PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 12, December 2009, Pages 4099–4107
S 0002-9939(09)09988-2
Article electronically published on July 15, 2009

ISOLATED SINGULARITIES FOR THE EXPONENTIAL TYPE SEMILINEAR ELLIPTIC EQUATION IN $\mathbb{R}^2$

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(Communicated by Matthew J. Gursky)

Abstract. In this article we study positive solutions of the equation $-\Delta u = f(u)$ in a punctured domain $\Omega' = \Omega \setminus \{0\}$ in $\mathbb{R}^2$ and show sharp conditions on the nonlinearity $f(t)$ that enables us to extend such a solution to the whole domain Ω and also preserve its regularity. We also show, using the framework of bifurcation theory, the existence of at least two solutions for certain classes of exponential type nonlinearities.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $0 \in \Omega$. Denote $\Omega^{'} = \Omega \setminus \{0\}$. Let $f : (0,\infty) \longrightarrow (0,\infty)$ be a locally Hölder continuous function which is nondecreasing for all large $t > 0$. In this article we study the following problem:

$$(P') \qquad \begin{cases} \left.\begin{array}{rcl} -\Delta u &=& f(u) \\ u &\geq& 0 \end{array}\right\} \text{ in } \Omega', \\ \quad u \in L^\infty_{loc}(\Omega'). \end{cases}$$

It is well-known from the works of Brezis-Lions [5] that if u solves (P'), then indeed u solves the following problem in the distributional sense in the whole domain Ω:

$$(P_\alpha) \qquad \begin{cases} \left.\begin{array}{rcl} -\Delta u &=& f(u) + \alpha\delta_0 \\ u &\geq& 0 \end{array}\right\} \text{ in } \Omega, \\ \alpha \geq 0, u, f(u) \in L^\infty_{loc}(\Omega') \cap L^1_{loc}(\Omega). \end{cases}$$

This leads us to the following two questions:

(Q1) Can we find a sharp condition on f that determines whether or not $\alpha = 0$ in (P_α)?

(Q2) If $\alpha = 0$, is it true that u is regular (say, C^2) in Ω?

We make the following

Definition 1.1. We say f is a sub-exponential type function if

$$\lim_{t\to\infty} f(t)e^{-\beta t} \leq C \quad \text{for} \quad \text{some} \quad \beta, \mathrm{C} > 0.$$

We say f is of super-exponential type if it is not a sub-exponential type function.

Received by the editors September 30, 2008.
2000 *Mathematics Subject Classification.* Primary 35B32, 35B65, 35J25, 35J60.
Key words and phrases. Isolated singularity, blow-up, Laplace equation.

As a complete answer to question (Q1) we show (Theorem 2.1) that if f is of super-exponential type, then $\alpha = 0$, and conversely (Theorem 2.2) that (P_α) has solutions for small $\alpha > 0$ if f is of sub-exponential type.

Similarly, we answer question (Q2) by showing that for any f of sub-exponential type, any solution u of (P_0) is regular(C^2) inside Ω (Theorem 3.1). Conversely, for f of super-exponential type with any prescribed growth at ∞ and behaviour for small $t > 0$, in Lemma 3.1 and Theorem 3.3 we construct solutions u of (P_0) that blow-up only at the origin. To our knowledge, the existence of such singular solutions has not been considered so far for super-exponential type problems. Theorem 3.2 should be contrasted with the results in [2] and [13]. Particularly in [13], the nonlinearity under study is of a model type, viz., $f(t) = e^{t^\mu}, \mu > 0$. These authors show that for a noncompact sequence of solutions to (P_0) posed on a ball, concentration phenomenon occurs for $1 < \mu < 2$ and total blow-up occurs for $\mu < 1$. Clearly, $\mu = 1$ appears as the borderline exponent between total blow up and concentration. In Theorem 3.2, when $\mu = 2$, for certain classes of nonmodel type nonlinearities we show that instead of concentration, convergence to a singular solution occurs. If the nonlinearity is closer to a model-type, more precisely, if $\liminf_{t\to+\infty} f(t)e^{-t^2}t = +\infty$, then only concentration takes place, as follows from the results in [1].

Definition 1.2. We denote by Γ the fundamental solution of $-\Delta$ in $\mathbb{R}^2$. That is, $\Gamma(x) = -\frac{1}{2\pi}\log|x|, x \in \mathbb{R}^2 \setminus \{0\}$.

