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ON THE SYZYGIES OF QUASI-COMPLETE INTERSECTION SPACE CURVES

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ABSTRACT. In this paper, we discuss minimal free resolutions of the homogeneous ideals of quasi-complete intersection space curves. We show that if X is a quasi-complete intersection curve in $\mathbb{P}^3$, then I_X has a minimal free resolution

$$0 \to \oplus_{i=1}^{\mu-3} S(d_{i+3}+c_1) \to \oplus_{i=1}^{2\mu-4} S(-e_i) \to \oplus_{i=1}^{\mu} S(-d_i) \to I_X \to 0,$$

where $d_i, e_i \in \mathbb{Z}$ and $c_1 = -d_1 - d_2 - d_3$. Therefore the ranks of the first and the second syzygy modules are determined by the number of elements in a minimal generating set of I_X. Also we give a relation for the degrees of syzygy modules of I_X. Using this theorem, one can construct a smooth quasi-complete intersection curve X such that the number of minimal generators of I_X is t for any given positive integer $t \in \mathbb{Z}^+$.

1. Preliminaries

Let X be a nondegenerate locally Cohen-Macaulay irreducible curve in $\mathbb{P}^3$, defined over an algebraically closed field K of characteristic 0. A curve $X \subset \mathbb{P}^3$ is said to be a monomial curve if it has a parametric representation of the form $(t_0^{n_3}, t_0^{n_3-n_1}t_1^{n_1}, t_0^{n_3-n_2}t_1^{n_2}, t_1^{n_3})$ where $n_1 < n_2 < n_3 \in \mathbb{Z}$ with g.c.d.$(n_1, n_2, n_3) = 1$. Let $I_X \subset S = K[x_0, x_1, x_2, x_3]$ be the homogeneous ideal of a monomial curve in $\mathbb{P}^3$ and let $\mu(I_X)$ be the number of elements in a minimal generating set of I_X. In his paper ([2]), Bresinsky proves that if $\mu(I_X) = \mu \geq 3$, then I_X has the following minimal free resolution:

$$0 \to \sum_{j=1}^{\mu-3} S(-d_{3,j}) \to \sum_{j=1}^{2\mu-4} S(-d_{2,j}) \to \sum_{j=1}^{\mu} S(-d_{1,j}) \to I_X \to 0,$$

where $d_{i,j} \in \mathbb{Z}$. Therefore the ranks of the first and the second syzygy modules of the homogeneous ideals of *monomial curves* in $\mathbb{P}^3$ are determined by the number of elements in a minimal generating set of I_X. In this paper, we give a similar result for quasi-complete intersection space curves. Here, a curve $X \subset \mathbb{P}^3$ is a

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quasi-complete intersection if it is locally Cohen-Macaulay and is cut out scheme-theoretically by three hypersurfaces in $\mathbb{P}^3$. In other words, the sheaf of the ideal $\mathcal{I}_X$ of a quasi-complete intersection space curve X has an exact sequence

$$0 \to \mathcal{E} \to \oplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-d_i) \xrightarrow{\varphi} \mathcal{I}_X \to 0, \tag{1.1}$$

where $\mathcal{E}$ is a kernel of φ and d_1, d_2, d_3 are the degrees of the hypersurfaces defining X. Note that $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^3$ because X is locally Cohen-Macaulay. A curve X is called a quasi-complete intersection curve of type (d_1, d_2, d_3) if X is defined scheme-theoretically by three surfaces f_1, f_2, f_3 of degrees d_1, d_2, d_3 in I_X with $d_1 \geq d_2 \geq d_3$.

