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MULTISECANT SUBSPACES TO SMOOTH PROJECTIVE VARIETIES IN ARBITRARY CHARACTERISTIC

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Abstract. Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n \geq 1$, degree d, and codimension e, not contained in any hyperplane, defined over an algebraically closed field $\Bbbk$ of arbitrary characteristic. We show that if a k-dimensional linear subspace M meets X at the smooth locus such that $X \cap M$ is finite and locally lies on a smooth curve, then the length $l(X \cap M)$ does not exceed $d - e + k - \min\{g, e-k\}$ for the sectional genus g of X.

1. Introduction

Let $X \subseteq \mathbb{P}^N$ $(N = n + e)$ be a *nondegenerate* (i.e., not contained in any hyperplane) projective variety of dimension $n \geq 1$, degree d, and codimension e defined over an algebraically closed field $\Bbbk$ of arbitrary characteristic. A linear subspace M of $\mathbb{P}^N$ is said to be *secant* (resp. *m-secant*) to X if $X \cap M$ is finite with length $l(X \cap M) := \text{length}(\mathcal{O}_{X \cap M})$ positive (resp. finite with length $l(X \cap M)$ at least m). In this paper, we study the maximal length of the intersection of X and a secant subspace M to X. A secant subspace M to X is said to be *curvilinear* if each component of $X \cap M$ locally lies on a smooth curve. So a secant line to X is always curvilinear. By g we denote the sectional genus of X, i.e., the arithmetic genus of a curve obtained by taking successive hyperplane sections of X (see [6, (I.2.1)]).

The purpose of this paper is to show the following:

Theorem 1. *Let $X \subseteq \mathbb{P}^N$ $(N = e + n)$ be a nondegenerate, projective variety of dimension n, degree d, codimension e, and sectional genus g. Let M be a curvilinear secant linear subspace of dimension k $(1 \leq k \leq e)$ that meets X on the smooth locus of X. Set $\ell = \min\{g, e-k\}$. Then*

$$l(X \cap M) \leq d - e + k - \ell. \tag{1.1}$$

The bound on $l(X \cap M)$ was studied in [2], [10] and [13]. In particular, the study of bounds on $l(X \cap M)$ for a line M was motivated by the open conjecture on Castelnuovo-Mumford regularity (see [3], [7, §4]). This conjecture suggests that $l(X \cap M)$ would be at most $d - e + 1$ for any secant line M. By Bertin's result in [2], this bound was proved and the boundary case was described when X is smooth in $\text{char}(\Bbbk) = 0$. The result of Bertin was generalized by Kwak in $\text{char}(\Bbbk) = 0$ in two

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directions (see [10, (1.6) and (3.6)]). First he obtained $l(X \cap M) \leq d - e + k$ for a secant k-dimensional subspace M under the assumption that $X \cap M$ is contained in the Cohen-Macaulay locus of X and that $\dim X = \dim \pi_M(X \setminus M)$ for the linear projection π_M from M. Second he showed that $l(X \cap M) \leq d - e + k - 1$ for a curvilinear secant subspace M to a smooth X of sectional genus $g \geq 1$, by describing the boundary case $l(X \cap M) = d - e + k$. The result here is a generalization of this second result of Kwak and of [13, (2.4)] where $\mathrm{char}(\Bbbk) = 0$ was assumed. On the other hand, a generalization of the first result of Kwak was studied in [13, (1.1)].

In Theorem 1, the inequality (1.1) is sharp. In fact, in [12, Theorem 1], curves with secant lines satisfying the equality in (1.1) were given (see Example 5). Also, in Proposition 6, we give a smooth curve X that admits a secant k-plane ($k \geq 2$) satisfying the equality in (1.1), but which for all λ ($1 \leq \lambda < k$), admits no secant λ-plane satisfying the equality. Moreover, in [13, (3.4) and (3.6)], smooth scrolls with secant lines satisfying the equality in (1.1) were classified (see Remark 7).

