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BURGHELEA-HALLER ANALYTIC TORSION FOR MANIFOLDS WITH BOUNDARY

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ABSTRACT. In this paper, we extend the complex-valued Ray-Singer torsion, introduced by Burghelea-Haller, to compact connected Riemannian manifolds with boundary. We also compare it with the refined analytic torsion.

1. INTRODUCTION

Let E be a unitary flat vector bundle on a closed Riemannian manifold M. In [20], Ray and Singer defined an analytic torsion associated to (M, E) and proved that it does not depend on the Riemannian metric on M. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on M (cf. [16]). This conjecture was later proved in the celebrated papers of Cheeger [11] and Müller [17]. Müller generalized this result in [18] to the case when E is a unimodular flat vector bundle on M. In [2], inspired by the considerations of Quillen [19], Bismut and Zhang reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology and proved an extension of it to the case of general flat vector bundle over M. The method used in [2] and [3] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [24] on the de Rham complex.

Braverman and Kappeler [4, 5] defined the refined analytic torsion for the flat vector bundle over odd dimensional manifolds and showed that it equals the Turaev torsion (cf. [12, 22]) up to a multiplication by a complex number of absolute value one. Burghelea and Haller [7, 8], following a suggestion of Müller, defined a generalized analytic torsion associated to a non-degenerate symmetric bilinear form on a flat vector bundle over an arbitrary dimensional manifold and made an explicit conjecture between this generalized analytic torsion and the Turaev torsion. This conjecture was proved up to sign by Burghelea-Haller [9] and in full generality by Su-Zhang [21].

Vertman [23] defined a different refinement of analytic torsion, similar to Braverman and Kappeler, which applied to compact manifolds with and without boundary. Inspired by this, in the present paper, we extend the Burghelea-Haller analytic torsion to compact connected Riemannian manifolds with boundary.

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The rest of this paper is organized as follows. In Section 2, we recall the definition of a Hilbert complex and some properties of it, in particular, the Hilbert complexes $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max},\nabla_{\max})$. In Section 3, we get some properties of the Hilbert complexes $(\mathscr{D}_{\min},\nabla_{\min})$, $(\mathscr{D}_{\max},\nabla_{\max})$ and extend the Burghelea-Haller analytic torsion to these complexes. In Section 4, we compare the extended Burghelea-Haller analytic torsion with the refined analytic torsion.

2. Fredholm complexes for compact manifolds

Let (M, g^{TM}) be a smooth n-dimensional compact connected Riemannian manifold with boundary ∂M, which may be empty. Let (E, ∇) be a flat complex vector bundle over M. The flat connection ∇ extends to $\Omega_0^*(M, E)$, which are E-valued differential forms with compact support in the interior of the manifold M. Since $\nabla^2 = 0$, we have the de Rham complex $(\Omega_0^*(M, E), \nabla)$. Assume that there is a fiberwise non-degenerate symmetric bilinear form b on E. By [8, Theorem 5.10], there exists a complex anti-linear involution $\nu : E \to E$ such that

$$\nu^2 = \mathrm{id}_E, \quad b(\nu x, y) = \overline{b(x, \nu y)}, \quad b(x, \nu x) \geq 0, \quad x, y \in E.$$

Then

$$\mu : E \otimes E \to \mathbb{C}, \quad \mu(x, y) = b(x, \nu y)$$

is a fiberwise positive definite Hermitian structure on E. The Riemannian metric g^{TM}, together with the Hermitian metric μ, defines an inner product in $\Omega_0^*(M, E)$ which we denote by $h_{g,\mu}$. We denote the L^2-completion of $\Omega_0^*(M, E)$ by $L_*^2(M, E)$. The Riemannian metric g^{TM}, together with the fiber wise non-degenerate symmetric bilinear form b^E, defines a non-degenerate symmetric bilinear form $\beta_{g,b}$ on $\Omega_0^*(M, E)$ such that if $u = \alpha f$, $v = \beta g \in \Omega_0^*(M, E)$ such that $\alpha, \beta \in \Omega^*(M)$, $f, g \in \Gamma(E)$, then

$$\beta_{g,b}(\omega, \eta) = \int_M (\alpha \wedge *_g \beta) b(f, g).$$

Then $\beta_{g,b}$ extends to a non-degenerate symmetric bilinear form on $L_*^2(M, E)$; we still denote its extension on $L_*^2(M, E)$ by $\beta_{g,b}$. Then we have $h_{g,\mu}(\omega, \eta) = \beta_{g,b}(\omega, \nu\eta)$.

Consider the differential operator ∇ and its formal adjoint ∇^t with respect to the inner product. The associated minimal closed extensions $\nabla_{\min}$ and $\nabla^t_{\min}$ are defined as the graph-closures in $L_*^2(M, E)$. The maximal closed extension of ∇ is defined by

$$\nabla_{\max} = \left(\nabla^t_{\min}\right)^*,$$

where $*$ denotes the adjoint operator with respect to the inner product in $L_*^2(M, E)$. We denote $\nabla^{\#}_{\min}$ and $\nabla^{\#}_{\max}$ to be the adjoint operators of $\nabla_{\min}$ and $\nabla_{\max}$ with respect to $\beta_{g,b}$ on $L_*^2(M, E)$. Let $\mathscr{D}_{\min}$, $\mathscr{D}^{\#}_{\min}$ be the domain of $\nabla_{\min}$, $\nabla^{\#}_{\min}$ respectively and let $\mathscr{D}_{\max}$, $\mathscr{D}^{\#}_{\max}$ denote the domain of $\nabla_{\max}$, $\nabla^{\#}_{\max}$ respectively. These extensions define Hilbert complexes in the following sense, as introduced in [10].

