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THE REVERSE ULTRA LOG-CONCAVITY OF THE BOROS-MOLL POLYNOMIALS

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ABSTRACT. We prove the reverse ultra log-concavity of the Boros-Moll polynomials. We further establish an inequality which implies the log-concavity of the sequence $\{i!d_i(m)\}$ for any $m \geq 2$, where $d_i(m)$ are the coefficients of the Boros-Moll polynomials $P_m(a)$. This inequality also leads to the fact that in the asymptotic sense, the Boros-Moll sequences are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We propose two conjectures on the log-concavity and reverse ultra log-concavity of the sequence $\{d_{i-1}(m)d_{i+1}(m)/d_i(m)^2\}$ for $m \geq 2$.

1. INTRODUCTION

This paper is concerned with the reverse ultra log-concavity of the Boros-Moll polynomials. A sequence $\{a_k\}_{k\geq 0}$ of real numbers is said to be log-concave if $a_k^2 \geq a_{k+1}a_{k-1}$ holds for all $k \geq 1$. A polynomial is said to be log-concave if the sequence of its coefficients is log-concave; see Brenti [4] and Stanley [11]. Furthermore, a sequence $\{a_k\}_{0\leq k\leq n}$ is called ultra log-concave if $\left\{a_k/\binom{n}{k}\right\}$ is log-concave; see Liggett [8]. This condition can be restated as

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \tag{1.1}$$

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. As noticed by Liggett [8], if a sequence $\{a_k\}_{0\leq k\leq n}$ is ultra log-concave, then the sequence $\{k!a_k\}_{0\leq k\leq n}$ is log-concave.

A sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (1.1), that is,

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0. \tag{1.2}$$

For example, it is easy to verify that for $n \geq 2$, the Bessel polynomial [6]

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^k$$

is log-concave and reverse ultra log-concave.

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The Boros-Moll polynomials, denoted by $P_m(a)$, arise in the following evaluation of a quartic integral:

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a),$$

where

$$P_m(a) = 2^{-2m}\sum_k 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k; \tag{1.3}$$

see [1, 2, 3, 9]. Write

$$P_m(a) = \sum_{i=0}^m d_i(m)a^i.$$

The sequence $\{d_i(m)\}_{0\le i\le m}$ is called a Boros-Moll sequence. The expression (1.3) gives the following formula for the coefficients $d_i(m)$:

$$d_i(m) = 2^{-2m}\sum_{k=0}^m 2^k\binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{i}.$$

Clearly, the coefficients $d_i(m)$ are positive. Moll conjectured that the sequence $\{d_i(m)\}_i$ is log-concave for $m\ge 2$, that is, $d_i(m)^2 \ge d_{i-1}(m)d_{i+1}(m)$ $(1\le i\le m-1)$. This conjecture has been proved by Kauers and Paule [7].

Despite the log-concavity of $\{d_i(m)\}$, we find that inverse ultra log-concavity holds.

Theorem 1.1. *For $m\ge 2$ and $1\le i\le m-1$, we have*

$$\left(\frac{d_{i-1}(m)}{\binom{m}{i-1}}\right)\cdot\left(\frac{d_{i+1}(m)}{\binom{m}{i+1}}\right) > \left(\frac{d_i(m)}{\binom{m}{i}}\right)^2 \tag{1.4}$$

or, equivalently,

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i}. \tag{1.5}$$

On the other hand, it can be shown that the coefficients $d_i(m)$ satisfy an inequality stronger than log-concavity. To be more specific, we will give a lower bound on $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$ which is very close to the upper bound in (1.5).

Theorem 1.2. *For $m\ge 2$ and $1\le i\le m-1$, we have*

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} > \frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)}. \tag{1.6}$$

This paper is organized as follows. We establish an upper bound for $d_i(m+1)/d_i(m)$ in Section 2, which leads to the reverse ultra log-concavity of $\{d_i(m)\}$. In Section 4 we give the proof of Theorem 1.2. We conclude this paper with two conjectures concerning the log-concavity and reverse ultra log-concavity of the sequence $\{d_{i-1}(m)d_{i+1}(m)/d_i^2(m)\}$ for $m\ge 2$.