On the other hand, in Theorem 1, the assumption that the secant space M is curvilinear with $M \cap X \subseteq \mathrm{Sm}\, X$ is necessary to reduce the general case to the curve case. The assumption is essential to prove Lemma 3 (see Remark 4). But I know of no example of a smooth projective variety with a non-curvilinear subspace which does not satisfy (1.1).

Theorem 1 is proved by taking hyperplane sections and by reducing to the curve case, based on an idea due to Bertin [2] and Kwak [10]. To this purpose, first we prove the inequality (1.1) for curves with secant lines (Proposition 2), without assuming that the secant lines meet on the smooth locus, by using a regularity bound. Next we show a Bertini-type result (Lemma 3) based on the usual Bertini Theorem, which reduces the proof of Theorem 1 to the curve case.

2. Proof of the main result

Proposition 2. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate, projective curve of degree d, codimension e, and arithmetic genus $p_a(X)$. Let L be a secant line to X. Set $\ell = \min\{p_a(X), e-1\}$. Then*

$$l(X \cap L) \leq d - e + 1 - \ell. \tag{2.1}$$

Proof. By [7, Theorem 1.1], X is $(d-e+1)$-regular, and hence the homogeneous ideal of X is generated in degree $\leq d-e+1$. Thus there is no $(d-e+2)$-secant line to X. This implies that $l(X \cap L) \leq d - e + 1$. If $p_a(X) = 0$ or $e = 1$, then $\ell = 0$ and hence we have (2.1). Thus we assume that $p_a(X) > 0$ and $e \geq 2$. Then $\ell > 0$ and hence, by [11], X is either $(d-e+1-\ell)$-regular or linearly normal (i.e., embedded by the complete linear system $|\mathcal{O}_X(1)|$) with $d \geq 2p_a(X) + 2$ and $\ell = p_a(X)$. In the former case, by the same argument as above, we have (2.1). In the latter case, $e - 1 = d - p_a(X) - 2 \geq p_a(X)$ by the Riemann-Roch Theorem. Hence $\ell = p_a(X)$ and $d - e + 1 - \ell = 2$. On the other hand, since $d \geq 2p_a(X) + 2$, by [5, Corollary 1.14], X is defined by quadrics. This implies that there is no 3-secant line to X, and hence we have (2.1) in the second case, too. □

The following lemma based on Bertini's Theorem (see [4, (3.4.10)] or [15, (I.6.3)]) will reduce the proof of Theorem 1 to the curve case.

Lemma 3. *Let $X \subseteq \mathbb{P}^N$ be a nondegenerate, projective variety of dimension n and codimension e. Let M be a curvilinear secant linear subspace of dimension k,* *meeting X on the smooth locus of X. Suppose $n \geq 2$ and $e > k \geq 1$. If $H \subseteq \mathbb{P}^N$ is a general hyperplane containing M, then $X \cap H$ is irreducible, reduced, nondegenerate in H, and smooth at every point of $X \cap M$. Moreover, the closure $\bar{X}$ of the image $\pi_M(X \setminus M)$ of the linear projection π_M from M to $\mathbb{P}^{N-k-1}$ has dimension n.*

