PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 12, December 2009, Pages 4157–4167
S 0002-9939(09)10008-4
Article electronically published on July 14, 2009

NONCOMMUTATIVE L_p-SPACE AND OPERATOR SYSTEM

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(Communicated by Marius Junge)

ABSTRACT. We show that noncommutative L_p-spaces satisfy the axioms of the (nonunital) operator system with a dominating constant $2^{\frac{1}{p}}$. Therefore, noncommutative L_p-spaces can be embedded into $B(H)$ $2^{\frac{1}{p}}$-completely isomorphically and complete order isomorphically.

1. INTRODUCTION

A unital involutive subspace of $B(H)$ had been abstractly characterized by Choi and Effros [CE]. Their axioms are based on observations of the relationship between the unit, the matrix order, and the matrix norm of the unital involutive subspace of $B(H)$. Indeed, the unit and the matrix order can be used to determine the matrix norm by applying

$$\|x\| = \inf\{\lambda > 0 : -\lambda I \leq \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \leq \lambda I\}$$

for a unital involutive subspace X of $B(H)$ and $x \in M_n(X)$.

The abstract characterization of a nonunital involutive subspace of $B(H)$ was completed by Werner [W]. The axioms are based on observations of the relationship between the matrix norm and the matrix order of the involutive subspace of $B(H)$. Since the unit may be absent, the above equality is replaced by

$$\|x\| = \sup\{|\varphi(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix})| : \varphi \in M_{2n}(X)^*_{1,+}\}$$

for an involutive subspace X of $B(H)$ and $x \in M_n(X)$. We denote by $M_{2n}(X)^*_{1,+}$ the set of positive contractive functionals on $M_{2n}(X)$.

A complex involutive vector space X is called a matrix ordered vector space if for each $n \in \mathbb{N}$, there is a set $M_n(X)_+ \subset M_n(X)_{sa}$ such that

(1) $M_n(X)_+ \cap [-M_n(X)_+] = \{0\}$ for all $n \in \mathbb{N}$,
(2) $M_n(X)_+ \oplus M_m(X)_+ \subset M_{n+m}(X)_+$ for all $m, n \in \mathbb{N}$,
(3) $\gamma^* M_m(X)_+ \gamma \subset M_n(X)_+$ for each $m, n \in \mathbb{N}$ and all $\gamma \in M_{m,n}(\mathbb{C})$.

One might infer from these conditions that $M_n(X)_+$ is actually a cone.

An operator space X is called a matrix ordered operator space iff X is a matrix ordered vector space and for every $n \in \mathbb{N}$,

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Received by the editors July 13, 2008, and, in revised form, February 16, 2009, and March 20, 2009.
2000 *Mathematics Subject Classification.* Primary 46L07, 46L52, 47L07.
This work was supported by the BK21 project of the Ministry of Education, Korea.

(1) the $*$-operation is an isometry on $M_n(X)$ and
(2) the cones $M_n(X)_+$ are closed.

Suppose X is a matrix ordered operator space. For $x \in M_n(X)$, the modified numerical radius is defined by

$$\nu_X(x) = \sup\{|\varphi(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix})| : \varphi \in M_{2n}(X)^*_{1,+}\}.$$

We call a matrix ordered operator space an operator system iff there is a $k > 0$ such that for all $n \in \mathbb{N}$ and $x \in M_n(x)$,

$$\|x\| \leq k\nu_X(x).$$

Since $\nu_X(x) \leq \|x\|$ always holds, we can say that an operator system is a matrix ordered operator space such that the operator space norm and the modified numerical radius are equivalent uniformly for all $n \in \mathbb{N}$.

Werner showed that X is an operator system if and only if there is a complete order isomorphism Φ from X onto an involutive subspace of $B(H)$, which is a complete topological onto-isomorphism [W]. Hence, the operator system is an abstract characterization of the involutive subspace of $B(H)$ in the completely isomorphic and complete order isomorphic sense.