Definition 2.1 ([10]). Let the Hilbert spaces H_i, $i = 0, \cdots, m$, $H_{m+1} = \{0\}$ be mutually orthogonal. For each $i = 0, \cdots, m$, let $D_i \in C(H_i, H_{i+1})$ be a closed operator with domain $\mathscr{D}(D_i)$ dense in H_i and range in H_{i+1}. Put $\mathscr{D}_i = \mathscr{D}(D_i)$ and $R_i = D_i(\mathscr{D}_i)$, and assume

$$R_i \subseteq \mathscr{D}_{i+1}, \quad D_{i+1} \circ D_i = 0.$$

This defines a complex $(\mathscr{D}_*, D_*)$,

$$0 \longrightarrow \mathscr{D}_0 \xrightarrow{D_0} \mathscr{D}_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{m-1}} \mathscr{D}_m \longrightarrow 0. \tag{2.1}$$

Such a complex is called a Hilbert complex. If the homology of the complex is finite, i.e. if R_i is closed and $\ker D_i/\mathrm{im} D_{i-1}$ is finite-dimensional for all $i = 0, \cdots, m$, the complex is referred to as a Fredholm complex.

For a Hilbert complex there is a dual Hilbert complex

$$0 \longrightarrow \mathscr{D}_m \xrightarrow{D^*_{m-1}} \mathscr{D}_{m-1} \xrightarrow{D^*_{m-2}} \cdots \xrightarrow{D^*_0} \mathscr{D}_0 \longrightarrow 0$$

defined using the Hilbert space adjoints of the differentials D_i^* and Laplacian $\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*$. We can compute the cohomology groups of the Hilbert complex (2.1) using the subcomplex $(\mathcal{D}^\infty \mathscr{D}_*, D_*)$, where $\mathcal{D}^\infty \mathscr{D}_i$ consists of all elements x that are in the domain of Δ_i^l for all $l \geq 0$.

Proposition 2.2 ([10, Theorem 2.12])**.** *The cohomology of the complex $(\mathscr{D}_*, D_*)$ is equal to the cohomology of the complex $(\mathcal{D}^\infty \mathscr{D}_*, D_*)$.*

By [10, Lemma 3.1] we have Hilbert complexes $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max}, \nabla_{\max})$, where $\mathscr{D}_{\min} = \mathscr{D}(\nabla_{\min})$ and $\mathscr{D}_{\max} = \mathscr{D}(\nabla_{\max})$. The following theorem [23, Theorem 3.2] is the twisted setup of [10, Theorem 4.1].

Theorem 2.3. *The Hilbert complexes $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max}, \nabla_{\max})$ are Fredholm with the associated Laplacians Δ_{rel} and Δ_{abs} being strongly elliptic in the sense of* [13]*. The de Rham isomorphism identifies the cohomology of the complexes with the relative and absolute cohomology with coefficients*

$$H^*(\mathscr{D}_{\min}, \nabla_{\min}) \cong H^*(M, \partial M, E),$$
$$H^*(\mathscr{D}_{\max}, \nabla_{\max}) \cong H^*(M, E).$$

Furthermore the cohomology of Fredholm complexes $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max}, \nabla_{\max})$ can be computed from the following smooth subcomplexes:

$$(\Omega^*_{\min}(M,E), \nabla),\ \Omega^*_{\min}(M,E) = \{\omega \in \Omega^*(M,E) | l^*(\omega) = 0\},$$
$$(\Omega^*_{\max}(M,E), \nabla),\ \Omega^*_{\max}(M,E) = \Omega^*(M,E),$$

respectively, where we denote by $l : \partial M \to M$ the natural inclusion of the boundary.

3. Ray-Singer symmetric bilinear torsion for $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max}, \nabla_{\max})$

In this section we define the Ray-Singer symmetric bilinear torsions for the Hilbert complexes $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max}, \nabla_{\max})$.

Proposition 3.1. *The restrictions of the non-degenerate symmetric bilinear form $\beta_{g,b}$ to $\mathscr{D}_{\min}$ and $\mathscr{D}_{\max}$ are all non-degenerate.*

Proof. Let $x \in \mathscr{D}_{\min}$; then there exist $\{x_n\} \subset \Omega_0^*(M,E)$ such that $x_n \to x$ in $L^2_*(M,E)$ and ∇x_n convergence in $L^2_*(M,E)$. Since ν is a bounded operator, we get $\nu(x_n) \to \nu x$ in $L^2_*(M,E)$. By

$$\nabla(\nu x_n) = (\nabla \nu) x_n + \nu(\nabla x_n)$$

and $\nabla\nu$ a bounded operator, we get $\nabla(\nu x_n)$ convergence in $L^2_*(M,E)$. Then by definition of $\nabla_{\min}$, we get $\nu x \in \mathscr{D}_{\min}$, so that if for any $y \in \mathscr{D}_{\min}$, $\beta_{g,b}(x,y) = 0$, then by

$$h_{g,\mu}(x,x) = \beta_{g,b}(x,\nu x) = 0$$

we get $x = 0$. Then the restriction of $\beta_{g,b}$ to $\mathscr{D}_{\min}$ is non-degenerate.

For the complex $(\mathscr{D}_{\max}, \nabla_{\max})$, we first recall that for $\sigma \in L^2_*(M,E)$, if there exists $\eta \in L^2_*(M,E)$ such that for any $\phi \in \Omega^*_0(M,E)$,

$$h_{g,\mu}\left(\sigma, \nabla^t\phi\right) = h_{g,\mu}(\eta,\phi),$$

then $\sigma \in \mathscr{D}_{\max}$ and $\nabla_{\max}\sigma = \eta$. If $\sigma \in \mathscr{D}_{\max}$, then for $\nu\sigma \in L^2_*(M,E)$, and for any $\phi \in \Omega^*_0(M,E)$ then $\nu\phi \in \Omega^*_0(M,E)$, we have