2. An upper bound for $d_i(m+1)/d_i(m)$

In this section, we establish an upper bound for the ratio $d_i(m+1)/d_i(m)$ that will lead to the reverse ultra log-concavity of the sequence of $\{d_i(m)\}$. For $m \geq 1$ and $0 \leq i \leq m$, set

$$T(m,i) = \frac{4m^2+7m+3+i\sqrt{4m+4i^2+1}-2i^2}{2(m-i+1)(m+1)}. \tag{2.1}$$

Theorem 2.1. *For all $m \geq 2$, $1 \leq i \leq m-1$, we have*

$$\frac{d_i(m+1)}{d_i(m)} < T(m,i), \tag{2.2}$$

and for $m \geq 1$, we have

$$\frac{d_0(m+1)}{d_0(m)} = T(m,0), \qquad \frac{d_m(m+1)}{d_m(m)} = T(m,m). \tag{2.3}$$

The following lemma will be needed in the proof of Theorem 2.1.

Lemma 2.2. *For $m \geq 2$ and $1 \leq i \leq m-1$,*

$$T(m,i) < F(m,i), \tag{2.4}$$

where

$$F(m,i) = \frac{(m+i+1)(4m+3)(4m+5)}{2(m+1)(4m^2-2i^2+9m+5-i\sqrt{4m+4i^2+5})}.$$

Proof. Let $A = \sqrt{4m+4i^2+1}$ and $B = \sqrt{4m+4i^2+5}$. It is easy to check that

$$F(m,i) - T(m,i) = \frac{i(X-Y)}{2(m+1)(m-i+1)(4m^2+9m+5-2i^2-iB)}, \tag{2.5}$$

where

$$X = (i-4i^3) + iAB,$$

$$Y = (5+4m^2+9m-2i^2)A - (3+4m^2+7m-2i^2)B.$$

Since $(4m^2+9m+5-2i^2)^2 - (iB)^2 = (4m+5)^2(m+i+1)(m-i+1) > 0$, it remains to show that the numerator of (2.5) is also positive. We claim that $X > 0$ and $X^2 > Y^2$.

Since $m > i$, we have $A > 2i+1$ and $B > 2i+1$. Moreover, since $i \geq 1$, we find that

$$X = (i-4i^3) + iAB \geq i - 4i^3 + i(2i+1)^2 = 4i^2 + 2i > 0.$$

It is routine to check that $X^2 - Y^2 = G(m,i) - H(m,i)$, where

$$\begin{aligned}
G(m,i) &= (32m^4 - 32m^2i^2 + 128m^3 - 64mi^2 + 190m^2 - 30i^2 + 124m + 30)AB,\\
H(m,i) &= 128m^5 + 608m^4 + 1128m^3 + 1014m^2 + 436m + 128m^4i^2 + 384m^3i^2\\
&\quad + 408m^2i^2 - 128m^2i^4 + 200mi^2 - 256mi^4 - 120i^4 + 50i^2 + 70.
\end{aligned}$$

Since $i < m$, it is easily seen that $G(m,i) > 0$ and $H(m,i) > 0$. To prove that $G(m,i) > H(m,i)$, it suffices to show that $G(m,i)^2 > H(m,i)^2$. In fact, for $1 \leq i \leq m-1$,

$$G(m,i)^2 - H(m,i)^2 = 16(4m+5)^2(16mi^2+12i^2-1)(m+i+1)^2(m-i+1)^2 > 0.$$

This yields $X^2 > Y^2$. Since $X > 0$, we see that $X > Y$, and hence (2.4) holds for $1 \leq i \leq m-1$. $\square$

Proof of Theorem 2.1. It is easy to check (2.3). To prove (2.2), we proceed by induction on m. For $m = 2$ and $i = 1$, we have $d_1(3)/d_1(2) = 43/15 < T(2,1) = (31+\sqrt{13})/12$. We now assume that (2.2) is true for m, that is,

$$d_i(m+1) < T(m,i)d_i(m), \quad 1 \le i \le m-1. \tag{2.6}$$

It will be shown that

$$d_i(m+2) < T(m+1,i)d_i(m+1), \quad 1 \le i \le m-1. \tag{2.7}$$

Using the recurrence relation (3.3), we may write (2.7) in the following form:

$$\begin{aligned} &\frac{-4i^2+8m^2+24m+19}{2(m-i+2)(m+2)}d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+1)(m+2)(m-i+2)}d_i(m) \\ &\qquad < T(m+1,i)d_i(m+1). \end{aligned} \tag{2.8}$$

Since $m > i$, we have $4m + 4i^2 + 5 < 12m + 4m^2 + 9$. It follows that

$$\begin{aligned} R(m,i) &= \frac{-4i^2+8m^2+24m+19}{2(m-i+2)(m+2)} - T(m+1,i) \\ &= \frac{4m^2+9m+5-2i^2-i\sqrt{4m+4i^2+5}}{2(m-i+2)(m+2)} \\ &\ge \frac{4m^2+9m+5-2i^2-i(2m+3)}{2(m-i+2)(m+2)} > 0. \end{aligned}$$

Therefore, (2.8) is equivalent to the inequality

$$\frac{d_i(m+1)}{d_i(m)} < F(m,i), \tag{2.9}$$

which is a consequence of (2.6) and Lemma 2.2.

