we can find a $\overline{q} > 0$ such that

$$|q(t)| \le \overline{q} \quad \text{for } t \ge 0. \tag{7}$$

Since $v(t)$ tends to a positive value v_0 as $t \to \infty$, there exists a $T_2 \ge t_0$ such that

$$0 < \frac{1}{2}v_0 < v(t) < \frac{3}{2}v_0 \quad \text{for } t \ge T_2. \tag{8}$$

Let ε be so small that

$$0 < \varepsilon < \min\left\{\frac{1}{2}\nu,\ \sqrt{\frac{\overline{p}e^{-2L}v_0}{4(\overline{q}+2/(b-a))^2+\overline{p}^2}},\ \sqrt{\frac{\underline{p}e^{-2L}v_0}{4(\overline{q}+2/(b-a))^2+\underline{p}^2}}\right\}, \tag{9}$$

where a and b are the numbers given in Lemma 4. Then, since $\liminf_{t\to\infty}|y(t)| = 0$, we can select two intervals $[\tau_n, \sigma_n]$ and $[t_n, s_n]$ with $[t_n, s_n] \subset [\tau_n, \sigma_n]$, $T_2 < \tau_n$ and $\tau_n \to \infty$ as $n \to \infty$ such that $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$, $|y(t_n)| = \nu/2$, $|y(s_n)| = 3\nu/4$ and

$$|y(t)| \ge \varepsilon \quad \text{for } \tau_n < t < \sigma_n, \tag{10}$$

$$0 \le |y(t)| \le \varepsilon \quad \text{for } \sigma_n < t < \tau_{n+1}, \tag{11}$$

$$\frac{1}{2}\nu < |y(t)| < \frac{3}{4}\nu \quad \text{for } t_n < t < s_n. \tag{12}$$

By (5), (8) and (11), we have

$$|x(t)| = \sqrt{2e^{-2R(t)}v(t) - y^2(t)} \ge \sqrt{e^{-2L}v_0 - \varepsilon^2} \tag{13}$$

for $\sigma_n \le t \le \tau_{n+1}$.

Claim. The sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfy $\tau_{n+1} - \sigma_n \le 2\omega$ for any integer n.

Suppose that there exists an $n_0 \in \mathbb{N}$ such that $\tau_{n_0+1} - \sigma_{n_0} > 2\omega$. We can choose an $m \in \mathbb{N}$ such that $(m-1)\omega < \sigma_{n_0} \le m\omega$. Hence, we have

$$\tau_{n_0+1} > \sigma_{n_0} + 2\omega > (m-1)\omega + 2\omega = (m+1)\omega,$$

and therefore $[m\omega, (m+1)\omega] \subset [\sigma_{n_0}, \tau_{n_0+1}]$. There are two cases to consider: (a) $\overline{p}+\underline{p} \ge 0$ and (b) $\overline{p}+\underline{p} < 0$. In case (a), by Lemma 4 and Remark 2, $p(t) \ge \overline{p}/2 > 0$ for $t \in [a+m\omega, b+m\omega] \subset [m\omega, (m+1)\omega]$. Hence, using the second equation in system (1) with (7), (11) and (13), we have

$$|y'(t)| \ge |p(t)||x(t)| - |q(t)||y(t)| = \frac{1}{2}\overline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon \tag{14}$$

for $a + m\omega < t < b + m\omega$. It follows from (9) that

$$\frac{1}{2}\overline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon > \frac{2}{b-a}\varepsilon. \tag{15}$$

From (11) and (14), we can estimate that

$$\begin{aligned}2\varepsilon \ge |y(b+m\omega)| + |y(a+m\omega)| &\ge \left|\int_{a+m\omega}^{b+m\omega} y'(s)ds\right| \\ &= \int_{a+m\omega}^{b+m\omega}|y'(s)|ds \ge (b-a)\left(\frac{1}{2}\overline{p}\sqrt{e^{-2L}v_0-\varepsilon^2} - \overline{q}\varepsilon\right).\end{aligned}$$

This contradicts (15). In case (b), by Lemma 4 and Remark 2, $p(t) \le \underline{p}/2 < 0$ for $t \in [a + m\omega, b + m\omega] \subset [m\omega, (m+1)\omega]$. Hence, combining this with (7), (11) and (13), we obtain

$$|y'(t)| \ge |p(t)||x(t)| - |q(t)||y(t)| = -\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon \tag{16}$$

for $a + m\omega < t < b + m\omega$. It follows from (9) that

$$-\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon > \frac{2}{b-a}\varepsilon. \tag{17}$$

From (11) and (16), we can estimate that

$$\begin{aligned} 2\varepsilon &\ge |y(b + m\omega)| + |y(a + m\omega)| \ge \left| \int_{a+m\omega}^{b+m\omega} y'(s)ds \right| \\ &= \int_{a+m\omega}^{b+m\omega} |y'(s)|ds \ge (b-a)\left(-\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon \right). \end{aligned}$$

This contradicts (17). Thus, the claim is proved.