$$\begin{aligned}
(3.1)\quad h_{g,\mu}\left(\nu\sigma, \nabla^t(\nu\phi)\right) &= \overline{h_{g,\mu}\left(\nabla^t(\nu\phi), \nu\sigma\right)} = \overline{\beta_{g,b}\left(\nabla^t(\nu\phi), \sigma\right)} \\
&= \overline{\beta_{g,b}\left(\nu\left(\nabla^t\phi\right) - \left(\nabla\nu^*\right)^*\phi, \sigma\right)} = \overline{\beta_{g,b}\left(\nu\left(\nabla^t\phi\right), \sigma\right)} - \overline{\beta_{g,b}\left(\left(\nabla\nu^*\right)^*\phi, \sigma\right)} \\
&= \beta_{g,b}\left(\left(\nabla^t\phi\right), \nu\sigma\right) - \overline{h_{g,\mu}\left(\left(\nabla\nu^*\right)^*\phi, \nu\sigma\right)} = h_{g,\mu}\left(\left(\nabla^t\phi\right), \sigma\right) - h_{g,\mu}\left(\nu\sigma, \left(\nabla\nu^*\right)^*\phi\right) \\
&= \overline{h_{g,\mu}\left(\sigma, \left(\nabla^t\phi\right)\right)} - h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma, \nu\phi\right) = \overline{h_{g,\mu}(\eta,\phi)} - h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma, \nu\phi\right) \\
&= h_{g,\mu}(\phi,\eta) - h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma, \nu\phi\right) = \beta_{g,b}(\phi,\nu\eta) - h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma, \nu\phi\right) \\
&= \overline{\beta_{g,b}(\nu\phi,\eta)} - h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma, \nu\phi\right) = \overline{h_{g,\mu}(\nu\phi,\nu\eta)} - h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma, \nu\phi\right) \\
&= h_{g,\mu}\left(\nu\eta - \nu^*\left(\nabla\nu^*\right)\nu\sigma, \nu\phi\right).
\end{aligned}$$

Then by definition we get $\nu\sigma \in \mathscr{D}_{\max}$. Then as in the case of $\mathscr{D}_{\min}$, we get that the restriction of $\beta_{g,b}$ to $\mathscr{D}_{\max}$ is non-degenerate. □

Proposition 3.2. *The following identities hold:*

$$(3.2)\qquad \nabla^{\#}_{\min} = \nabla^*_{\min} + \left(\nu\left(\nabla\nu\right)\right)^*, \ \nabla^{\#}_{\max} = \nabla^*_{\max} + \left(\nabla\nu^*\right)^*\nu.$$

Particularly, $\mathscr{D}^{\#}_{\min}$ *equals the domain of* $\nabla^*_{\min}$ *and* $\mathscr{D}^{\#}_{\max}$ *equals the domain of* $\nabla^*_{\max}$.

Proof. Let $y \in \mathscr{D}^{\#}_{\min}$; then there exists $z \in L^2_*(M,E)$ such that for any $x \in \mathscr{D}_{\min}$, we have

$$\beta_{g,b}\left(\nabla_{\min}x, y\right) = \beta_{g,b}(x,z).$$

By Proposition 3.1, we have $\nu x \in \mathscr{D}_{\min}$. Then

$$\begin{aligned}
(3.3)\quad h_{g,\mu}\left(\nabla_{\min}(\nu x), y\right) &= h_{g,\mu}\left((\nabla\nu)x + \nu\nabla_{\min}x, y\right) \\
&= h_{g,\mu}\left((\nabla\nu)x, y\right) + h_{g,\mu}\left(\nu\nabla_{\min}x, y\right) = h_{g,\mu}\left((\nabla\nu)\nu\nu x, y\right) + \overline{\beta_{g,b}\left(\nabla_{\min}x, y\right)} \\
&= h_{g,\mu}\left(\nu x, \left((\nabla\nu)\nu\right)^* y\right) + \overline{\beta_{g,b}(x,z)} = h_{g,\mu}\left(\nu x, \left((\nabla\nu)\nu\right)^* y\right) + \beta_{g,b}(\nu x, \nu z) \\
&= h_{g,\mu}\left(\nu x, \left((\nabla\nu)\nu\right)^* y\right) + h_{g,\mu}(\nu x, z).
\end{aligned}$$

Then by definition we have $y \in \mathscr{D}(\nabla^*_{\min})$ and

$$(3.4)\qquad \nabla^*_{\min}y = \left((\nabla\nu)\nu\right)^* y + \nabla^{\#}_{\min}y.$$

Since $\nu^2 = Id$, we have that $\nu(\nabla\nu) = -(\nabla\nu)\nu$. Then by (3.4) we get

$$\nabla^{\#}_{\min} = \nabla^*_{\min} + \left(\nu\left(\nabla\nu\right)\right)^*.$$

For $\sigma \in \mathscr{D}_{\max}$, by (3.1) we have $\nu\sigma \in \mathscr{D}_{\max}$ and

$$\nabla_{\max}(\nu\sigma) = \nu\left(\nabla_{\max}\sigma\right) - \nu^*\left(\nabla\nu^*\right)\nu\sigma.$$

Let $\eta \in \mathscr{D}^{\#}_{\max}$; then there exists $\omega \in L^2_*(M,E)$ such that for any $\sigma \in \mathscr{D}_{\max}$, we have

$$\beta_{g,b}\left(\nabla_{\max}\sigma, \eta\right) = \beta_{g,b}(\sigma, \omega).$$

Then

$$\begin{aligned}(3.5)\quad & h_{g,\mu}\left(\nabla_{\max}(\nu\sigma), \eta\right) = h_{g,\mu}\left(\nu\nabla_{\max}\sigma - \nu^*\left(\nabla\nu^*\right)\nu\sigma, \eta\right)\\ &= h_{g,\mu}\left(\nu\nabla_{\max}\sigma, \eta\right) - h_{g,\mu}\left(\nu\sigma, \left(\nabla\nu^*\right)^*\nu\eta\right) = \overline{\beta_{g,b}\left(\nabla_{\max}\sigma, \eta\right)} - h_{g,\mu}\left(\nu\sigma, \left(\nabla\nu^*\right)^*\nu\eta\right)\\ &= h_{g,\mu}(\nu\sigma, \omega) - h_{g,\mu}\left(\nu\sigma, \left(\nabla\nu^*\right)^*\nu\eta\right).\end{aligned}$$

By definition we have $\eta \in \mathscr{D}(\nabla^*_{\max})$ and

$$\nabla^*_{\max}\eta = \nabla^{\#}_{\max}\eta - \left(\nabla\nu^*\right)^*\nu\eta.$$

The proof of Proposition 3.2 is complete. □

We consider the operator

$$\Delta_{b,\mathrm{rel/abs}} = \left(\nabla_{\min/\max} + \nabla^{\#}_{\min/\max}\right)^2 = \nabla_{\min/\max}\nabla^{\#}_{\min/\max} + \nabla^{\#}_{\min/\max}\nabla_{\min/\max}.$$

Next we will construct the symmetric bilinear torsions for $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max}, \nabla_{\max})$. We will write the construction for the complex $(\mathscr{D}_{\min}, \nabla_{\min})$ explicitly. The construction for the complex $(\mathscr{D}_{\max}, \nabla_{\max})$ is exactly the same.