In this paper, noncommutative L_p-space is meant in the sense of Haagerup, which is based on Tomita-Takesaki theory [Te1]. While the noncommutative L_p-spaces arising from semifinite von Neumann algebras are a natural generalization of classical L_p-spaces [S, Ta], noncommutative L_p-spaces arising from type III von Neumann algebras are quite complicated. Recently, the reduction method approximating a general noncommutative L_p-space by tracial noncommutative L_p-spaces has been developed [HJX].

Noncommutative L_p-spaces can also be obtained by the complex interpolation method. There exist a canonical matrix order and a canonical operator space structure on each noncommutative L_p-space. The matrix order is given by the positive cones $L_p(M_n(\mathcal{M}))^+$, and the operator space structure is given by the complex interpolation

$$M_n(L_p(\mathcal{M})) = (M_n(\mathcal{M}), M_n(\mathcal{M}_*^{op}))_{\frac{1}{p}}.$$

We refer to [K, Pi1, Te2] for the details, and the reference [JRX, Section 4] is recommended for the summary. In particular, its discrete noncommutative vector valued L_p-space $S_p^n(L_p(\mathcal{M}))$ is given by

$$S_p^n(L_p(\mathcal{M})) = L_p(M_n(\mathcal{M})).$$

We refer to [Pi2] for general information on noncommutative vector valued L_p-spaces.

The purpose of this paper is to show that noncommutative L_p-spaces satisfy the axioms of the operator system with a dominating constant $2^{\frac{1}{p}}$. As a corollary, we obtain the following embedding theorem: noncommutative L_p-spaces can be embedded into $B(H)$ $2^{\frac{1}{p}}$-completely isomorphically and complete order isomorphically.

2. Noncommutative L_p-space and operator system

If X is a matrix ordered operator space, we regard $S_p^n(X)$ as an operator space having the same matrix order structure with $M_n(X)$. That is, we do not distinguish between $S_p^n(X)$ and $M_n(X)$ as matrix ordered vector spaces.

For a noncommutative L_p-space $L_p(\mathcal{M})$, it is more efficient to describe $S_p^n(L_p(\mathcal{M}))$ rather than $M_n(L_p(\mathcal{M}))$. So, we change the modified numerical radius into a form more adequate to noncommutative L_p-spaces. For a matrix ordered operator space X and $x \in M_n(X)$, we define $\nu_X^p(x)$ as

$$\nu_X^p(x) = 2^{-\frac{1}{p}} \sup\{|\varphi(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix})| : \varphi \in S_p^{2n}(X)_{1,+}^*\},$$

where $S_p^{2n}(X)_{1,+}^*$ denotes the set of positive contractive functionals on $S_p^{2n}(X)$. Note that a functional φ is positive on $S_p^{2n}(X)$ if and only if it is positive on $M_{2n}(X)$ because $S_p^{2n}(X)$ and $M_{2n}(X)$ have the same order structure. According to the following lemma, the uniform equivalence of a matrix norm and ν_X can be confirmed by showing the uniform equivalence of the S_p^n-norm and ν_X^p.

Lemma 2.1. *Suppose X is a matrix ordered operator space and there is a constant $k > 0$ satisfying $\|x\|_{S_p^n(X)} \le k\nu_X^p(x)$ for any $x \in M_n(X)$. Then we have*

$$\|x\|_{M_n(X)} \le k\nu_X(x)$$

for any $x \in M_n(X)$. Hence, X is an operator system.

Proof. For $a, b \in (S_{2p}^n)_1$, we have

$$\| \begin{pmatrix} a & 0 \\ 0 & b^* \end{pmatrix} \|_{S_{2p}^{2n}} = (\|a\|_{S_{2p}^n}^{2p} + \|b^*\|_{S_{2p}^n}^{2p})^{\frac{1}{2p}} \le 2^{\frac{1}{2p}}.$$