Proof. Consider the linear system $\mathfrak{D}$ of hyperplane sections of X by the hyperplanes containing M. So $X \cap H$ is a general member of $\mathfrak{D}$. First note that $X \cap H$ is smooth at every point of $X \cap M$, since a general hyperplane H containing M does not contain any embedded tangent space to X at points of $X \cap M$ by the curvilinearity and the finiteness of $X \cap M$. Moreover $X \cap H \setminus X \cap M$ satisfies Serre's condition S_1 by a Bertini-type theorem (see [4, (3.4.6)]), since X is reduced and hence satisfies S_1. Thus $X \cap H$ satisfies S_1 by the smoothness at $X \cap M$. Now we assume, for a moment, that the image of the rational map associated with $\mathfrak{D}$ has dimension ≥ 2, and we will prove the first part of the lemma. By Bertini's Theorem (see [4, (3.4.10)] or [15, (I.6.3)]), $X \cap H$ is irreducible. Hence its dense open subset $\mathrm{Sm}(X) \cap H$ is a prime divisor or a multiple of a prime divisor on the smooth locus $\mathrm{Sm}(X)$ of X. Recalling that $X \cap H$ is smooth at $X \cap M$, we know that $\mathrm{Sm}(X) \cap H$ is a prime divisor on $\mathrm{Sm}(X)$ and hence that $X \cap H$ is generically reduced (i.e., $X \cap H$ satisfies Serre's condition R_0). Consequently $X \cap H$ satisfies S_1 and R_0, and $X \cap H$ is reduced (see [1, (VII.2.2)]). Moreover $X \cap H$ is nondegenerate in H (see [4, p.116] or [8, (18.10)]).

To obtain that the image of the rational map associated with $\mathfrak{D}$ has dimension ≥ 2, we have to show that the closure $\bar{X}$ of $\pi_M(X \setminus M)$ for the linear projection $\pi_M : \mathbb{P}^N \setminus M \to \mathbb{P}^{N-k-1}$ has dimension at least 2, since $\mathfrak{D}$ defines $\pi_M | X \setminus M$. Let $\pi : X \setminus M \to \bar{X}$ be the restriction of π_M to $X \setminus M$. Since the closure of a general fibre of π lies on a $(k+1)$-dimensional linear subspace containing M and since $X \cap M$ is finite, a general fibre of π has dimension at most one and hence $\dim \bar{X} = n$ or $n-1$. Now suppose, to the contrary, that $\dim \bar{X} \leq 1$. Then $n = 2$ and $\dim \bar{X} = 1$. Note that $\bar{X}$ is nondegenerate, since X is. Thus if $\deg \bar{X} = 1$, then $\bar{X} = \mathbb{P}^{N-k-1}$ and $e = k$, a contradiction. Hence $\deg \bar{X} \geq 2$. Since $\bar{H} := \pi_M(H \setminus M)$ is a general hyperplane of $\mathbb{P}^{N-k-1}$ by the generality of $H \supseteq M$, the intersection $\bar{X} \cap \bar{H}$ consists of $\bar{d}$ distinct points $\bar{x}_1, \ldots, \bar{x}_{\bar{d}}$ for $\bar{d} := \deg \bar{X} \geq 2$ such that $\bar{x}_i = \pi_M(x_i)$ for some $x_i \in X \setminus M$, by Bézout's Theorem. Hence $X \cap H = C_{\bar{x}_1} \cup \cdots \cup C_{\bar{x}_{\bar{d}}}$ for the closure $C_{\bar{x}_i}$ of the fibre $\pi^{-1}(\bar{x}_i)$. On the other hand, $X \cap H$ is connected by a Lefschetz type theorem [9, (II.6.2)]. But $C_{\bar{x}_1}, \ldots, C_{\bar{x}_{\bar{d}}}$ meet only at $X \cap M$, since $\langle M, x_i \rangle \cap \langle M, x_j \rangle = M$ for $i \neq j$. This contradicts the smoothness of $X \cap H$ at $X \cap M$. Thus $\dim \bar{X} \geq 2$, as required.

To show the second part of the lemma, $\dim \bar{X} = \dim X$, by taking the hyperplane section $X \cap H$ for a general hyperplane H containing M and applying the first part, we may assume that $\dim X = 2$ and have to show that $\dim \bar{X} = 2$. This has already been done in the paragraph above. □

Remark 4. We cannot relax the assumption of Lemma 3: If $X \cap M$ is not curvilinear or if X is not smooth at $X \cap M$, then the conclusion of Lemma 3 is not true in general (see [13, Example 2.2]).