It remains to consider the case $i = m$. We aim to show that

$$\frac{d_m(m+2)}{d_m(m+1)} < T(m+1,m). \tag{2.10}$$

By an easy computation, we find that

$$\begin{aligned} \frac{d_m(m+2)}{d_m(m+1)} &= \frac{(m+1)(4m^2+18m+21)}{2(2m+3)(m+2)}, \\ T(m+1,m) &= \frac{2m^2+15m+14+m\sqrt{4m^2+4m+5}}{4(m+2)}. \end{aligned}$$

Thus (2.10) can be rewritten as

$$(2m^2+3m)\sqrt{4m^2+4m+5} > 4m^3+8m^2+5m. \tag{2.11}$$

Denote by U and V the left hand side and the right hand side of (2.11), respectively. Then $U^2 - V^2 = 4m^2(4m+5) > 0$, and so (2.10) is verified. This completes the proof. $\square$

3. The reverse ultra log-concavity

In this section, we give the proof of Theorem 1.1. Our approach can be described as follows. Let $f(x) = ax^2 + bx + c$ be a quadratic function with $a > 0$. Suppose that the equation $f(x) = 0$ has two distinct real zeros x_1 and x_2, where $x_1 < x_2$. Then $f(x) > 0$ if $x > x_2$ or $x < x_1$, and $f(x) < 0$ if $x_1 < x < x_2$. The key step is to transform the inequality (1.5), that is,

$$\frac{d_i(m)^2}{d_{i-1}(m)d_{i+1}(m)} < \frac{(m-i+1)(i+1)}{(m-i)i},$$

into a quadratic inequality in the ratio $d_i(m+1)/d_i(m)$.

We will need the following recurrence relations for the coefficients $d_i(m)$. For $m \geq 1$ and $0 \leq i \leq m+1$,

$$2(m+1)d_i(m+1) = 2(m+i)d_{i-1}(m) + (4m+2i+3)d_i(m), \tag{3.1}$$

$$\begin{aligned} 2(m+1)(m+1-i)d_i(m+1) = {} & (4m-2i+3)(m+i+1)d_i(m) \\ & - 2i(i+1)d_{i+1}(m), \end{aligned} \tag{3.2}$$

$$\begin{aligned} & 4(m+2-i)(m+1)(m+2)d_i(m+2) \\ & \quad = 2(m+1)(-4i^2+8m^2+24m+19)d_i(m+1) \\ & \qquad - (m+i+1)(4m+3)(4m+5)d_i(m). \end{aligned} \tag{3.3}$$

These recurrence relations were derived by Kauers and Paule [7]. The relation (3.3) was also derived independently by Moll [10]. Based on these recurrence relations, Kauers and Paule [7] derived the following lower bound for $d_i(m+1)/d_i(m)$ in their proof of the log-concavity of the Boros-Moll polynomials:

$$\frac{d_i(m+1)}{d_i(m)} \geq Q(m,i), \quad 0 \leq i \leq m, \tag{3.4}$$

where

$$Q(m,i) = \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}. \tag{3.5}$$

Note that Chen and Xia [5] have shown that inequality (3.4) becomes strict, that is,

$$\frac{d_i(m+1)}{d_i(m)} > Q(m,i), \tag{3.6}$$

for $m \geq 2$ and $1 \leq i \leq m-1$.

Now we are ready to prove the reverse ultra log-concavity of $\{d_i(m)\}$.

Proof of Theorem 1.1. Applying (3.1) and (3.2), we may reformulate (1.5) in the form

$$\begin{aligned} & 4(m-i+1)^2(m+1)^2\left(\frac{d_i(m+1)}{d_i(m)}\right)^2 \\ & - 4(m-i+1)(m+1)(4m^2-2i^2+7m+3)\left(\frac{d_i(m+1)}{d_i(m)}\right) \\ & - (32mi^2-56m^3-73m^2-42m+13i^2-9-16m^4+16i^2m^2) < 0. \end{aligned} \tag{3.7}$$