Let $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$. Then, by means of Lemma 2 with (6) and (10), we get

$$\begin{aligned} \infty > \int_{t_0}^{\infty} |v'(s)|ds &= \frac{1}{2}\int_{t_0}^{\infty} |\psi(s)|e^{2R(s)}y^2(s)ds \\ &\ge \frac{1}{2}e^{-2L}\int_{t_0}^{\infty} \psi_+(s)y^2(s)ds \ge \frac{1}{2}\varepsilon^2 e^{-2L}\int_I \psi_+(s)ds. \end{aligned}$$

Hence, it follows from assumption (i) and the Claim that $\liminf_{n\to\infty}(\sigma_n - \tau_n) = 0$. Since $[t_n, s_n] \subset [\tau_n, \sigma_n]$, it follows that

$$\liminf_{n\to\infty}(s_n - t_n) = 0. \tag{18}$$

By (5), (8) and (12), we have

$$|x(t)| = \sqrt{2e^{-2R(t)}v(t) - y^2(t)} \le \sqrt{3e^{2L}v_0 - \frac{\nu^2}{4}}$$

for $t_n \le t \le s_n$. Let $K = \max\{|\overline{p}|, |\underline{p}|\}$. Then, from (7) and (12), we see that

$$|y'(t)| \le |p(t)||x(t)| + |q(t)||y(t)| < K\sqrt{3e^{2L}v_0 - \frac{\nu^2}{4}} + \frac{3}{4}\overline{q}\nu$$

for $t_n \le t \le s_n$. Letting $N = K\sqrt{3e^{2L}v_0 - \nu^2/4} + 3\overline{q}\nu/4$ and integrating this inequality from t_n to s_n, we obtain

$$\begin{aligned} \frac{1}{4}\nu &= |y(s_n)| - |y(t_n)| \le |y(s_n) - y(t_n)| \\ &= \left| \int_{t_n}^{s_n} y'(s)ds \right| \le \int_{t_n}^{s_n} |y'(s)|ds \le N(s_n - t_n). \end{aligned}$$

This contradicts (18). We therefore conclude that $\limsup_{t\to\infty} |y(t)| = \nu = 0$.

In summary, $y(t)$ tends to zero as $t \to \infty$. Hence, there exists a $T_3 \ge T_2$ such that

$$|y(t)| < \varepsilon \quad \text{for } t \ge T_3. \tag{19}$$

Let l be an integer satisfying $l\omega > T_3$. Using (19) instead of (11) and following the same process as in the proof of the Claim, we see that if $\overline{p} + \underline{p} \geq 0$, then

$$2\varepsilon \geq |y(b+l\omega)| + |y(a+l\omega)| \geq \left| \int_{a+l\omega}^{b+l\omega} y'(s)ds \right|$$
$$= \int_{a+l\omega}^{b+l\omega} |y'(s)|ds \geq (b-a)\left(\frac{1}{2}\overline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon\right) > 2\varepsilon,$$

which is a contradiction; if $\overline{p} + \underline{p} < 0$, then

$$2\varepsilon \geq |y(b+l\omega)| + |y(a+l\omega)| \geq \left| \int_{a+l\omega}^{b+l\omega} y'(s)ds \right|$$
$$= \int_{a+l\omega}^{b+l\omega} |y'(s)|ds \geq (b-a)\left(-\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon\right) > 2\varepsilon,$$

which is again a contradiction. Thus, the case of $v_0 > 0$ cannot happen.

The proof of Theorem 1 is thus complete. □

4. Examples

We illustrate our main result with simple examples in which $p(t)$, $q(t)$ and $r(t)$ are periodic. It is well-known that if the zero solution of a linear periodic system is attractive, then it is uniformly asymptotically stable (for example, see [5, 18]).

Example 1. Let $\lambda > 0$. Consider system (1) with

$$p(t) = \cos t, \quad q(t) = \frac{\lambda}{2 - \sin t} \quad \text{and} \quad r(t) = 0. \tag{20}$$

Then the zero solution is attractive.

Since $\lambda/3 \leq q(t) \leq \lambda$ and $R(t) \equiv 0$, it is clear that $q(t)$ and $R(t)$ are bounded for $t \geq 0$. Also, assumptions (i) and (ii) are satisfied. In fact, we have

$$\psi(t) = 2(q(t) - r(t)) = \frac{2\lambda}{2 - \sin t},$$

and therefore

$$\psi_+(t) = \frac{2\lambda}{2 - \sin t} \quad \text{and} \quad \psi_-(t) = 0$$

for $t \geq 0$. Hence, $\psi_+(t)$ is weakly integrally positive and

$$\int_0^\infty \psi_-(t)dt = 0.$$

Thus, by means of Theorem 1, we conclude that the zero solution is attractive.