2. Extendability of the solution from the punctured domain to the entire domain

In this section, we will discuss the extension of a solution of (P') to the whole domain Ω.

Theorem 2.1 (Removable singularity)**.** *Let f be of super-exponential type. Then any solution u of (P') extends to a distributional solution of (P_0).*

Proof. As noted before for some $\alpha \geq 0, u$ solves (P_α). Therefore, $-\Delta(u - f(u) * \Gamma - \alpha\Gamma) = 0$. Since $f(u) * \Gamma \geq 0$ it follows that $u(x) \geq \alpha\Gamma(x) - C$ for all $x \in \Omega$ for some constant $C > 0$. Since $f(t)$ is nondecreasing for all large $t > 0$, we obtain, for any $\delta > 0$ small enough, $f(\alpha\Gamma(x) - C) \leq f(u(x))$ for all $|x| < \delta$. If $\alpha > 0$, we choose $\beta = \frac{4\pi}{\alpha}$ and apply Definition 1.1 to obtain $|x|^{-2} \leq f(u(x))$ for all $|x|$ small. This contradicts the fact that $f(u) \in L^1_{loc}(\Omega)$. Hence, necessarily, $\alpha = 0$. ☐

Define

$$\beta^* = \inf\{\beta > 0 \ \text{ occurring in Definition 1.1}\}.$$

Then, we can show the following:

Theorem 2.2. *Let f be a sub-exponential type nonlinearity. Then for all $\alpha \in (0, \frac{2\pi}{\beta^*})$ the problem (P_α) admits a solution. Furthermore, if $f(t) \geq Ce^{\overline{\beta}t}, \forall t \geq 0$, for some $\overline{\beta} > 0$, then (P_α) has no solution for all $\alpha \geq 4\pi(\overline{\beta})^{-1}$.*

Proof. We will prove the above statements using the monotone iteration technique. In fact, we will construct a solution u of (P_α) which vanishes on $\partial\Omega$ for all suitable α. Without loss of generality, for this purpose we assume that f is nondecreasing for all $t > 0$ (if not, we replace $f(t)$ by $f(t) + kt$ for some large k, use the argument below and recover the result for f). It is clear that $u_0 = 0$ is a sub-solution of (P_α) for all $\alpha > 0$.

Let us define, for any $\beta, C > 0$ given by Definition 1.1,

$$v_\beta(x) = -\beta^{-1} \log \left(4|x|(1 + \frac{\beta C}{4}|x|)^2 \right).$$

Then a simple computation gives $-\Delta v_\beta = 2\pi\beta^{-1}\delta_0 + g$, where

$$g(x) = C \left(2|x|(1 + \frac{\beta C}{4}|x|)^2 \right)^{-1}.$$

It can be easily checked that $g \in L^r(\Omega)$ for all $1 < r < 2$ and $g \geq f(v_\beta)$ in Ω. Hence v_β is a supersolution of (P_α) for all $\alpha \leq 2\pi\beta^{-1}$. Now, for any such α, consider the following sequence of problems:

$$(P_n) \qquad \begin{cases} -\Delta u_n &= f(u_{n-1}) + \alpha\delta_0 & \text{in} & \Omega, \\ u_n &= 0 & \text{on} & \partial\Omega, \\ u_n &\in L^p(\Omega) & \forall & 1 < p < \infty, \\ f(u_n) &\in L^r(\Omega) & \forall & 1 < r < 2. \end{cases}$$

We construct a solution u_n of the problem (P_n) inductively as follows. Let $w_1 \in C^2(\overline{\Omega})$ be the solution of the problem

$$\begin{cases} -\Delta w_1 &= f(u_0) & \text{in} & \Omega, \\ w_1 &= -\alpha\Gamma & \text{on} & \partial\Omega. \end{cases}$$

Define $u_1 = w_1 + \alpha\Gamma$. It can be easily seen that u_1 is a solution for (P_1) with $f(u_1) \in L^r(\Omega)$ for $1 < r < 2$. Now assume that there exists a solution for (P_{n-1}). Let $w_n \in W^{2,r}(\Omega)$ be a solution of

$$\begin{cases} -\Delta w_n &= f(u_{n-1}) & \text{in} & \Omega, \\ w_n &= -\alpha\Gamma & \text{on} & \partial\Omega. \end{cases}$$