The domain of $\Delta_{b,\mathrm{rel}}$ is the following:

$$\mathscr{D}(\Delta_{b,\mathrm{rel}}) = \left\{x \in \mathscr{D}_{\min} \cap \mathscr{D}^{\#}_{\min} | \nabla_{\min}x \in \mathscr{D}^{\#}_{\min} \text{ and } \nabla^{\#}_{\min}x \in \mathscr{D}_{\min}\right\}.$$

By (3.2), we see that the domain of $\Delta^l_{b,\mathrm{rel}}$ equals the domain of Δ^l_{rel} for all $l \geq 0$. By Proposition 3.2, $\Delta_{b,\mathrm{rel}}$ has the same leading symbol with $\Delta_{\mathrm{rel}} = (\nabla_{\min} + \nabla^*_{\min})^2$. Then the spectral of $\Delta_{b,\mathrm{rel}}$ is discrete. Let $\lambda \in \mathrm{Spec}(\Delta_{b,\mathrm{rel}})$ and denote by $P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ the spectral projection of $\Delta_{b,\mathrm{rel}}$ corresponding to λ. Then

$$P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} = \frac{i}{2\pi}\int_{C(\lambda)} \left(\Delta_{b,\mathrm{rel}} - x\right)^{-1} dx,$$

with $C(\lambda)$ being any closed counterclockwise circle surrounding λ with no other spectrum inside. The image of $P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ is finite dimensional. In particular $P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ is a bounded operator in $L^2_*(M,E)$. Then by [14, Section 4, p. 155] the decomposition

$$(3.6)\qquad L^2_*(M,E) = \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \oplus \mathrm{Im}\left(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}\right)$$

is a direct sum decomposition into closed subspaces of the Hilbert space $L^2_*(M,E)$.

Proposition 3.3. *The decomposition* $L^2_*(M,E) = \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \oplus \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ *is* $\beta_{g,b}$*-orthogonal.*

Proof. Let N_λ be the multiplicity of the generalized eigenvalue λ. Then we have $(\Delta_{b,\mathrm{rel}} - \lambda)^{N_\lambda}|_{\mathscr{D}(\Delta_{b,\mathrm{rel}})\cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}} = 0$, and $\Delta_{b,\mathrm{rel}} - \lambda : \mathscr{D}(\Delta_{b,\mathrm{rel}}) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}) \to \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ has an everywhere defined bounded inverse. By the decomposition (3.6), for $\Omega^*_0(M,E)$ we have

$$(3.7)\quad \Omega^*_0(M,E) = \Omega^*_0(M,E) \cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \oplus \Omega^*_0(M,E) \cap \mathrm{Im}\left(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}\right).$$

In particular $(\Delta_{b,\mathrm{rel}} - \lambda)^{N_\lambda}|_{\Omega^*_0(M,E)\cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}} = 0$, and $\Delta_{b,\mathrm{rel}} - \lambda : \Omega^*_0(M,E) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}) \to \Omega^*_0(M,E) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ is bijective. So the decomposition (3.7) is $\beta_{g,b}$-orthogonal. In fact, for $\omega \in \Omega^*_0(M,E) \cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and

$\eta \in \Omega_0^*(M,E) \cap \mathrm{Im}(1-P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ there exists $\eta_{N_\lambda} \in \Omega_0^*(M,E) \cap \mathrm{Im}(1-P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ such that

$$(\Delta_{b,\mathrm{rel}} - \lambda)^{N_\lambda} \eta_{N_\lambda} = \eta.$$

Then we have

$$\beta_{g,b}(\omega,\eta) = \beta_{g,b}\left(\omega, (\Delta_{b,\mathrm{rel}} - \lambda)^{N_\lambda} \eta_{N_\lambda}\right) = \beta_{g,b}\left((\Delta_{b,\mathrm{rel}} - \lambda)^{N_\lambda} \omega, \eta_{N_\lambda}\right) = 0.$$

For $x \in \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and $y \in \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ there exist $\{x_n\} \subset \Omega_0^*(M,E)$ and $\{y_n\} \subset \Omega_0^*(M,E)$ such that $x_n \to x$ and $y_n \to y$. Since $\mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and $\mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ are closed subspaces of $L^2_*(M,E)$, for sufficient large n, $\beta_{g,b}(x_n,y_n) = 0$. Then we have

$$\beta_{g,b}(x,y) = \lim_{n\to\infty} \beta_{g,b}(x_n,y_n) = 0.$$

The proof of Proposition 3.3 is complete. □

Using the analytic Fredholm theorem, one finds $\mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \subset \mathscr{D}(\Delta_{b,\mathrm{rel}})$; therefore $\mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \subset \mathscr{D}_{\min}$. Then by the decomposition (3.6) we can decompose $\mathscr{D}_{\min}$ as follows:

$$\mathscr{D}_{\min} = \mathscr{D}_{\min} \cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \oplus \mathscr{D}_{\min} \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}).$$

By Proposition 3.1 and Proposition 3.3, we get that the restrictions of $\beta_{g,b}$ to $\mathscr{D}_{\min} \cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and $\mathscr{D}_{\min} \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ are all non-degenerate. For any $a \geq 0$, let $P_{[0,a],\Delta_{b,\mathrm{rel}}}$ be the spectral projection of $\Delta_{b,\mathrm{rel}}$ corresponding to the spectral with absolute value in $[0,a]$. Then we have the $\beta_{g,b}$-orthogonal decomposition

$$\mathscr{D}_{\min} = \mathscr{D}_{\min,[0,a]} \oplus \mathscr{D}_{\min,(a,\infty)},$$

where $\mathscr{D}_{\min,[0,a]} = \mathscr{D}_{\min} \cap \mathrm{Im}P_{[0,a],\Delta_{b,\mathrm{rel}}}$ and $\mathscr{D}_{\min,(a,\infty)} = \mathscr{D}_{\min} \cap \mathrm{Im}(1 - P_{[0,a],\Delta_{b,\mathrm{rel}}})$. By Proposition 3.1, we get that the restrictions of $\beta_{g,b}$ to $\mathscr{D}_{\min,[0,a]}$ and $\mathscr{D}_{\min,(a,\infty)}$ are all non-degenerate. Since $\nabla_{\min}$ commutes with $\Delta_{b,\mathrm{rel}}$, we get two subcomplexes $(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]})$ and $(\mathscr{D}_{\min,(a,\infty)}, \nabla_{\min,(a,\infty)})$ such that

$$(\mathscr{D}_{\min}, \nabla_{\min}) = \left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) \oplus \left(\mathscr{D}_{\min,(a,\infty)}, \nabla_{\min,(a,\infty)}\right).$$