From [Pi2, Theorem 1.5], the map

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto 2^{-\frac{1}{p}} \varphi(\begin{pmatrix} a & 0 \\ 0 & b^* \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} a^* & 0 \\ 0 & b \end{pmatrix})$$

for $\varphi \in S_p^{2n}(X)_{1,+}^*$ defines a positive contractive functional on $M_{2n}(X)$. It follows that

$$\begin{aligned}
\nu_X(x) &= \sup\{|\varphi(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix})| : \varphi \in M_{2n}(X)_{1,+}^*\} \\
&\ge 2^{-\frac{1}{p}} \sup\{|\varphi(\begin{pmatrix} a & 0 \\ 0 & b^* \end{pmatrix} \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a^* & 0 \\ 0 & b \end{pmatrix})| : \varphi \in S_p^{2n}(X)_{1,+}^*, a, b \in (S_{2p}^n)_1\} \\
&= 2^{-\frac{1}{p}} \sup\{|\varphi(\begin{pmatrix} 0 & axb \\ b^*x^*a^* & 0 \end{pmatrix})| : \varphi \in S_p^{2n}(X)_{1,+}^*, a, b \in (S_{2p}^n)_1\} \\
&= \sup\{\nu_X^p(axb) : a, b \in (S_{2p}^n)_1\} \\
&\ge k^{-1} \sup\{\|axb\|_{S_p^n(X)} : a, b \in (S_{2p}^n)_1\} \\
&= k^{-1} \|x\|_{M_n(X)}
\end{aligned}$$

for $x \in M_n(X)$. For the last equality, see [Pi2, Lemma 1.7]. □

Lemma 2.2. *Suppose a is a bounded linear operator on a Hilbert space and $a = v|a|$ is its polar decomposition. Let f be a continuous function defined on $[0, \infty)$ with*

$f(0) = 0$. *Then* $\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}$ *is a positive operator and the equality*

$$f(\frac{1}{2}\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}) = \frac{1}{2}\begin{pmatrix} f(|a^*|) & f(|a^*|)v \\ v^*f(|a^*|) & f(|a|) \end{pmatrix}$$

holds.

Proof. The positivity follows from [Pa, Exercise 8.8 (vi)]. By using the Weierstrass theorem and the right polar decomposition $a = |a^*|v$, it is sufficient to show that the equality

$$\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} |a^*|^n & |a^*|^{n-1}a \\ a^*|a^*|^{n-1} & |a|^n \end{pmatrix}$$

holds for $n \geq 1$. We proceed by mathematical induction. Since $a^*(aa^*)^k a = (a^*a)^{k+1}$ for $k \geq 0$, we have

$$a^*|a^*|^{n-1}a = (a^*a)^{\frac{n-1}{2}+1} = |a|^{n+1}.$$

Similarly, $a(a^*a)^k = (aa^*)^k a$ implies

$$a|a|^n = |a^*|^n a.$$

It follows that

$$\begin{aligned} &\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix} \cdot 2^{n-1}\begin{pmatrix} |a^*|^n & |a^*|^{n-1}a \\ a^*|a^*|^{n-1} & |a|^n \end{pmatrix} \\ &= 2^{n-1}\begin{pmatrix} |a^*|^{n+1} + |a^*|^{n+1} & |a^*|^n a + a|a|^n \\ a^*|a^*|^n + |a|a^*|a^*|^{n-1} & a^*|a^*|^{n-1}a + |a|^{n+1} \end{pmatrix} \\ &= 2^n \begin{pmatrix} |a^*|^{n+1} & |a^*|^n a \\ a^*|a^*|^n & |a|^{n+1} \end{pmatrix}. \end{aligned}$$

□

In order to prove that the Schatten p-class S_p is an operator system, Lemma 2.2 is sufficient. However, in order to prove that a general noncommutative L_p-space is an operator system, an unbounded version of Lemma 2.2 is needed.

We denote the Dirac measure massed at the zero point by δ_0.

Lemma 2.3. *Let a be a closed densely defined operator on a Hilbert space. Suppose $a = v|a|$ is a polar decomposition and $|a| = \int_0^\infty t dE$ is a spectral decomposition. Then*

$$\widetilde{E} = \frac{1}{2}\begin{pmatrix} vEv^* & vE \\ Ev^* & v^*vE \end{pmatrix} + \delta_0 \begin{pmatrix} I - \frac{1}{2}vv^* & -\frac{1}{2}v \\ -\frac{1}{2}v^* & I - \frac{1}{2}v^*v \end{pmatrix}$$

is a spectral measure corresponding to $\frac{1}{2}\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}$.