By standard elliptic regularity w_n is a Hölder continuous function in $\overline{\Omega}$. Then $u_n = w_n + \alpha\Gamma$ solves $-\Delta u_n = f(u_{n-1}) + \alpha\delta_0$ in Ω, and $u_n = 0$ on $\partial\Omega$. Also $u_n \in L^p(\Omega)$ for every $1 \leq p < \infty$. Next we notice that $u_n - v_\beta$ solves $-\Delta(u_n - v_\beta) \leq f(u_{n-1}) - g$ a.e. in Ω and $u_n - v_\beta \leq 0$ on $\partial\Omega$ (for C large enough). Hence by the maximum principle $u_n \leq v_\beta$ in Ω. Also we notice that $f(v_\beta) \in L^r(\Omega)$ for $1 \leq r < 2$. Using the monotonicity of f we conclude that $f(u_n) \in L^r(\Omega)$ for $1 \leq r < 2$. Hence we have obtained a sequence $\{u_n\}$ solving (P_n) and

$$u_n \leq v_\beta \quad \text{in } \Omega \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

It can also be shown easily that $0 \leq u_1 \leq u_2 \cdots \leq u_{n-1} \leq u_n \cdots$. Now define $u(x) = \lim_{n\to\infty} u_n(x)$. Then it follows that u is a solution to the problem (P_α) for any $\alpha \leq 2\pi(\beta)^{-1}$. Since $\beta \geq \beta^*$ we indeed have a solution to (P_α) for all $\alpha < 2\pi(\beta^*)^{-1}$.

Let us now take $f(t) \geq Ce^{\overline{\beta}t}, \forall t \geq 0$, for some $\overline{\beta} > 0$. Suppose there exists a solution u of (P_α). We then have $u \geq -\frac{\alpha}{2\pi}\log|x| - C_1$ in Ω, which, if $\alpha \geq \frac{4\pi}{\overline{\beta}}$, contradicts the basic conclusion that $f(u) \in L^1_{loc}(\Omega)$. □

We then have the following:

Corollary 2.1. *If $f(t) \leq Ce^{\beta t^\mu}, \forall t \geq 0$, for some $0 < \mu < 1$, then $\beta^* = 0$, and hence (P_α) admits a solution for every $\alpha > 0$.*

3. Regularity and the lack of it for the extendable solution

In this section we discuss question (Q2) and show that regularity or the lack of it for the solution to (P_0) is determined by whether f is of sub-exponential type. As an application of results in Brezis-Merle [6] we have the following:

Theorem 3.1. *Let f be a sub-exponential type nonlinearity. Then any solution u of the problem (P_0) is regular in Ω.*

Proof. Let u solve (P_0). By Corollary 5.2 in [6], $e^{k|u|} \in L^1(\Omega)$ for every $k > 0$. Therefore, since f is of sub-exponential type, we obtain that $f(u) \in L^r(\Omega)$ for every $1 < r < \infty$. Let u_1 be the solution of

$$\begin{cases} -\Delta u_1 &= f(u) \quad \text{in } \Omega, \\ u_1 &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then, $u_1 \in W^{2,r}(\Omega)$ for all $r > 1$. Therefore, by Sobolev embedding $u_1 \in C^{1,\theta}(\overline{\Omega})$ for every $0 < \theta < 1$. But in the interior of Ω we have, in the sense of distributions, $\Delta(u - u_1) = 0$. Then, it is well-known that $u = u_1 + h$ a.e. for some harmonic function h. Therefore, u is Hölder continuous in Ω, and by standard elliptic regularity, it is C^2 inside Ω. $\square$

In the next two proofs we construct solutions of (P_0) which blow-up at the origin when f is of super-exponential type. Let B_R denote the open ball of radius R centered at the origin.

Lemma 3.1. *Given any $\mu > 1$ there exists an f of super-exponential type satisfying $\lim_{t\to\infty} f(t)e^{-t^\mu} = 0$ such that the corresponding problem (P_0) posed on the unit ball B_1 admits a solution that blows-up at the origin.*

Proof. Given $\mu > 1$, define $f(t) = 4(\mu - 1)\mu^{-2}t^{1-2\mu}e^{t^\mu}, t > 0$. Clearly f satisfies the requirements stated in the lemma. It can be checked that if we define $u(x) = (-2|\log|x||)^{\frac{1}{\mu}}, x \in B_1$, then, thanks to Theorem 2.1, u solves (P_0) with the above choice of f. $\square$