Proof of Theorem 1. By Lemma 3, upon taking general hyperplanes H_i ($i = 1, \ldots, n-1$) containing M and replacing X by $X \cap H_1 \cap \cdots \cap H_{n-1}$, we may assume that $X \subseteq \mathbb{P}^N$ ($N = e+1$) is a nondegenerate projective curve of degree d, arithmetic genus $p_a(X) = g$, and codimension e, with a secant k-space M meeting it on the smooth locus of X. Let Λ be a general $(k-2)$-dimensional subspace of M. So $\Lambda \cap \langle x_i, x_j \rangle = \emptyset$ and $\Lambda \cap T_{x_i}(X) = \emptyset$ for all $x_i \neq x_j \in X \cap M$, and in particular $X \cap \Lambda = \emptyset$. Consider the linear projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{N-k+1}$ from Λ. Set $\bar{X} = \pi_\Lambda(X)$ and $\bar{M} = \pi_\Lambda(M \setminus \Lambda)$. At each point of $\bar{X} \cap \bar{M}$, the induced morphism $\pi = \pi_\Lambda|X : X \to \bar{X}$ is an isomorphism by the choice of Λ. Thus π is a birational morphism, and $\bar{M}$ is a secant line to $\bar{X}$ with $l(X \cap M) = l(\bar{X} \cap \bar{M})$. Moreover $\bar{X}$ is a nondegenerate projective curve of $\deg \bar{X} = \deg X$ and codimension $e - k + 1$, with $p_a(\bar{X}) \geq p_a(X) = g$. Applying Proposition 2 to $\bar{X}$ and $\bar{M}$, we have

$$l(\bar{X} \cap \bar{M}) \leq d - (e-k+1) + 1 - \min\{p_a(\bar{X}), (e-k+1) - 1\}.$$

Consequently we have (1.1). □

3. Examples of the boundary case

In this section, we will give examples of projective curves that admit secant k-planes attaining the equality in Theorem 1.

Example 5 ([12, Theorem 1]). There exists a smooth projective curve $X \subseteq \mathbb{P}^N$ of degree d and genus g with secant line L satisfying one of the following conditions.

(1) $N \geq g+2$, $d \geq N+g+1$ and $l(X \cap L) = d - N + 2 - g$. In this case $\ell = \min\{g, N-2\} = g$.

(2) $N = g+1 \geq 3$, $d \geq 2g+2$ and $l(X \cap L) = d - N + 2 - (g-1) = d - N + 2 - (N-2)$. In this case $\ell = \min\{g, N-2\} = N-2 = g-1$.

(3) X is hyperelliptic, $g+1 > N \geq 3$, $d \geq 2g+2$ and $l(X \cap L) = d - N + 2 - (N-2) \geq 6$. In this case $\ell = \min\{g, N-2\} = N-2$.

In the next proposition, by an elementary argument similar to the proof of [12, Theorem 1], we will give a smooth curve X that admits a secant k-plane ($k \geq 2$) reaching the bound in Theorem 1 and Proposition 2, but which for all λ ($1 \leq \lambda < k$) does not admit any secant λ-plane reaching the bound.

Proposition 6. *Let g, k, N and d be integers with*

$$(6.0) \qquad g \geq 0,\ k \geq 2,\ N \geq g + 2k + 1 \text{ and } 2(N-k) \geq d \geq g + N + 1.$$

There exists a smooth, nondegenerate, projective curve $X \subseteq \mathbb{P}^N$ of genus g that admits a k-plane M with $l(X \cap M) = d - N + 1 + k - g$, but which for all λ $(1 \leq \lambda < k)$ admits no λ-plane L with $l(X \cap L) = d - N + 1 + \lambda - g$.