Theorem 1.1. *Let $X \subset \mathbb{P}^3$ be a quasi-complete intersection space curve of type (d_1, d_2, d_3) and assume that the homogeneous ideal I_X of X has a minimal free resolution*

$$0 \to L_2 \to L_1 \to L_0 \to I_X \to 0, \tag{1.2}$$

where L_i is a free S-module of rank μ_i for each $i = 0, 1, 2$. Then $\mu_2 = \mu_0 - 3$ and $\mu_1 = 2\mu_0 - 4$. Furthermore, $L_0 = Hom(L_2, S)(c) \oplus (\oplus_{i=1}^{3} S(-d_i))$ and $L_1 \simeq Hom(L_1, S)(c)$ where $c = -d_1 - d_2 - d_3$.

To prove Theorem 1.1, we use results on the first cohomology group of a rank two vector bundle on $\mathbb{P}^3$. Let $\mathcal{E}$ be a rank two vector bundle on $\mathbb{P}^3$ and let $M_{\mathcal{E}} = H^1_*(\mathcal{E})$. Assume that $M_{\mathcal{E}}$ has the following minimal free resolution over $S = K[x_0, x_1, x_2, x_3]$:

$$0 \to M_4 \to M_3 \to M_2 \to M_1 \to M_0 \to M_{\mathcal{E}} \to 0,$$

where rank $M_0 = s$ and rank $M_1 = t$. Decker and Rao ([5], Proposition 1, and [12], Corollary 2.3) independently show that $t = 2s + 2$; i.e., the rank of the first syzygy module is determined by the number of minimal generators of $M_{\mathcal{E}}$. Furthermore Decker gives in his paper ([5]) sufficient conditions for having a cohomology module of a rank 2 vector bundle, i.e., that there exist isomorphisms $\Phi : M_1^*(c_1) \simeq M_1$ and $M_2 \simeq M_2' \oplus M_0^*(c_1)$ such that the composition $M_0^*(c_1) \xrightarrow{\sigma} M_1^*(c_1) \overset{\Phi}{\simeq} M_1 \to M_0$ is a trivial morphism. We show in Theorem 1.1 that the morphism σ induces a syzygy module morphism $L_2 \to L_1$. Here, $c_1 (:= c_1(\mathcal{E}))$ is the first Chern class of a vector bundle $\mathcal{E}$ and $M_0^* (:= Hom(M_0, S))$ is an S-dual module of M_0.

This paper is organized in the following way: In section 2, we prove Theorem 1.1 and give several corollaries. One of these is the characterization of quasi-complete intersections whose ideals are generated by four homogeneous polynomials. We show that the ideal of X is generated by four elements if and only if the Hartshorne-Rao module M_X is a "complete intersection" graded S-module with one generator, i.e., $M \simeq S(-d)/(f_1, f_2, f_3, f_4)$ where f_1, f_2, f_3, f_4 is a regular sequence and $d \in \mathbb{Z}$.

In section 3, we present a couple of examples and open questions. In particular, for any positive integer $t \in \mathbb{Z}$, one can construct a smooth quasi-complete intersection curve X such that the number of minimal generators of I_X is t.

Notation. Let $\mu(M)$ be the number of minimal generators of a finitely generated S-module M. For a curve $X \subset \mathbb{P}^3$, we denote the Hartshorne-Rao module by M_X, i.e., $M_X := H^1_*(\mathcal{I}_X)$; and for a coherent sheaf $\mathcal{F}$, we write $H^i_*(\mathcal{F}) = \oplus_{k \in \mathbb{Z}} H^i(\mathcal{F}(k))$. Also, note that $M^* = Hom(M, S)$ and $M^\vee = \oplus_{j \in \mathbb{Z}} Hom([M]_{-j}, K)$.

2. Theorem and corollaries

To prove Theorem 1.1, we need a nice theorem of Rao.

Theorem 2.1. *Let X be a curve in $\mathbb{P}^3$ and let M_X be its deficiency module, i.e., $M_X = H^1_*(\mathcal{I}_X)$. Assume that M_X has a minimal free resolution*

$$0 \to L_4 \stackrel{\sigma_4}{\to} L_3 \to L_2 \to L_1 \to L_0 \to M_X \to 0.$$

Then I_X has a minimal free resolution of the form

$$0 \to L_4 \stackrel{(\sigma_4,0)}{\to} L_3 \oplus \oplus_{i=1}^{r} S(-l_i) \to \oplus_{i=1}^{m} S(-e_i) \to I_X \to 0$$

for some integers e_i, l_i, r, m.