In the above result, though we could choose f satisfying any prescribed super-exponential type growth at infinity, the behaviour for small $t > 0$ is of singular type. In the next result we exhibit super-exponential type nonlinearities whose growth rate at infinity is fixed (in fact it grows like e^{t^2} as $t \to \infty$) but has regular behaviour for small $t > 0$. For this we need to use the nonexistence results proved in [3], which together with Theorem 3.3 stated below helps us to show the following:

Theorem 3.2. *Let $f : [0, \infty) \to [0, \infty)$ be a C^3 super-exponential type nonlinearity with $f(0) = 0$, which has the form $f(t) = h(t)e^{t^2}$, where $h(t) = e^{-t^\mu}|\log t|^p, \mu \in (0, 2), p \geq 0$, or $h(t) = t^{-\theta}, \theta \geq 1$, for all large $t > 0$. Then there exists $R_* > 0$ such that (P_0) posed on B_{R_*} admits a radial solution blowing up at the origin.*

Proof. We first assume the proof of Theorem 3.3. Then the nonexistence results contained in Theorems A and B of [3] imply that the assumptions of Theorem 3.3 hold and therefore the existence of a radial solution blowing-up at the origin. $\square$

Consider the following problem which is a regular version of (P_0) posed on B_R:

$$(P_R) \qquad \begin{cases} \left.\begin{aligned} -\Delta u &= f(u) \\ u &> 0 \end{aligned}\right\} \text{ in } B_R, \\ u = 0 \ \text{ on } \partial B_R, u \in C^2_{loc}(B_R). \end{cases}$$

Theorem 3.3. *Let $f : [0,\infty) \to [0,\infty)$ be a C^3 super-exponential type nonlinearity such that $g \triangleq \log f$ is convex for all large $t > 0$. Suppose there exist a sequence $\{R_n\}$ of positive real numbers with $R_* \triangleq \liminf_{n\to\infty} R_n > 0$ and a sequence $\{u_n\}$ of solutions to (P_{R_n}) such that $\sup_{B_{R_n}} u_n \to \infty$ as $n \to \infty$. Then the problem (P_0) posed on B_{R_*} admits a solution that blows up only at the origin.*

Proof. In order to prove the theorem, we perform some transformations that will put (P_R) into the equivalent form of the classical Emden-Fowler equations. First, we observe that thanks to the symmetry result of Gidas-Ni-Nirenberg, any solution of (P_R) is radially symmetric and, in fact, strictly radially decreasing about the origin. Therefore, (P_R) can be rewritten as the following ODE boundary value problem via the transformation $w(r) = u(|x| = r)$ for $r \in (0, R)$:

$$(P_R) \qquad \begin{cases} \left.\begin{array}{rcl} -(rw')' & = & rf(w) \\ w & > & 0 \end{array}\right\} \text{ in } (0,R), \\ \quad w'(0) = w(R) = 0. \end{cases}$$

We finally make the following Emden-Fowler transformation:

$$y(t) = w(r), \text{ where } r = 2e^{-\frac{t}{2}}, \ \ t \in (2\log(2R^{-1}), \infty).$$

Then it can be checked that (P_R) is equivalent to the following problem with $T = 2\log(\frac{2}{R})$:

$$\left.\begin{array}{rcl} -y'' & = & e^{-t}f(y) \\ y & > & 0 \end{array}\right\} \text{ in } (T,\infty),$$
$$y(T) = y'(\infty) = 0.$$

For our purpose, instead of the above boundary value problem, it will be more convenient to consider the following initial-value problem depending upon a parameter $\gamma > 0$:

$$(P_\gamma) \qquad \begin{cases} -y'' = e^{-t}f(y), \\ y(\infty) = \gamma, y'(\infty) = 0. \end{cases}$$

Since $f(y(t)) > 0$ as long as $y(t) > 0$, it follows from (P_γ) that y is a strictly concave function as long as it is positive. Therefore, there exists $T_0(\gamma) > -\infty$ such that $y(T_0(\gamma)) = 0$ and $y(t) > 0$ for all $t > T_0(\gamma)$. $T_0(\gamma)$ thus defined is clearly the first zero of the solution y of (P_γ) as we move left from infinity. Let $y_0 > 0$ be such that g is convex for all $t > y_0$. We also define the point $t_0(\gamma) > T_0(\gamma)$ to be such that $y(t_0(\gamma)) = y_0$ for each $\gamma > 0$.