Proposition 3.4. *The inclusion* $(\mathscr{D}_{\min,\{0\}}, \nabla_{\min,\{0\}}) \to (\mathscr{D}_{\min}, \nabla_{\min})$ *induces an isomorphism on cohomology. In particular, the subcomplex* $(\mathscr{D}_{\min,(a,\infty)}, \nabla_{\min,(a,\infty)})$ *is acyclic for any* $a \geq 0$, *and*

$$H^*(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}) \cong H^*(\mathscr{D}_{\min}, \nabla_{\min}). \tag{3.8}$$

Proof. By Proposition 2.2, in order to compute the cohomology of $(\mathscr{D}_{\min,\{0\}}, \nabla_{\min,\{0\}})$ and $(\mathscr{D}_{\min}, \nabla_{\min})$ we need only to compute the cohomology of $(\mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}, \nabla_{\min,\{0\}})$ and $(\mathcal{D}^\infty \mathscr{D}_{\min}, \nabla_{\min})$. Then we only need to prove that the inclusion

$$(\mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}, \nabla_{\min,\{0\}}) \to (\mathcal{D}^\infty \mathscr{D}_{\min}, \nabla_{\min})$$

induces an isomorphism of cohomology groups. Since $\Delta_{b,\mathrm{rel}} : \mathcal{D}^\infty \mathscr{D}_{\min} \to \mathcal{D}^\infty \mathscr{D}_{\min}$, then $\Delta_{b,\mathrm{rel}}$ induces an isomorphism

$$\Delta_{b,\mathrm{rel}} : \mathcal{D}^\infty \mathscr{D}_{\min} / \mathcal{D}^\infty \mathscr{D}_{\min,\{0\}} \to \mathcal{D}^\infty \mathscr{D}_{\min} / \mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}.$$

It therefore induces an isomorphism on the cohomology group

$$H^*\left(\mathcal{D}^\infty \mathscr{D}_{\min} / \mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}, \nabla_{\min}\right) \to H^*\left(\mathcal{D}^\infty \mathscr{D}_{\min} / \mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}, \nabla_{\min}\right). \tag{3.9}$$

For $[x] \in H^*\left(\mathcal{D}^\infty \mathscr{D}_{\min}/\mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}, \nabla_{\min}\right)$ we have $x = z + \mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}$ with $\nabla_{\min} z \in \mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}$. Then by $\Delta_{b,\mathrm{rel}} z = \nabla^\#_{\min}\nabla_{\min} z + \nabla_{\min}\nabla^\#_{\min} z$ and $\nabla^\#_{\min}\Delta_{b,\mathrm{rel}} = \Delta_{b,\mathrm{rel}}\nabla^\#_{\min}$ we get $\Delta_{b,\mathrm{rel}} z - \nabla_{\min}\nabla^\#_{\min} z \in \mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}$. Then by definition we get $\Delta_{b,\mathrm{rel}}[x] = 0$. Since (3.9) is an isomorphism, we get

$$H^*(\mathcal{D}^\infty \mathscr{D}_{\min}/\mathcal{D}^\infty \mathscr{D}_{\min,\{0\}}, \nabla_{\min}) \cong \{0\},$$

so that $H^*(\mathscr{D}_{\min,\{0\}}, \nabla_{\min,\{0\}}) \cong H^*(\mathscr{D}_{\min}, \nabla_{\min})$. □

For a finite-dimensional complex vector space V, we define

$$\det V = \Lambda^{\max} V.$$

Then for the complex $(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]})$, we define the complex determinant lines

$$\det\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) = \bigotimes_{k=0}^{n} \left(\det\left(\mathscr{D}_{\min,[0,a],k}\right)\right)^{(-1)^k} \tag{3.10}$$

and

$$\det H^*\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) = \bigotimes_{k=0}^{n} \left(\det H^k\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right)\right)^{(-1)^k}.$$

Let $\mathscr{D}_{\min,[0,a],k} = \mathscr{D}_{\min,[0,a]} \cap L^2_k(M,E)$ and let the induced non-degenerate symmetric bilinear form be denoted by $b_{\min,[0,a],k}$. Then by $b_{\min,[0,a],k}$ and (3.10) we get a non-degenerate symmetric bilinear form on $\det(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]})$ and denote it by $b_{\det(\mathscr{D}_{\min,[0,a]})}$. By the canonical isomorphism (cf. [15] and [1, Section 1a])

$$\det H^*\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) \cong \det\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right)$$

and the isomorphism (3.8), we get a non-degenerate symmetric bilinear form on the determinant line $\det H^*(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}) = \det H^*(\mathscr{D}_{\min}, \nabla_{\min})$ and denote it by $b_{\det H(\mathscr{D}_{\min,[0,a]})}$.

For the subcomplex $(\mathscr{D}_{\min,(a,\infty)}, \nabla_{\min,(a,\infty)})$, we define the Laplace operator by

$$\Delta_{b,\mathrm{rel},(a,\infty)} = \nabla_{\min,(a,\infty)}\nabla^\#_{\min,(a,\infty)} + \nabla^\#_{\min,(a,\infty)}\nabla_{\min,(a,\infty)},$$

where $\nabla^\#_{\min_{(a,\infty)}}$ is the adjoint of $\nabla_{\min,(a,\infty)}$ with respect to the induced non-degenerate symmetric bilinear form on $\mathscr{D}_{\min,(a,\infty)}$. For $0 \le k \le n$, let $\Delta_{b,\mathrm{rel},(a,\infty),k}$ be the restriction of $\Delta_{b,\mathrm{rel},(a,\infty)}$ to $\mathscr{D}(\Delta_{b,\mathrm{rel},(a,\infty)}) \cap L^2_k(M,E)$. Since $\Delta_{b,\mathrm{rel}}$ has the same leading symbol with Δ_{rel}, the following regularized zeta determinant is well defined (cf. [13]):

$$\det{}'\left(\Delta_{b,\mathrm{rel},(a,\infty),k}\right) = \exp\left(-\left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Tr}\left[\left(\Delta_{b,\mathrm{rel},(a,\infty),k}\right)^{-s}\right]\right).$$

The above constructions are also valid for the complex $(\mathscr{D}_{\max}, \nabla_{\max})$.