Proof. Let

$$E' = \frac{1}{2}\begin{pmatrix} vEv^* & vE \\ Ev^* & v^*vE \end{pmatrix}.$$

Since $v^*v = E((0,\infty))$, v^*v commutes with $E(\Delta)$ for any Borel set Δ in $[0,\infty)$. For all Borel sets Δ_1 and Δ_2 in $[0,\infty)$, we have

$$\begin{aligned}
&E'(\Delta_1)E'(\Delta_2)\\
=&\frac{1}{4}\begin{pmatrix} vE(\Delta_1)v^* & vE(\Delta_1)\\ E(\Delta_1)v^* & v^*vE(\Delta_1)\end{pmatrix}\begin{pmatrix} vE(\Delta_2)v^* & vE(\Delta_2)\\ E(\Delta_2)v^* & v^*vE(\Delta_2)\end{pmatrix}\\
=&\frac{1}{4}\begin{pmatrix} vE(\Delta_1)v^*vE(\Delta_2)v^*+vE(\Delta_1)E(\Delta_2)v^* & vE(\Delta_1)v^*vE(\Delta_2)+vE(\Delta_1)v^*vE(\Delta_2)\\ E(\Delta_1)v^*vE(\Delta_2)v^*+v^*vE(\Delta_1)E(\Delta_2)v^* & E(\Delta_1)v^*vE(\Delta_2)+v^*vE(\Delta_1)v^*vE(\Delta_2)\end{pmatrix}\\
=&\frac{1}{2}\begin{pmatrix} vE(\Delta_1\cap\Delta_2)v^* & vE(\Delta_1\cap\Delta_2)\\ E(\Delta_1\cap\Delta_2)v^* & v^*vE(\Delta_1\cap\Delta_2)\end{pmatrix}\\
=&E'(\Delta_1\cap\Delta_2).
\end{aligned}$$

Taking $\Delta_1 = \Delta_2$, we see that E' is projection valued. The countable additivity with respect to the strong operator topology is obvious. Since $E(\{0\}) = I - v^*v$, we have

$$E'(\{0\}) = 0.$$

From these, it is easy to check that $\widetilde{E}$ satisfies all the conditions of a spectral measure.

We still need to show that

$$\frac{1}{2}\langle\begin{pmatrix} |a^*| & a\\ a^* & |a|\end{pmatrix}\begin{pmatrix} h\\ k\end{pmatrix} | \begin{pmatrix} h\\ k\end{pmatrix}\rangle = \int_0^\infty t d\widetilde{E}_{\xi,\xi}$$

for $\xi = \begin{pmatrix} h\\ k\end{pmatrix}$ and $h \in D(a^*), k \in D(a)$. For the left-hand side, we compute

$$\begin{aligned}
\langle\begin{pmatrix} |a^*| & a\\ a^* & |a|\end{pmatrix}\begin{pmatrix} h\\ k\end{pmatrix} | \begin{pmatrix} h\\ k\end{pmatrix}\rangle &= \langle |a^*|h|h\rangle + \langle ak|h\rangle + \langle a^*h|k\rangle + \langle |a|k|k\rangle\\
&= \langle v|a|v^*h|h\rangle + \langle v|a|k|h\rangle + \langle |a|v^*h|k\rangle + \langle v^*v|a|k|k\rangle\\
&= \int_0^\infty tdE_{v^*h,v^*h} + \int_0^\infty tdE_{k,v^*h}\\
&\qquad + \int_0^\infty tdE_{v^*h,k} + \int_0^\infty tdE_{k,v^*vk}.
\end{aligned}$$