Our idea is to obtain the blow-up solution of (P_0) posed on B_{R_*} as the upper envelope of the sequence of solutions $\{u_n\}$. Let $\gamma_n = u_n(0)$ and $\{y_n\}$ be the corresponding sequence of solutions to (P_{γ_n}). Thanks to our assumptions it follows that $\gamma_n \to \infty$ as $n \to \infty$ and $T^* \triangleq \limsup_{n\to\infty} T_0(\gamma_n) < \infty$ (we remark that $\liminf_{n\to\infty} T_0(\gamma_n)$ can be $-\infty$). By definition, $T^* > -\infty$. We make the following claim.

Claim. $\{y_n\}$ is a uniformly bounded sequence on compact subsets of $[T_*, \infty)$.

Proof of claim. We define the following energy function associated to (P_{γ_n}):

$$E_n(t) = y_n' - \frac{1}{2}(y_n')^2 g'(y_n) - e^{g(y_n)-t}, \ t \geq T_0(\gamma_n).$$

Hence, $E_n'(t) = -\frac{1}{2}(y_n')^3 g''(y_n) \leq 0$, $\forall t \geq t_0$, since y_n is strictly increasing and g is convex for this range of t. Since $\lim_{t\to\infty} E_n(t) = 0$ we obtain that E_n is a nonnegative function on $(t_0(\gamma_n), \infty)$. This immediately implies that

$$y_n'(t)g'(y_n(t)) < 2, \ \forall t \geq t_0(\gamma_n). \tag{3.1}$$

Now, integrating the ODE in (P_{γ_n}) we have

$$\int_{t_0(\gamma_n)}^{\infty} f(y_n(t))e^{-t}dt = y_n'(t_0(\gamma_n)).$$

From (3.1) and recalling that $y_n(t_0(\gamma_n)) = y_0$, we get

$$\sup_n \int_{t_0(\gamma_n)}^{\infty} f(y_n(t))e^{-t}dt < \infty. \tag{3.2}$$

If now $t_0(\gamma_n) \to \infty$ as $n \to \infty$, then clearly the claim holds for any interval $[a, b] \subset [T_*, \infty)$. Suppose for some subsequence of $\{\gamma_n\}$, denoted again by $\{\gamma_n\}$ for convenience, we have $\limsup_{n\to\infty} t_0(\gamma_n) < \infty$. It is enough to show, in view of the monotonicity of y_n, that $\{y_n(t)\}$ is a bounded sequence of real numbers for any $t \in [T_*, \infty)$. If this is not true, then for some subsequence of $\{y_n(t)\}_{n\geq 1}$, we will have $\lim_{n\to\infty} y_n(t) = \infty$. Clearly, such a t has to be larger than $t_0(\gamma_n)$ for all large n. In view of monotonicity of y_n again, it follows that $y_n \to \infty$ uniformly on $[t, t+1]$, which contradicts (3.2). Thus we prove the claim in this case also.

Define

$$y(t) = \sup_{n\geq 1} y_n(t), \quad t > T_*. \tag{3.3}$$

Clearly, y is positive and nondecreasing on $[T_*, \infty)$. For each n, choose $T_1(\gamma_n) > T_*$ by the rule $y_n(T_1(\gamma_n)) = \frac{\gamma_n}{2}$. Clearly, $T_1(\gamma_n) \to \infty$ as $n \to \infty$ and $y(T_1(\gamma_n)) \geq \frac{\gamma_n}{2}$. Hence $y(T_1(\gamma_n)) \to \infty$ as $n \to \infty$. By the monotonicity of y we conclude that $y(t) \to \infty$ as $t \to \infty$. By Helly's theorem, up to a subsequence, $y_n \to y$ pointwise a.e. in $[T_*, \infty)$. Integrating the ODE satisfied by y_n twice, we get

$$y_n(t) - y_n(s) = \int_s^t (\rho - s) f(y_n(\rho))e^{-\rho}d\rho, \ \ T_* < s < t < \infty.$$

Passing to the limit as $n \to \infty$ on either side of the above equation we obtain that y also satisfies the same integral equation for a.e. t in (T_*, ∞). From (3.2) and Fatou's Lemma, we obtain that $\int_{T_*}^{\infty} f(y(t))e^{-t}dt < \infty$. Thus, y solves the differential equation $-y'' = e^{-t}f(y)$ in (T_*, ∞) with $\int_{T_*}^{\infty} f(y(t))e^{-t}dt < \infty$. Going back to our original variable $x \in B_{R_*}$ and defining $u(x) = y(2\log(\frac{2}{|x|}))$, we obtain that u solves the following problem:

$$\begin{cases} \left.\begin{array}{rcl} -\Delta u & = & f(u) \\ u & > & 0 \end{array}\right\} \text{ in } B_{R_*} \setminus \{0\}, \\ \lim_{|x|\to 0} u(x) = \infty, \\ \int_{B_{R_*}} f(u) < \infty. \end{cases}$$