Theorem 3.5. *The symmetric bilinear forms on* $\det H^*(\mathscr{D}_{\min/\max}, \nabla_{\min/\max})$ *defined by*

$$b_{\det H^*(\mathscr{D}_{\min/\max,[0,a]})} \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel/abs},(a,\infty),k}\right)\right)^{(-1)^k k} \tag{3.11}$$

are independent of the choice of $a \ge 0$.

Proof. Let $0 \leq a < c < \infty$. We have

$$(3.12)\quad \left(\mathscr{D}_{\min/\max,[0,c]}, \nabla_{\min/\max,[0,c]}\right) = \left(\mathscr{D}_{\min/\max,[0,a]}, \nabla_{\min/\max,[0,a]}\right) \oplus \left(\mathscr{D}_{\min/\max,(a,c]}, \nabla_{\min/\max,(a,c]}\right)$$

and

$$(3.13)\quad \left(\mathscr{D}_{\min/\max,(a,\infty)}, \nabla_{\min/\max,(a,\infty)}\right) = \left(\mathscr{D}_{\min/\max,(a,c]}, \nabla_{\min/\max,(a,c]}\right) \oplus \left(\mathscr{D}_{\min/\max,(c,\infty)}, \nabla_{\min/\max,(c,\infty)}\right).$$

Then we have

$$\det{}'\left(\Delta_{b,\mathrm{rel/abs},(a,\infty),k}\right) = \det{}'\left(\Delta_{b,\mathrm{rel/abs},(a,c],k}\right) \cdot \det{}'\left(\Delta_{b,\mathrm{rel/abs},(c,\infty),k}\right).$$

In particular,

$$(3.14)\quad \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel/abs},(a,\infty),k}\right)\right)^{(-1)^k k} = \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel/abs},(a,c],k}\right)\right)^{(-1)^k k} \cdot \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel/abs},(c,\infty),k}\right)\right)^{(-1)^k k}.$$

Applying [8, Lemma 3.3] to (3.12), we get

$$b_{\det H(\mathscr{D}_{\min/\max,[0,a]})} \cdot \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel/abs},(a,c],k}\right)\right)^{(-1)^k k} = b_{\det H(\mathscr{D}_{\min/\max,[0,c]})}.$$

The proof of Theorem 3.5 is complete. □

Definition 3.6. The bilinear forms defined by (3.11) are called the Ray-Singer symmetric bilinear torsions on $\det H^*(\mathscr{D}_{\min/\max}, \nabla_{\min/\max})$ and are denoted by $b^{\mathrm{RS}}_{\mathrm{rel/abs}}$.

Remark 3.7. By Theorem 2.3, we have

$$\det H^*\left(\mathscr{D}_{\min}, \nabla_{\min}\right) \cong \det H^*(M, \partial M, E),\ \det H^*\left(\mathscr{D}_{\max}, \nabla_{\max}\right) \cong \det H^*(M, E),$$

so that $b^{\mathrm{RS}}_{\mathrm{rel/abs}}$ can be viewed as the extension of the Burghelea-Haller analytic torsion to compact manifolds with relative/absolute boundary condition.

Now we investigate the dependence of the torsions $b^{\mathrm{RS}}_{\mathrm{rel/abs}}$ on the Riemannian metric on M and the non-degenerate symmetric bilinear form on E.

Theorem 3.8. *If* $\dim M$ *is odd, the smooth family of Riemannian metrics* g_u^{TM} *and non-degenerate symmetric bilinear forms* b_u *vary only in a compact subset of the interior of* M*. Then the torsion* $b^{\mathrm{RS}}_{u,\mathrm{rel/abs}}$ *is independent of* u*.*

Proof. First we note that the formula [8, (54)] is also valid in our case. Since the metrics g_u^{TM} and the symmetric bilinear forms b_u are independent of u in an open neighborhood of the boundary, the right hand term of [8, (54)] can be computed in the interior of M, which equals the right hand term of [8, (55)]. Since the integrand of [8, (55)] vanishes near the boundary and $\dim M$ is odd, we get the theorem. □

4. Compare with the refined analytic torsion

In this section we will only consider the case where the manifold is odd dimensional. Assume the Riemannian metric g^{TM} and the symmetric bilinear form b to be product near the boundary ∂M. More precisely, we identify, using the inward geodesic flow, a collar neighborhood $U \subset M$ of the boundary ∂M diffeomorphically with $[0,\varepsilon) \times \partial M$, $\varepsilon > 0$. Explicitly we have the diffeomorphism

$$\phi^{-1} : [0,\varepsilon) \times M \longrightarrow U,$$
$$(t,p) \longmapsto \gamma_p(t),$$

where γ_p is the geodesic flow starting at $p \in \partial M$ and $\gamma_p(t)$ is the geodesic from p of length $t \in [0,\varepsilon)$. The metric g^{TM} is product near the boundary if over U it is given under the diffeomorphism $\phi : U \to [0,\varepsilon) \times \partial M$ by

$$\phi_* g^{TM}|_U = dx^2 \oplus g^{TM}|_{\partial M}.$$

The diffeomorphism $U \cong [0,\varepsilon) \times \partial M$ shall be covered by a bundle isomorphism $\widetilde{\phi} : E|_U \to [0,\varepsilon) \times E|_{\partial M}$. The non-degenerate symmetric bilinear form b is product near the boundary if it is preserved by the bundle isomorphism, i.e.

$$\widetilde{\phi}_* b|_{\{x\}\times\partial M} = b|_{\partial M}.$$

Then the closed double manifold $\mathbb{M} = M \cup_{\partial M} M$ is a smooth closed Riemannian manifold with Riemannian metric $g^{T\mathbb{M}}$ and the bundle E extends to a smooth vector bundle $\mathbb{E}$ over $\mathbb{M}$ with a non-degenerate symmetric bilinear form b. Also the Hermitian metric h induced by b as in Section 2 is product near the boundary. Moreover we assume the flat connection ∇ on E to be temporal gauge (cf. [23]). Then ∇ extends to a smooth flat connection $\mathbb{D}$ on $\mathbb{E}$ with $\mathbb{D}|_M = \nabla$.