Proof. If we set $t := d - g - N - 1$, it is easy to see that giving the integers g, k, N and d in (6.0) is equivalent to giving integers g, k, t, and d such that

$$g \geq 0,\ k \geq 2,\ t \geq 0, \text{ and } d \geq 2g + 2k + 2t + 2.$$

Thus setting $b := k + t + 1$, we will construct an embedding of a smooth projective curve X into $\mathbb{P}^N$ $(N = d - g - t - 1)$, where X is nondegenerate in $\mathbb{P}^N$, of degree d and genus g, and such that it admits a k-plane M with $l(X \cap M) = d - (N-1) + k - g = b + 1$ but for all λ $(1 \leq \lambda < k)$ admits no λ-plane L with $l(X \cap L) = d - (N-1) + \lambda - g = t + \lambda + 2$. To this purpose, let X be a smooth projective curve of genus g, and first embed X into $\mathbb{P}^{d-g}$ by the complete linear system associated with a line bundle $\mathcal{O}_X(1)$ of degree d. Note that $\dim H^0(\mathcal{O}_X(1)) = d - g + 1$ since $d \geq 2g+1$. To obtain the required embedding of X, we will project X down to $\mathbb{P}^N$ from a suitable t-dimensional linear space Λ.

Before showing how to choose Λ, we will look at the linear span of divisors on X in $\mathbb{P}^{d-g}$. For a subset $Z \subseteq \mathbb{P}^{d-g}$, the linear span $\langle Z \rangle$ of Z in $\mathbb{P}^{d-g}$ is the intersection of all hyperplanes of $\mathbb{P}^{d-g}$ that contain Z. Thus for an effective divisor D on X, $\langle D \rangle$ is defined by $H^0(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D))$. In this case, it is clear by the Riemann-Roch Theorem that

$$\dim\langle D \rangle = \deg D - 1 \quad \text{if} \quad \deg D \leq 2b+1, \tag{6.1}$$

since $d \geq 2g + 2b$ by assumption. On the other hand, for effective divisors D_1 and D_2 on X, we define $D_1 \vee D_2$ to be the smallest effective divisor D' with $D' \geq D_1$ and $D' \geq D_2$, where we say that $D' \geq D_1$ if $D' - D_1$ is effective, and we also define $D_1 \wedge D_2$ to be the biggest effective divisor D'' with $D'' \leq D_1$ and $D'' \leq D_2$. In this case, $\deg D_1 \vee D_2 + \deg D_1 \wedge D_2 = \deg D_1 + \deg D_2$. With this notation, we have

$$\langle \langle D_1 \rangle \cup \langle D_2 \rangle \rangle = \langle D_1 \vee D_2 \rangle, \tag{6.2}$$

since as subspaces of $H^0(\mathcal{O}_X(1))$,

$$H^0(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D_1)) \cap H^0(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D_2)) = H^0(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-(D_1 \vee D_2))).$$

Take any points $x_1, \ldots, x_{b+1} \in X$ and set $D := x_1 + \cdots + x_{b+1}$. By (6.1), we have $\dim\langle D \rangle = b$. Let $\Lambda \subseteq \langle D \rangle$ be a t-plane such that

$$\Lambda \not\subseteq \bigcup_{0 \leq D' \leq D, \deg D' = b} \langle D' \rangle \quad \text{and} \quad \Lambda \cap \bigcup_{0 \leq D'' \leq D, \deg D'' = 2} \langle D'' \rangle = \emptyset. \tag{6.3}$$

We claim that for any effective divisors E and B with $\deg E = t + \lambda + 2 (\leq b)$ and $\deg B = 2$,

$$\Lambda \not\subseteq \langle E \rangle \quad \text{and} \quad \Lambda \cap \langle B \rangle = \emptyset. \tag{6.4}$$