For $1 \leq i \leq m-1$, the discriminant of the above quadratic function in $d_i(m+1)/d_i(m)$ is

$$\triangle = 16i^2(m+1)^2(4i^2+4m+1)(m-i+1)^2 > 0.$$

We see that the quadratic function on the left hand side of (3.7) has two real roots:

$$x_1 = \frac{4m^2-2i^2+7m+3-i\sqrt{4m+4i^2+1}}{2(m-i+1)(m+1)},$$

$$x_2 = \frac{4m^2-2i^2+7m+3+i\sqrt{4m+4i^2+1}}{2(m-i+1)(m+1)}.$$

Clearly, $Q(m,i) > x_1$. In view of (3.4), we deduce that $d_i(m+1)/d_i(m) \geq Q(m,i) > x_1$. Observe that x_2 coincides with the upper bound $T(m,i)$ in Theorem 2.1. Thus we have $d_i(m+1)/d_i(m) < x_2$. So we have shown that for $1 \leq i \leq m-1$,

$$x_1 < \frac{d_i(m+1)}{d_i(m)} < x_2,$$

which implies (3.7). This completes the proof of Theorem 1.1. □

4. A lower bound for $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$

In this section, we give the proof of Theorem 1.2, providing a lower bound on $d_i(m)^2/(d_{i-1}(m)d_{i+1}(m))$. As will be seen, this lower bound is very close to the upper bound in (1.5) for the reverse ultra log-concavity. So in the asymptotic sense, we may say that the Boros-Moll polynomials are just on the borderline between ultra log-concavity and reverse ultra log-concavity. We conclude this paper with two conjectures.

Proof of Theorem 1.2. Utilizing the recurrence relations (3.1) and (3.2), the inequality (1.6) can be restated as

$$\begin{aligned}
&4(m+1)^2(m-i+1)^2\left(\frac{d_i(m+1)}{d_i(m)}\right)^2\\
&-4(m-i+1)(m+1)(4m^2+7m-2i^2+3)\frac{d_i(m+1)}{d_i(m)}\\
&+(4m^2+7m+3)(-4i+3+4m)(m+i+1) > 0.
\end{aligned}$$

For $1 \leq i \leq m-1$, the discriminant of the above quadratic function in $d_i(m+1)/d_i(m)$ is

$$\delta = 16i^2(2i+1)^2(m+1)^2(m-i+1)^2 > 0. \tag{4.1}$$

Hence the above quadratic function has two real roots,

$$x_1 = \frac{4m^2+7m-4i^2-i+3}{2(m+1)(m-i+1)},$$

$$x_2 = \frac{4m^2+7m+i+3}{2(m+1)(m-i+1)}.$$

As $x_2 = Q(m,i)$, it follows from (3.6) that $d_i(m+1)/d_i(m) > x_2$. So we arrive at (1.6). This completes the proof. □

Notice that for $1 \le i \le m-1$,

$$\frac{(m-i+1)(i+1)(m+i)}{(m-i)i(m+i+1)} > \frac{i+1}{i}.$$

As a consequence of Theorem 1.2, we obtain the log-concavity of the sequence $\{i!d_i(m)\}$.

Corollary 4.1. *For $m \ge 2$ and $1 \le i \le m-1$,*

$$\frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} > \frac{i+1}{i} \tag{4.2}$$

or, equivalently, the sequence $\{i!d_i(m)\}$ is log-concave.

Corollary 4.2. *For $1 \le i \le m-1$, let*

$$c_i(m) = \frac{d_i^2(m)}{d_{i-1}(m)d_{i+1}(m)} \quad \text{and} \quad u_i(m) = \left(1+\frac{1}{i}\right)\left(1+\frac{1}{m-i}\right).$$

Then for any $i \ge 1$,

$$\lim_{m\to\infty} \frac{c_i(m)}{u_i(m)} = 1. \tag{4.3}$$

Proof. By Theorems 1.1 and 1.2, we find that

$$\frac{m+i}{m+i+1} < \frac{c_i(m)}{u_i(m)} < 1,$$

which implies (4.3). □

We remark that even when m is small, $c_i(m)$ is quite close to $u_i(m)$ for any $1 \le i \le m-1$. Numerical evidence indicates that $c_i(m)/u_i(m)$ is increasing for any given m. For example, when $m = 8$, the values of $c_i(m)/u_i(m)$ for $1 \le i \le 7$ are given below:

0.956593, 0.969751, 0.978293, 0.983956, 0.987811, 0.990507, 0.992445.

We propose the following two conjectures on the log-concavity and reverse ultra log-concavity of the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}$.

Conjecture 4.3. *For $m \ge 2$, the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2\le i\le m-2}$ is log-concave.*

Conjecture 4.4. *For $m \ge 2$, the sequence $\{d_{i+1}(m)d_{i-1}(m)/d_i(m)^2\}_{2\le i\le m-2}$ is reverse ultra log-concave.*

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