Denote by $(\mathscr{D}, \mathbb{D})$ the twisted de Rham complex $(\Omega^*(\mathbb{M}, \mathbb{E}), \mathbb{D})$. The Riemannian metric $g^{T\mathbb{M}}$ and the non-degenerate symmetric bilinear form b on $\mathbb{E}$ define a non-degenerate symmetric bilinear form on $(\mathscr{D}, \mathbb{D})$. So we have two Hilbert complexes:

$$\left(\widetilde{\mathscr{D}}, \widetilde{\nabla}\right) = (\mathscr{D}_{\min}, \nabla_{\min}) \oplus (\mathscr{D}_{\max}, \nabla_{\max})$$

(cf. [23]) associated to $(M, \partial M)$ and $(\mathscr{D}, \mathbb{D})$ associated to $\mathbb{M}$. Then we have the Burghelea-Haller analytic torsion $b^{\mathrm{RS}}_{\det H^*(\mathscr{D},\mathbb{D})}$ (cf. [7, 8, 9, 21]) on the determinant line $\det H^*(\mathscr{D}, \mathbb{D})$ and the refined analytic torsion $\rho_{\mathrm{an}}(\mathbb{D})$ (cf. [4, 5]) as an element of the determinant line $\det H^*(\mathscr{D}, \mathbb{D})$ for the closed manifold $\mathbb{M}$, the symmetric bilinear form $b^{\mathrm{RS}}_{\det H^*(\widetilde{\mathscr{D}},\widetilde{\nabla})}$ on the determinant line

$$\det H^*\left(\widetilde{\mathscr{D}}, \widetilde{\nabla}\right) = \det H^*\left(\mathscr{D}_{\min}, \nabla_{\min}\right) \otimes \det H^*\left(\mathscr{D}_{\max}, \nabla_{\max}\right)$$

defined by $b^{\mathrm{RS}}_{\mathrm{rel}} \otimes b^{\mathrm{RS}}_{\mathrm{abs}}$ and the refined analytic torsion $\rho_{\mathrm{an}}(\widetilde{\nabla})$ (we use the notation $\rho_{\mathrm{an}}(\widetilde{\nabla})$ instead of $\rho_{\mathrm{an}}(\nabla)$ in [23]) associated to $(\widetilde{\mathscr{D}}, \widetilde{\nabla})$. Let h be the Hermitian metric induced by b as in Section 2.

As in [23], we can decompose $(\mathscr{D}, \mathbb{D})$ as

$$(\mathscr{D}, \mathbb{D}) = \left(\mathscr{D}^+, \mathbb{D}^+\right) \oplus \left(\mathscr{D}^-, \mathbb{D}^-\right).$$

We also have an isomorphism of the complexes $\Phi : (\mathscr{D}, \mathbb{D}) \to (\widetilde{\mathscr{D}}, \widetilde{\nabla})$, defined in [23], which extends to an isometry with respect to the natural L^2-structures. By (g^{TM}, h) one can also construct the Ray-Singer analytic torsion $h^{\mathrm{RS}}_{\det H^*(\widetilde{\mathscr{D}},\widetilde{\nabla})}$ as an inner product on $\det H^*(\widetilde{\mathscr{D}}, \widetilde{\nabla})$. Denote by Δ and $\widetilde{\triangle}$ respectively the Laplacians of the complexes $(\mathscr{D}, \mathbb{D})$ and $(\widetilde{\mathscr{D}}, \widetilde{\nabla})$. Then from [23] one can find

$$\Phi \circ \Delta \circ \Phi^{-1} = \widetilde{\triangle} \text{ and } \Phi\mathscr{D}(\Delta) = \mathscr{D}\left(\widetilde{\triangle}\right). \tag{4.1}$$

Let $\mathbb{D}^{\#}$ and $\widetilde{\nabla}^{\#}$ be the adjoint operators of $\mathbb{D}$ and $\widetilde{\nabla}$ with respect to the non-degenerate symmetric bilinear forms respectively. From the definition of Φ in [23], we see that Φ preserves the non-degenerate symmetric bilinear forms, so from [23, (5.13)] we get $\Phi \circ \mathbb{D}^{\#} \circ \Phi^{-1} = \widetilde{\nabla}^{\#}$. Let

$$\Delta_b = \mathbb{D}\mathbb{D}^{\#} + \mathbb{D}^{\#}\mathbb{D} \text{ and } \widetilde{\triangle}_b = \widetilde{\nabla}\widetilde{\nabla}^{\#} + \widetilde{\nabla}^{\#}\widetilde{\nabla}.$$

Then from (4.1) and $\mathscr{D}(\Delta) = \mathscr{D}(\Delta_b), \mathscr{D}(\widetilde{\triangle}) = \mathscr{D}(\widetilde{\triangle}_b)$ we get

$$\Phi \circ \Delta_b \circ \Phi^{-1} = \widetilde{\triangle}_b \text{ and } \Phi\mathscr{D}(\Delta_b) = \mathscr{D}\left(\widetilde{\triangle}_b\right),$$

so that Δ_b and $\widetilde{\triangle}_b$ are spectrally equivalent. Consider the spectral projections $\Pi_{\Delta_b,[0,\lambda]}$ and $\Pi_{\widetilde{\triangle}_b,[0,\lambda]}$, $\lambda \geq 0$ of Δ_b and $\widetilde{\triangle}_b$ respectively, associated to eigenvalues of absolute value in $[0, \lambda]$. By the spectral equivalent of Δ_b and $\widetilde{\triangle}_b$ we find

$$\Phi \circ \Pi_{\Delta_b,[0,\lambda]} = \Pi_{\widetilde{\triangle}_b,[0,\lambda]} \circ \Phi.$$