By the result of Brezis and Lions [5], in fact u solves the problem (P_α) posed on B_{R_*} for some $\alpha \geq 0$. Since f is of super-exponential type from Theorem 2.1 we obtain that $\alpha = 0$. $\square$

4. Bifurcation analysis of the branch converging to a singular solution

Let $f : [0,\infty) \to (0,\infty)$ (in particular, $f(0) > 0$) be a C^3 nondecreasing convex nonlinearity which has one of the following forms for $m \in \mathbb{R}$ and all large $t > 0$:

(f1) $f(t) = t^m e^{t^2 - t^\mu}, 1 < \mu < 2,$

(f2) $f(t) = t^m e^{t^2 - t^\mu}, 0 < \mu < 1$ or $f(t) = t^m e^{t^2 + t^\mu}, 0 < \mu < 2.$

Consider the following problem depending on a parameter $\lambda > 0$:

$$(P_\lambda) \qquad \left\{ \begin{array}{rcl} -\Delta u &=& \lambda f(u) \\ u &>& 0 \\ u &=& 0 \text{ on } \partial B_1. \end{array} \right. \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{ in } B_1,$$

Let $\mathcal{S} = \{(\lambda, u) \in \mathbb{R}^+ \times C^{2,\gamma}(\overline{\Omega}) \,|\, u \text{ solves } (P_\lambda)\}$ denote the set of solutions of (P_λ). Using tools from bifurcation theory and Theorem 3.2 we describe qualitative properties of a branch of solutions to the problem (P_λ) with the above choice of f. In particular, we highlight the fact that we obtain at least two solutions to (P_λ) when f is of the form $(f1)$ for certain small ranges of λ (see property (3) in Theorem 4.1 below).

Theorem 4.1. *Let f be of the form $(f1)$ or $(f2)$. Then there exists a connected branch of solutions $\mathcal{C}$ in $\mathcal{S}$ and a positive real number Λ with the following properties:*

1. *$\mathcal{C} \subset (0, \Lambda] \times C^{2,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$.*
2. *For $0 \leq \lambda \leq \Lambda$, $(\lambda, w_\lambda) \in \mathcal{C}$, where w_λ is the minimal solution to (P_λ).*
3. *(Bending) $\exists\, \delta > 0$ such that for $\lambda \in (\Lambda - \delta, \Lambda)$, there exists another solution u_λ with $(\lambda, u_\lambda) \in \mathcal{C}$. If f is of the form $(f2)$, in fact we can choose $\delta = \Lambda$.*
4. *If f is of the form $(f1)$, $\exists\, \epsilon > 0$ such that $(\lambda, u_\lambda) \in \mathcal{C}$, $\lambda \leq \epsilon \Rightarrow u_\lambda = w_\lambda$.*
5. *(Convergence to singular solution) If f is of the form $(f1)$, there exists a pair (λ^*, u^*) with $0 < \lambda^* \leq \Lambda$, u^* a singular solution to (P_{λ^*}) and a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that $\lambda_n \to \lambda^*$, $u_n(0) \to \infty$ and $u_n \to u^*$ in $C^2_{loc}(B_1 \setminus \{0\})$.*
6. *(Concentration) If f is of the form $(f2)$, there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that $\lambda_n \to 0$, $u_n(0) \to \infty$ and $|\nabla u_n|^2 dx \rightharpoonup 4\pi\delta_0$ in the sense of measure.*

Proof. From the Gidas-Ni-Nirenberg symmetry result, we see that all solutions of (P_λ) are radially symmetric. The existence of the connected branch $\mathcal{C}$ follows from the Crandall-Rabinowitz Theorem (see [7]). First, observe that we can get the existence and the uniqueness of a branch of minimal solutions to (P_λ) near $(0,0)$ using the Implicit Function Theorem (since $f(0) > 0$). In fact, using sub- and supersolution techniques, we can extend this local branch to a maximal branch of minimal solutions $\{(\lambda, w_\lambda)\}$ for $\lambda \in (0, \Lambda)$. We can show easily that $0 < \Lambda < \infty$ since f is superlinear at infinity and also that there is no solution to (P_λ) for $\lambda > \Lambda$. By elliptic regularity, we can show that w_λ belongs to $C^{2,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. This proves (1)-(2).