Proof. See [13], Theorem 2.5. □

Remark 2.2. Note that if X is minimal in its even liaison class with respect to degree, then it can furthermore be shown that the direct summand $\oplus_{i=1}^{r} S(-l_i)$ in Theorem 2.1 does not occur ([10]). One can see in the following proof of Theorem 1.1 that this is also true for a quasi-complete intersection curve X in $\mathbb{P}^3$ (Corollary 2.6).

Proof of Theorem 1.1. Since X is a quasi-complete intersection, we have an exact sequence

$$0 \to \mathcal{E} \to \oplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-d_i) \stackrel{\varphi}{\to} \mathcal{I}_X \to 0,$$

where $\mathcal{E}$ is a kernel of φ which is a rank 2 vector bundle. By the results of ([1], Proposition 1) and ([3], Theorem 1.6), the set of scheme-theoretic generators of X always extends to a set of minimal generators of the homogeneous ideal I_X of X. This set of minimal generators of I_X gives a surjective homomorphism $\oplus_{i=1}^{\mu} S(-d_i) \to I_X \to 0$ where μ is the number of minimal generators of I_X. Taking a sheafification, we obtain an exact sequence

$$0 \to \mathcal{F} \to \oplus_{i=1}^{\mu} \mathcal{O}_X(-d_i) \to \mathcal{I}_X \to 0,$$

with $H^1_*(\mathcal{F}) = 0$. Since X is locally Cohen-Macaulay, the coherent sheaf $\mathcal{F}$ is locally free of rank $\mu - 1$. These two exact sequences give the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E} & \to & \oplus_{i=1}^{3} \mathcal{O}_X(-d_i) & \to & \mathcal{I}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{F} & \to & \oplus_{i=1}^{\mu} \mathcal{O}_X(-d_i) & \to & \mathcal{I}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \oplus_{i=4}^{\mu} \mathcal{O}_X(-d_i) & \to & \oplus_{i=4}^{\mu} \mathcal{O}_X(-d_i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

If we take global sections on the first column, we get the exact sequence

$$0 \to E \to H^0_*(\mathcal{F}) \to \oplus_{i=4}^{\mu} S(-d_i) \stackrel{\psi}{\to} M_{\mathcal{E}} \to 0,$$

where $E = H^0_*(\mathcal{E})$ and $M_{\mathcal{E}} = H^1_*(\mathcal{E})$.

By Peterson's theorem ([11], Theorem 6.4.17), $\mu(I_X) - \mu(M_X^\vee) = 3$ where $M_X^\vee = \oplus_j \ Hom([H^2_*(\mathcal{E})]_{-j}, K)$. Since $M_X^\vee = \oplus_j H^1(\mathcal{E}(-c_1 - 4 + j)) = M_{\mathcal{E}}(-c_1 - 4)$ and

$\mu(I_X) = \mu$, we have $\mu(M_{\mathcal{E}}) = (\mu - 3)$. Therefore the morphism ψ is minimal and one has a minimal free resolution,

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \to \oplus_{i=4}^{\mu} S(-d_i) \to M_{\mathcal{E}} \to 0. \tag{2.1}$$

By the theorem of Rao and Decker ([5], [12]), the rank of the free module L_1 is $2\mu-4$ and $L_1^*(c_1) \simeq L_1$ where c_1 is the first Chern class of the vector bundle $\mathcal{E}$. Also, in the exact sequence in ([12], Corollary 2.3), one has a surjection $L_1^*(c_1) \to H_*^0(\mathcal{F}) \to 0$.