Hence the isomorphism Φ reduces to an isomorphism of finite-dimensional complexes

$$\Phi_\lambda : \left(\mathscr{D}_{b,[0,\lambda]}, \mathbb{D}_{[0,\lambda]}\right) \to \left(\widetilde{\mathscr{D}}_{b,[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}\right),$$

where $\mathscr{D}_{b,[0,\lambda]} = \mathscr{D} \cap \mathrm{Image}\Pi_{\Delta_b,[0,\lambda]}$ and $\widetilde{\mathscr{D}}_{b,[0,\lambda]} = \widetilde{\mathscr{D}} \cap \mathrm{Image}\Pi_{\widetilde{\triangle}_b,[0,\lambda]}$. Moreover, Φ_λ preserves the non-degenerate symmetric bilinear forms of the corresponding determinant lines, which we denote again by Φ_λ:

$$\Phi_\lambda : \det\left(\mathscr{D}_{b,[0,\lambda]}, \mathbb{D}_{[0,\lambda]}\right) \to \det\left(\widetilde{\mathscr{D}}_{b,[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}\right).$$

By Proposition 3.4, we have the canonical identifications of determinant lines

$$\det\left(\mathscr{D}_{b,[0,\lambda]}, \mathbb{D}_{[0,\lambda]}\right) \cong \det H^*(\mathscr{D}, \mathbb{D}) \text{ and } \det\left(\widetilde{\mathscr{D}}_{b,[0,\lambda]}, \widetilde{\nabla}_{[0,\lambda]}\right) \cong \det H^*\left(\widetilde{\mathscr{D}}, \widetilde{\nabla}\right),$$

so Φ_λ preserves the non-degenerate symmetric bilinear forms

$$\Phi_\lambda : \det H^*(\mathscr{D}, \mathbb{D}) \to \det H^*\left(\widetilde{\mathscr{D}}, \widetilde{\nabla}\right).$$

Then from the definition and the above discussion one finds that

$$(\Phi_\lambda)_* b^{\mathrm{RS}}_{\det H^*(\mathscr{D},\mathbb{D})} = b^{\mathrm{RS}}_{\det H^*(\widetilde{\mathscr{D}},\widetilde{\nabla})}. \tag{4.2}$$

Also from [23] we have

$$\Phi_\lambda\left(\rho_{\mathrm{an}}(\mathbb{D})\right) = \rho_{\mathrm{an}}\left(\widetilde{\nabla}\right). \tag{4.3}$$

Then from (4.2) and (4.3), we get

$$b^{\mathrm{RS}}_{\det H^*(\mathscr{D},\mathbb{D})}(\rho_{\mathrm{an}}(\mathbb{D})) = b^{\mathrm{RS}}_{\det H^*(\widetilde{\mathscr{D}},\widetilde{\nabla})}(\rho_{\mathrm{an}}(\widetilde{\nabla})). \tag{4.4}$$

By [6, Theorem 1.2], we have

$$b^{\mathrm{RS}}_{\det H^*(\mathscr{D},\mathbb{D})}\left(\rho_{\mathrm{an}}(\mathbb{D})\right) = \pm e^{-2\pi i(\eta(\mathbb{D}) - \mathrm{rank}(E)\cdot\eta(\mathbb{B}_{\mathrm{trivial}}))}, \tag{4.5}$$

where $\mathbb{B}_{\mathrm{trivial}}$ is the odd-signature operator of the trivial line bundle over $\mathbb{M}$ and $\eta(\mathbb{D})$ is defined as in [6, (2.17)].

Theorem 4.1. *If* $\dim M$ *is odd, the Riemannian metric* g^{TM} *and the non-degenerate symmetric bilinear form* b *are product near the boundary* ∂M*, and the flat connection* ∇ *on* E *is in temporal gauge, then*

$$b^{\mathrm{RS}}_{\det H^*\left(\widetilde{\mathscr{D}},\widetilde{\nabla}\right)}\left(\rho_{\mathrm{an}}\left(\widetilde{\nabla}\right)\right)=\pm e^{-2\pi i\left(\eta\left(\widetilde{\nabla}\right)-\mathrm{rank}(E)\cdot\eta(\mathcal{B}_{\mathrm{trivial}})\right)} \tag{4.6}$$

and

$$\left|\frac{b^{\mathrm{RS}}_{\det H^*\left(\widetilde{\mathscr{D}},\widetilde{\nabla}\right)}}{h^{\mathrm{RS}}_{\det H^*\left(\widetilde{\mathscr{D}},\widetilde{\nabla}\right)}}\right|=1, \tag{4.7}$$

where $\mathcal{B}_{\mathrm{trivial}}$ *is the odd-signature operator of the trivial line bundle over* M*.*

Proof. From (4.4) and (4.5), we have

$$b^{\mathrm{RS}}_{\det H^*(\widetilde{\mathscr{D}},\widetilde{\nabla})}(\rho_{\mathrm{an}}(\widetilde{\nabla}))=\pm e^{-2\pi i(\eta(\mathbb{D})-\mathrm{rank}(E)\cdot\eta(\mathbb{B}_{\mathrm{trivial}}))}. \tag{4.8}$$

From [23, (5.13)], one finds

$$\eta(\mathbb{D})=\eta(\widetilde{\nabla})\text{ and }\eta(\mathbb{B}_{\mathrm{trivial}})=\eta(\mathcal{B}_{\mathrm{trivial}}). \tag{4.9}$$

So by (4.8) and (4.9), we get (4.6).

From (4.6), one finds that

$$\left|b^{\mathrm{RS}}_{\det H^*(\widetilde{\mathscr{D}},\widetilde{\nabla})}\left(\rho_{\mathrm{an}}\left(\widetilde{\nabla}\right)\right)\right|=\exp\left(2\pi\mathrm{Im}(\eta(\widetilde{\nabla}))\right). \tag{4.10}$$

By [23, Theorem 5.3], we get

$$h^{\mathrm{RS}}_{\det H^*\left(\widetilde{\mathscr{D}},\widetilde{\nabla}\right)}\left(\rho_{\mathrm{an}}\left(\widetilde{\nabla}\right)\right)=\exp\left(2\pi\mathrm{Im}\eta\left(\widetilde{\nabla}\right)\right), \tag{4.11}$$

so by (4.10) and (4.11), we get (4.7). The proof of Theorem 4.1 is complete. □

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