Assuming the claim, we will complete the proof. Let $\pi_\Lambda \colon \mathbb{P}^{d-g} \dashrightarrow \mathbb{P}^N$ be the linear projection from Λ. From the latter condition of (6.4), it follows that X is isomorphic onto its image $\pi_\Lambda(X)$. Clearly the image $M := \pi_\Lambda(\langle D \rangle)$ is a linear subspace of dimension $b-t-1 = k$ such that $l(\pi_\Lambda(X) \cap M) = b+1$. Suppose to the contrary that there exists a λ-plane L $(1 \leq \lambda < k)$ with $l(\pi_\Lambda(X) \cap L) = t + \lambda + 2$. Then $\pi_\Lambda(X) \cap L$ can be seen an effective divisor E on X by the isomorphism $X \cong \pi_\Lambda(X)$. In this case, $\pi_\Lambda(\langle E \rangle) \subseteq L$ and $\dim\langle E \rangle = t + \lambda + 1$ by (6.1). Therefore Λ must be contained in $\langle E \rangle \subseteq \mathbb{P}^{d-g}$, which contradicts (6.4).

Finally we prove the claim (6.4). To prove the first part, suppose to the contrary that $\Lambda \subseteq \langle E \rangle$ for an effective divisor E of $\deg E = h \leq b$. If necessary, upon replacing E by an effective divisor less than E we may assume that $\Lambda \subseteq E$ but $\Lambda \not\subseteq \langle E' \rangle$ for any effective divisor $E' \leq E$ of degree $h-1$. Thus $D \wedge E \neq E$ by the first part of (6.3). Let E_1 be an effective divisor with $D \wedge E \leq E_1 < E$ and $\deg E_1 = \deg E - 1$. Hence $\Lambda \not\subseteq \langle E_1 \rangle$. Since $D \wedge E = D \wedge E_1$, we have $\deg(D \vee E) = \deg(D \vee E_1) + 1 \leq 2b+1$, and hence $\langle D \vee E \rangle \neq \langle D \vee E_1 \rangle$ by (6.1). On the other hand, $\Lambda \subseteq \langle E \rangle$ but $\Lambda \not\subseteq \langle E_1 \rangle$ by the choice of E and E_1, and hence $\langle \Lambda \cup \langle E_1 \rangle \rangle = \langle E \rangle$. This implies that $\langle \langle D \rangle \cup \langle E \rangle \rangle = \langle \langle D \rangle \cup \Lambda \cup \langle E_1 \rangle \rangle = \langle \langle D \rangle \cup \langle E_1 \rangle \rangle$, which contradicts $\langle D \vee E \rangle \neq \langle D \vee E_1 \rangle$ by (6.2).

To prove the second part of (6.4), suppose to the contrary that $\Lambda \cap \langle B \rangle \neq \emptyset$ for an effective divisor B of $\deg B = 2$. Then $\dim\langle \langle D \rangle \cup \langle B \rangle \rangle \leq b+1$ since $\Lambda \subseteq \langle D \rangle$. Hence $\deg(D \vee B) \leq b+1$ by (6.1) and (6.2). Thus there exists a point $y \in \operatorname{Supp} D \cap \operatorname{Supp} B$. By the choice of Λ in (6.3), we have $y \notin \Lambda$, and hence $\langle \Lambda \cup \{y\} \rangle = \langle \Lambda \cup \langle B \rangle \rangle$ by comparing the dimension. This implies that

$\langle\langle D\rangle \cup \langle B\rangle\rangle = \langle\langle D\rangle \cup \Lambda \cup \langle B\rangle\rangle = \langle\langle D\rangle \cup \Lambda \cup \{y\}\rangle = \langle D\rangle$. Hence $D \geq B$ by (6.1) and (6.2). This contradicts the choice of Λ in (6.3). □

Remark 7. In [13, (3.4) and (3.6)], scrolls with secant lines satisfying the equality in (1.1) were classified. Also, in [14], smooth Del Pezzo varieties with secant lines satisfying the equality in (1.1) were classified. But much is not known yet for $n \geq 2$ and $k \geq 2$.

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