Since $H_*^1(\mathcal{F}) = 0$, from the exact sequence in the second row of the above diagram, we have

$$0 \to F_2 \to L_1^*(c_1) \xrightarrow{\phi} \oplus_{i=1}^{\mu} S(-d_i) \to I_X \to 0,$$

where F_2 is a kernel of the morphism ϕ. If depth $S/I_X = 2$, $F_2 = 0$. Otherwise, depth $S/I_X = 1$ and the projective dimension of I_X is two by the Auslander-Buchsbaum theorem. Therefore F_2 is a free module. By Serre duality, we have $M_{\mathcal{E}}^{\vee} \simeq M_X(-c_1 - 4) \simeq H_*^2(\mathcal{E})(-c_1 - 4) \simeq Ext_S^4(M_{\mathcal{E}}, S)(-4)$, and dualizing the exact sequence (2.1), M_X has a minimal free resolution,

$$0 \to (\oplus_{i=4}^{\mu} S(-d_i))^* \to L_1^* \to L_2^* \to L_3^* \to L_4^* \to M_X(-c_1) \to 0.$$

By Theorem 2.1, the homogeneous ideal I_X of X has a minimal free resolution,

$$0 \to (\oplus_{i=4}^{\mu} S(-d_i))^*(c_1) \to L_1^*(c_1) \to \oplus_{i=1}^{\mu} S(-d_i) \to I_X \to 0.$$

□

Our first application of Theorem 1.1 is a characterization of quasi-complete intersections whose ideals are generated by four elements. A quasi-complete intersection whose ideal is generated by two elements is a complete intersection, and a quasi-complete intersection whose ideal is generated by three elements is arithmetically Cohen-Macaulay. The geometry of these curves is well-known. Now we characterize quasi-complete intersections whose ideals are generated by four elements.

Lemma 2.3. *Assume that M is a "complete intersection" graded S-module with one generator, i.e., $M \simeq S(-d_{0,1})/(f_1, f_2, f_3, f_4)$ where f_1, f_2, f_3, f_4 is a regular sequence and $d_{0,1} \in \mathbb{Z}$. Then, M has the minimal free resolution*

$$0 \to S(d_{4,1}) \to \oplus_{i=1}^{4} S(-d_{3,i}) \to \oplus_{i=1}^{6} S(-d_{2,i})$$
$$\to \oplus_{i=1}^{4} S(-d_{1,i}) \to S(-d_{0,1}) \xrightarrow{\psi} M \to 0$$

for some $d_{i,j} \in \mathbb{Z}$.

Proof. This is a Koszul complex and is exact since f_1, f_2, f_3, f_4 is a regular sequence. □

Corollary 2.4. *Among all quasi-complete intersections with $\mu(X) \geq 4$, $\mu(X) = 4$ if and only if the Hartshorne-Rao module M_X is a "complete intersection" module.*

Proof. Assume that X is a quasi-complete intersection of type (d_1, d_2, d_3) with $\mu(X) = 4$. By Theorem 1.1, I_X has a minimal free resolution,

$$0 \to S(d_4 - d_1 - d_2 - d_3) \to \oplus_{i=1}^{4} S(-e_i) \to \oplus_{i=1}^{4} S(-d_i) \to I_X \to 0,$$

for some integers e_i and d_4. Then, by Theorem 2.1, the Hartshorne-Rao module M_X of X has a minimal free resolution,

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \to L_0 \to M_X \to 0,$$

with $\operatorname{rank} L_4 = 1$ and $\operatorname{rank} L_3 = 4$. Dualizing the exact sequence, we get

$$0 \to L_0^* \to L_1^* \to L_2^* \to L_3^* \to L_4^* \to Ext_S^4(M_X, S) \to 0.$$

This gives that $Ext_S^4(M_X, S)$ is a complete intersection module with one generator because $Ext_S^4(M_X, S)$ is a finite module. By Lemma 2.3, $\operatorname{rank} L_0 = 1$ and $\operatorname{rank} L_1 = 4$, and it follows that M_X is a complete intersection module with one generator.