For the right-hand side, we compute

$$\begin{aligned}
E'_{\xi,\xi}(\Delta) &= \frac{1}{2}\langle\begin{pmatrix} vE(\Delta)v^* & vE(\Delta)\\ E(\Delta)v^* & v^*vE(\Delta)\end{pmatrix}\begin{pmatrix} h\\ k\end{pmatrix} | \begin{pmatrix} h\\ k\end{pmatrix}\rangle\\
&= \frac{1}{2}(\langle vE(\Delta)v^*h|h\rangle + \langle vE(\Delta)k|h\rangle + \langle E(\Delta)v^*h|k\rangle + \langle v^*vE(\Delta)k|k\rangle)\\
&= \frac{1}{2}(E_{v^*h,v^*h}(\Delta) + E_{k,v^*h}(\Delta) + E_{v^*h,k}(\Delta) + E_{k,v^*vk}(\Delta)).
\end{aligned}$$

□

Lemma 2.4. *Suppose $\mathcal{M}$ is a semifinite von Neumann algebra with a faithful semifinite normal trace τ and a is a τ-measurable operator with its polar decomposition $a = v|a|$. Let f be a continuous function defined on $[0,\infty)$ with $f(0) = 0$.*

Then $\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}$ *is a positive self-adjoint* τ_2*-measurable operator and the equality*

$$f(\frac{1}{2}\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}) = \frac{1}{2}\begin{pmatrix} f(|a^*|) & f(|a^*|)v \\ v^*f(|a^*|) & f(|a|) \end{pmatrix}$$

holds.

Proof. Because we have

$$\begin{aligned}\tau_2(\tilde{E}((n,\infty)) &= \tau(\frac{1}{2}vE((n,\infty))v^*) + \tau(\frac{1}{2}v^*vE(n,\infty)) \\ &= \tau(E(n,\infty)) \to 0,\end{aligned}$$

$\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}$ is τ_2-measurable.

Suppose E is a spectral measure corresponding to $|a|$. We let

$$a_n = aE([0,n]).$$

Then we have

$$|a_n| = (E([0,n]a^*aE([0,n])^{\frac{1}{2}} = |a|E([0,n]).$$

Since $a_nv^* = v|a|E([0,n])v^* \geq 0$ and $(a_nv^*)^2 = aE([0,n])v^*vE([0,n])a^* = aE([0,n])a^*$, we have

$$a_nv^* = (aE([0,n])a^*)^{\frac{1}{2}} = |a_n^*|.$$

We compute

$$\begin{aligned}&\frac{1}{2}\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}\widetilde{E}([0,n]) \\ &\quad= \frac{1}{4}\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}\begin{pmatrix} vE([0,n])v^* & vE([0,n]) \\ E([0,n])v^* & v^*vE([0,n]) \end{pmatrix} \\ &\quad= \frac{1}{4}\begin{pmatrix} |a^*|vE([0,n])v^* + aE([0,n])v^* & |a^*|vE([0,n]) + av^*vE([0,n]) \\ a^*vE([0,n])v^* + |a|E([0,n])v^* & a^*vE([0,n]) + |a|v^*vE([0,n]) \end{pmatrix} \\ &\quad= \frac{1}{2}\begin{pmatrix} |a_n^*| & a_n \\ a_n^* & |a_n| \end{pmatrix}.\end{aligned}$$

Since $|a_n|$ is bounded, we can apply Lemma 2.2. The partial isometry in the polar decomposition of a_n is $v_n := vE([0,n])$. It follows that

$$\begin{aligned}f(\frac{1}{2}\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix})\widetilde{E}([0,n]) &= f(\frac{1}{2}\begin{pmatrix} |a_n^*| & a_n \\ a_n^* & |a_n| \end{pmatrix}) \\ &= \frac{1}{2}\begin{pmatrix} f(|a_n^*|) & f(|a_n^*|)v_n \\ v_n^*f(|a_n^*|) & f(|a_n|) \end{pmatrix}.\end{aligned}$$