Moreover, since f' is a nondecreasing function and w_λ is the minimal solution (thus, stable), $\lambda_1(-\Delta - \lambda f'(w_\lambda)) > 0$ for $0 \leq \lambda < \Lambda$ (which implies that the map $\lambda \to w_\lambda$ is C^2 and w_λ is locally unique for $\lambda \in [0, \Lambda)$). It follows that for $\lambda \in (0, \Lambda)$,

$$\int_\Omega |\nabla w_\lambda|^2 - \lambda \int_\Omega f'(w_\lambda) w_\lambda^2 \geq 0.$$

From Vitali's Convergence Theorem, we get that there exists a weak solution u_Λ to (P_Λ) such that $w_\lambda \to u_\Lambda$ in $H_0^1(\Omega)$ as $\lambda \to \Lambda$. Then, from elliptic regularity and Schauder estimates, we get that $u_\Lambda \in C^{2,\gamma}(\overline{\Omega})$. From the above, it follows that

$$\lambda_1(-\Delta - \Lambda f'(u_\Lambda)) = 0.$$

Using the Fredholm Alternative and letting $L = -\Delta - \Lambda f'(u_\Lambda)$, it is easy to see that $C_0^{2,\gamma}(\Omega) = \mathrm{N}(L) \oplus \mathrm{R}(L)$, where $\mathrm{N}(L)$ (resp. $\mathrm{R}(L) \subset C^{0,\gamma}(\overline{\Omega})$) denotes the kernel of L (resp. the range of L). From the Krein-Rutman Theorem, it follows that $\mathrm{N}(L)$ is one dimensional, spanned by a positive function ϕ_1. Moreover, since L is self-adjoint, $\mathrm{R}(L) = \{\phi_1\}^\perp$. Then the transversality condition is satisfied since

$$-\int_\Omega (f'(u_\Lambda) + \Lambda f''(u_\Lambda)\frac{dw_\lambda}{d\lambda}(\Lambda))\phi_1^2 < 0.$$

Therefore, we can apply Theorem 1.7 in [7], and there exists $\nu > 0$ such that the solutions to (P_λ) near (Λ, u_Λ) form a twice continuously differentiable curve $\mathcal{B} = \{(\lambda(s), \tilde{u}(s)) | \, |s| < \nu\}$ with $\lambda(0) = \Lambda$, and from computation of Theorem 4.8 in [8] (see also Theorem 1.1 in [9]), $\lambda'(0) = 0$, $\lambda''(0) < 0$. Therefore, the curve $\mathcal{C}$ bends to the left at $\lambda = \Lambda$. Appealing to the uniqueness and multiplicity result in [11] (see Theorems 1.2,1.3 and Proposition 8.3), we complete the proof of (3).

If f is of the form (f1), from property (4) and the global bifurcation theory of Rabinowitz (see [14]) we see that there exists $(\lambda_n, u_n) \in \mathcal{C}$ and $\lambda_* > 0$ such that $\lambda_n \to \lambda_*$ and $u_n(0) \to \infty$ (since $\mathcal{C}$ cannot "cross" the minimal solutions branch which is locally unique). Making the preliminary reductions as in Section 7 in [11] and from Theorem 3.3, (5) follows.

If f is of the form (f2), from property (3) and the global bifurcation result of Crandall-Rabinowitz again, we get that $\lambda = 0$ is the unique asymptotic bifurcation line for $\mathcal{C}$. Let $u_\lambda, \lambda \in (0, \lambda)$ be as in (3). Clearly, we have that $u_\lambda(0) \to \infty$ as $\lambda \to 0$. We obtain (6) by using Theorem B in [2]. □

Remark 4.2. We guess that if f is of the form (f1), $\mathcal{C}$ has infinitely many turning points similar to the problems studied in [10] and [12].

Remark 4.3. From properties (5) and (6) in Theorem 4.1, we get two different situations determined by the asymptotic behaviour of f. For the detailed microscopic blow-up analysis of u_λ see [2] and [4], where more general cases are considered.

Acknowledgement

One of the authors would like to thank Professor Adimurthi for helpful discussions on some of the material in this work.

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