Assume that M_X is a complete intersection module with one generator. Then M_X has a minimal free resolution,

$$0 \to L_0^* \to L_1^* \to L_2^* \to \oplus_{i=1}^4 S(e_i) \to S(-d) \to M_X \to 0.$$

By Lemma 2.3, $\operatorname{rank} L_0 = 1$, $\operatorname{rank} L_1 = 4$ and $\operatorname{rank} L_2 = 6$. By Theorem 1.1,

$$0 \to L_0^* \to L_1^* \to \oplus_{i=1}^4 S(-f_i) \to I_X \to 0$$

for some $f_i \in \mathbb{Z}$. Therefore, we get $\mu(X) = 4$. □

If X is a subcanonical quasi-complete intersection and M_X is generated by one element, then M_X is a complete intersection module with one generator and by Corollary (2.4), $\mu(I_X) = 4$. One can compare this result to the following theorem of Bresinsky, Schenzel, and Vogel.

Theorem 2.5 ([4], Theorem 2)**.** *Let X be a Buchsbaum, non-arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$. Let $N = \dim M_X$. If X is a quasi-complete intersection, then $\mu(I_X) = 4$ and $N = 1$.*

Also, note that by Rao's theorem ([12], Corollary 3.1) and Decker's theorem ([5], Proposition 1), $Ext_S^4(M_X, S)$ in the proof of Corollary 2.4 is a first cohomology module of a rank two vector bundle.

Another application of Theorem 1.1 is that if X is a quasi-complete intersection curve of type (d_1, d_2, d_3), then one can bound the degree of minimal generators of I_X in terms of (d_1, d_2, d_3). This bound is sharp as the homogeneous ideal of a twisted cubic curve is generated by quadric polynomials.

Corollary 2.6. *Let X be a quasi-complete intersection curve of type (d_1, d_2, d_3) in $\mathbb{P}^3$ with deficiency module M_X. Assume that M_X has a minimal free resolution*

$$0 \to L_4 \xrightarrow{\sigma_4} L_3 \to L_2 \to L_1 \to L_0 \to M_X \to 0. \tag{2.2}$$

Then I_X has a minimal free resolution of the form

$$0 \to L_4 \xrightarrow{\sigma_4} L_3 \to L_4^*(c_1) \oplus (\oplus_{i=1}^3 S(-d_i)) \to I_X \to 0,$$

where $c_1 = -(d_1 + d_2 + d_3)$.

Proof. This is clear from the proof of Theorem 1.1 and Theorem 2.1. □

Corollary 2.7. *If X is a quasi-complete intersection curve of type (d_1, d_2, d_3) in $\mathbb{P}^3$, then the homogeneous ideal I_X is generated by homogeneous polynomials of degree $\leq d_1 + d_2 - 2$.*

Proof. By Theorem 1.1, I_X has a minimal free resolution,

$$0 \to \oplus_{i=1}^{\mu-3} S(d_{i+3} + c_1) \to \oplus_{i=1}^{2\mu-4} S(-e_i) \to \oplus_{i=1}^{\mu} S(-d_i) \to I_X \to 0.$$

Let $m_1 = \max\{d_i\}$, $m_2 = \min\{d_i\}$ and $n_1 = \max\{e_i\}$. It is well-known that the minimal degree of the hypersurface containing a quasi-complete intersection X of type (d_1, d_2, d_3) is d_3, i.e., $m_2 = d_3$ ([1]). By ([7], page 304), we have $m_1 < n_1$. On the other hand, we get $n_1 < -(c_1 + m_2)$ since the dual of the exact sequence (2.2) is also a minimal free resolution of $\mathrm{Ext}^4_S(M_X, S)$ and the minimal twisting numbers of syzygy modules are increasing. Therefore, we have

$$m_1 \le -c_1 - d_3 - 2 = (d_1 + d_2) - 2.$$

□

Finally, one can consider Theorem 1.1 as a generalization of the following corollary.