Figure 1(a) shows a positive orbit of (1) with (20) and $\lambda = 0.1$. The starting point $\mathbf{x}_0$ is $(-1, 0)$ and the initial time t_0 is 0. The positive orbit moves around the origin $\mathbf{0}$ in a clockwise and a counter-clockwise direction alternately, because $p(t)$ changes its sign. The positive orbit approaches the origin $\mathbf{0}$ as it goes up and down.

Example 2. Let $\lambda \geq 1$. Consider system (1) with

$$p(t) = \cos \lambda t, \quad q(t) = \cos^2 t + \sin t \quad \text{and} \quad r(t) = \sin t. \tag{21}$$

Then the zero solution is attractive.

It is easy to check that $q(t)$ and $R(t)$ are bounded for $t \geq 0$ and that assumptions (i) and (ii) are satisfied. We omit the details.

In Figure 1(b), we show a positive orbit of (1) with (21) and $\lambda = 4$. The positive orbit starts from the point $(-1, 0)$ at the initial time 0. The positive orbit goes to the right and then goes to the left, and it repeats such a movement regularly. Although the positive orbit displays intricate behavior, it approaches the origin $\mathbf{0}$ ultimately.

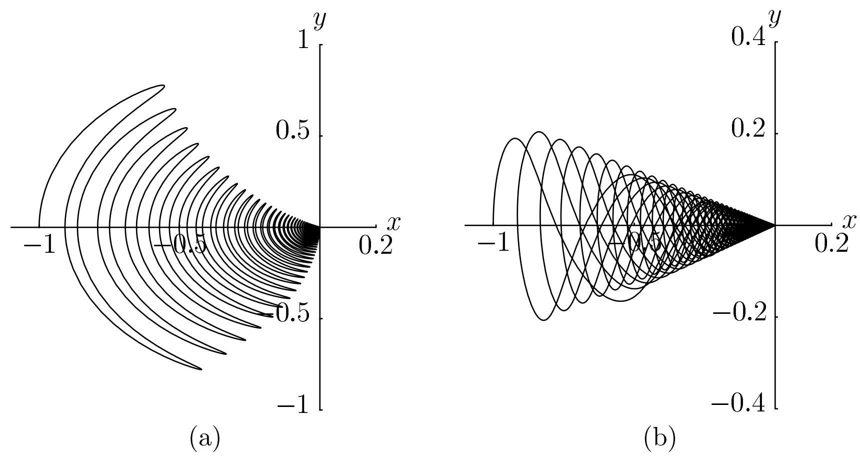

FIGURE 1. (a) A positive orbit of (1) with (20); (b) a positive orbit of (1) with (21)

In Examples 1 and 2, all coefficients of (1) are periodic functions with period 2π. However, we cannot find the monodromy matrix $\Phi(2\pi)$. It is particularly hard to estimate the absolute value of the trace of $\Phi(2\pi)$. For this reason, we cannot apply Floquet's theorem to Examples 1 and 2 directly. Theorem 1 has the advantage of being applicable to cases where the monodromy matrix of (1) cannot be found and cases where $q(t)$ or $r(t)$ is not periodic.

λ	μ_1	μ_2
1	0.3351718550789	0.0793024028529
0.1	0.8888872982404	0.7827240687567
0.01	0.9882826823640	0.9758079535053
0.001	0.9988220356864	0.9975540561378

TABLE 1. Floquet multipliers of (1) with (20)

λ	μ_1	μ_2
1	0.5569470757759	0.0775907086028
10	0.9845517600942	0.0438919719768
100	0.9998429464892	0.0432207062297
1000	0.9999986933319	0.0432139974342

TABLE 2. Floquet multipliers of (1) with (21)

Fortunately, in Examples 1 and 2, the Floquet multipliers μ_1 and μ_2 can be calculated by a numerical scheme. As shown in Tables 1 and 2, $|\mu_1| < 1$ and $|\mu_2| < 1$. Hence, we see that the zero solution of (1) is attractive.

Remark 3. The zero solution of system (1) with (20) is attractive if and only if $\lambda > 0$. In fact, if $\lambda \le 0$, then

$$\exp\left(-\int_0^\omega (q(s)+r(s))ds\right) = \exp\left(-\int_0^\omega \frac{\lambda}{2-\sin t}ds\right) \ge 1.$$

Hence, as mentioned in Section 1, the zero solution is not attractive in this case.

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