Corollary 2.8. *If X is a quasi-complete intersection curve in $\mathbb{P}^3$ defined by three surfaces f_1, f_2, f_3 in I_X of degrees d_1, d_2, d_3, then we have*

$$M_X^\vee(4) \simeq Ext^4_S(M_X, S) \simeq I_X/(f_1, f_2, f_3)(d_1 + d_2 + d_3)$$

as an S-module.

Proof. See ([13], page 215) or ([4], page 289). □

Corollary 2.8 just gives an isomorphism between two S-modules $M_X^\vee(4)$ and $I_X/(f_1, f_2, f_3)(d_1 + d_2 + d_3)$. Theorem 1.1 states that if X is a quasi-complete intersection, then a minimal free resolution of the homogeneous ideal I_X is determined by a minimal free resolution of the Hartshorne-Rao module M_X.

3. Examples and open questions

Example 3.1. Let X be a Buchsbaum curve in $\mathbb{P}^3$ with $H^1(\mathcal{I}_X(k)) \neq 0$ for some integer k. If X is defined scheme-theoretically by three surfaces f_1, f_2, f_3 of degrees d_1, d_2, d_3 in I_X, then $\dim(M_X) = 1$ and the number of minimal generators of I_X is 4 ([11]). Therefore, by Theorem 1.1, the homogeneous ideal I_X has a minimal free resolution,

$$0 \to S(d_4 - d_1 - d_2 - d_3) \to \oplus_{i=1}^4 S(-e_i) \to \oplus_{i=1}^4 S(-d_i) \to I_X \to 0,$$

for some integers e_i and d_4.

Example 3.2. Let X be a smooth elliptic curve in $\mathbb{P}^3$ of degree $t + 4$. Since $\omega_X = \mathcal{O}_X$, X is the zero locus of a section of a rank two vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ with $c_1(\mathcal{F}) = 4$ and $c_2(\mathcal{F}) = t + 4$ ([12], Example 3.5, and [8], (4.3.3)). If $\mathcal{E} = \mathcal{F}(-2)$, we have

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{E}(-2) \to \mathcal{I}_X \to 0.$$

Since $H^1(\mathcal{P}^3, \mathcal{I}_X(l)) = 0$ for $l \le 0$, $\mathcal{E}$ is a mathematical instanton bundle. Since $H^1(\mathcal{E}(-1)) = H^1(\mathcal{I}_X(1)) = t$, $M_{\mathcal{E}}$ has a minimal free resolution,

$$\oplus S(-d_i) \to S^{\oplus 2t+2} \to S(1)^{\oplus t} \to M_{\mathcal{E}} \to 0.$$

By Theorem 2.5 in [6], a residual Y of X in a complete intersection of two surfaces of degrees d_1 and d_2 is a quasi-complete intersection of type $(d_1 + d_2 - 4, d_1, d_2)$. By Theorem 1.1, the homogeneous ideal I_Y has a minimal free resolution,

$$0 \to \oplus_{i=1}^t S(f_i + c_1) \to L_1$$
$$\to \oplus_{i=1}^t S(-f_i) \oplus S(-d_1) \oplus S(-d_2) \oplus S(-d_1 - d_2 + 4) \to I_Y \to 0,$$

where $c_1 = -(2d_1 + 2d_2 - 4)$ and $f_i \in \mathbb{Z}$. Therefore, for any given number t, one can construct a quasi-complete intersection curve Y in $\mathbb{P}^3$ such that the number of minimal generators of I_Y is t.

Prof. Schenzel raises the following questions:

Question 3.3. Bresinsky's work ([2]) and Theorem 1.1 determine the structure of the minimal free resolutions of homogeneous ideals I_X of monomial curves and quasi-complete intersections. What is the common generalization of monomial curves and quasi-complete intersections? How do we enlarge the classes of varieties which have the same type of minimal free resolutions as in Theorem 1.1?

For example, one of the nice properties of monomial curves is the following:

Theorem 3.4 ([4], Theorem 3)**.** *Let X be a monomial curve in $\mathbb{P}^3$ over an algebraically closed field K. Then the following conditions are equivalent:*

(1) $\mu(I_X) = 4$ *and X lies on a quadric.*
(2) *There is an integer $n \geq 1$ such that the minimal free resolution of S/I_X has the form*

$$0 \to S(-2n-3) \to S^4(-2n-2) \to S(-2) \oplus S^3(-2n-1) \to S \to S/I_X \to 0.$$

On the other hand, by Corollary 2.4, if X is a quasi-complete intersection of type (d_1, d_2, d_3) with $\mu(I_X) = 4$, then I_X has the following minimal free resolution:

$$0 \to S(d_4 - d_1 - d_2 - d_3) \to \oplus_{i=1}^4 S(-e_i) \to \oplus_{i=1}^4 S(-d_i) \to I_X \to 0$$

for some integers e_i and d_4.

Therefore, it would also be very interesting to know whether there exists a monomial curve X that has $\mu(I_X) = 4$ but is not a quasi-complete intersection.

Question 3.5. How can we characterize the monomial curves in $\mathbb{P}^3$ that are quasi-complete intersections?

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References

[1] V. Beorchia, Ph. Ellia, On the equations defining quasicomplete intersection space curves, *Arch. Math. (Basel)* **70** (1998) 244-249. MR1604080 (98k:14042)
[2] H. Bresinsky, Minimal free resolutions of monomial curves in $\mathbb{P}^3$, *Linear Alg. Appl.* **59** (1984) 121-129. MR743050 (85d:14042)
[3] H. Bresinsky, P. Schenzel, J. Stückrad, Quasi-complete intersection ideals of height 2, *J. Pure Appl. Algebra* **127** (1998) 137–145. MR1620704 (99d:13014)
[4] H. Bresinsky, P. Schenzel, W. Vogel, On liaison, arithmetical Buchsbaum curves and monomial curves in $\mathbb{P}^3$, *J. Algebra* **86** (1984) 283-301. MR732252 (85c:14031)
[5] W. Decker, Monads and cohomology modules of rank 2 vector bundles, *Compositio Math.* **76** (1990) 7–17. MR1078855 (91k:14030)
[6] D. Franco, S. Kleiman, A. Lascu, Gherardelli linkage and complete intersections, *Michigan Math. J.* **48** (2000) 271–279. MR1786490 (2002a:14057)
[7] M. Green, R. Lazarsfeld, Some results on the syzygies of finite sets and algebraic curves, *Compositio Math.* **67** (1988) 301–314. MR959214 (90d:14034)
[8] R. Hartshorne, Stable vector bundles of rank 2 on $\mathbb{P}^3$, *Math. Ann.* **238** (1978) 229–280. MR514430 (80c:14011)
[9] F. Horrocks, Vector bundles on the punctured spectrum of a local ring, *Proc. London Math. Soc.* **14** (3) (1964) 689–713. MR0169877 (30:120)
[10] M. Martin-Deschamps, D. Perrin, Sur la classification des courbes gauches, *Astérisque* **184-185** (1990). MR1073438 (91h:14039)
[11] J. Migliore, Introduction to Liaison Theory and Deficiency Modules, Birkhäuser Boston, 1998. MR1712469 (2000g:14058)

[12] P. Rao, A note on cohomology modules of rank two bundles, *J. Algebra* **86** (1984) 23–34. MR727366 (85b:14022)

[13] P. Rao, Liaison among curves in $\mathbb{P}^3$, *Invent. Math.* **50** (1978/79) 205–217. MR520926 (80e:14023)

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