Since $v^*|a_n^*|^kv = v^*(a_nv^*)^kv = |a_n|^k$ for any $k \geq 1$, we have

$$v^*f(|a_n^*|)v = f(|a_n|).$$

Similarly, we also have

$$vf(|a_n|)v^* = f(|a_n^*|).$$

The spectral measure for $|a^*|$ is $vEv^* + \delta_0(I - vv^*)$. It follows that

$$\begin{aligned}
&\frac{1}{2}\begin{pmatrix} f(|a^*|) & f(|a^*|)v \\ v^*f(|a^*|) & f(|a|) \end{pmatrix}\widetilde{E}([0,n]) \\
=&\frac{1}{4}\begin{pmatrix} 2f(|a^*|)vE([0,n])v^* & 2f(|a^*|)vE([0,n]) \\ v^*f(|a^*|)vE([0,n])v^* + f(|a|)E([0,n])v^* & v^*f(|a^*|)vE([0,n]) + f(|a|)v^*vE([0,n]) \end{pmatrix} \\
=&\frac{1}{4}\begin{pmatrix} 2f(|a^*|vE([0,n])v^*) & 2f(|a^*|vE([0,n])v^*)v_n \\ v_n^*f(|a^*|vE([0,n])v^*) + v_n^*vf(|a|E([0,n]))v^* & v^*f(|a^*|vE([0,n])v^*)v + f(|a|E([0,n])) \end{pmatrix} \\
=&\frac{1}{2}\begin{pmatrix} f(|a_n^*|) & f(|a_n^*|)v_n \\ v_n^*f(|a_n^*|) & f(|a_n|) \end{pmatrix}.
\end{aligned}$$

Because the set $\bigcup_{n=1}^{\infty} \operatorname{ran}\widetilde{E}([0,n])$ is τ_2-dense, we obtain the desired result from [Te1, Proposition I.12]. □

Lemma 2.5. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful semifinite normal weight φ. If a is an element of Haagerup's noncommutative L_p-space $L_p(\mathcal{M})$, then we have*

$$\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix} \in L_p(M_2(\mathcal{M}))_+ \qquad \text{and} \qquad \| \begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix} \|_{L_p(M_2(\mathcal{M}))} = 2\|a\|_{L_p(\mathcal{M})}.$$

Proof. Let θ be the dual action of $\mathbb{R}$ on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. Then $id_{M_2} \otimes \theta$ is the dual action of $\mathbb{R}$ on $M_2(\mathcal{M}) \rtimes_{\sigma^{tr\otimes\varphi}} \mathbb{R}$. Because

$$\begin{aligned}
(id_{M_2} \otimes \theta_s)\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix} &= \begin{pmatrix} \theta_s(|a^*|) & \theta_s(a) \\ \theta_s(a^*) & \theta_s(|a|) \end{pmatrix} \\
&= e^{-\frac{s}{p}}\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix},
\end{aligned}$$

we have

$$\begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix} \in L_p(M_2(\mathcal{M}))_+.$$

From Lemma 2.4, it follows that

$$\begin{aligned}
\| \begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix} \|_{L_p(M_2(\mathcal{M}))}^p &= \| \begin{pmatrix} |a^*| & a \\ a^* & |a| \end{pmatrix}^p \|_{L_1(M_2(\mathcal{M}))} \\
&= 2^{p-1}\| \begin{pmatrix} |a^*|^p & |a^*|^p v \\ v^*|a^*|^p & |a|^p \end{pmatrix} \|_{L_1(M_2(\mathcal{M}))} \\
&= 2^{p-1}(\||a^*|^p\|_{L_1(\mathcal{M})} + \||a|^p\|_{L_1(\mathcal{M})}) \\
&= 2^p\|a\|_{L_p(\mathcal{M})}^p.
\end{aligned}$$

□

Theorem 2.6. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful semifinite normal weight φ. Its noncommutative L_p-space $L_p(\mathcal{M})$ is an operator system with*

$$\|a\|_{M_n(L_p(\mathcal{M}))} \leq 2^{\frac{1}{p}}\nu_{L_p(\mathcal{M})}(a), \qquad a \in M_n(L_p(\mathcal{M})).$$

Proof. Let $a \in L_p(M_n(\mathcal{M}))$ with its right polar decomposition $a = |a^*|v$. Because

$$\begin{aligned}
\|a\|_{L_p(M_n(\mathcal{M}))} &= \||a^*|v\|_{L_p(M_n(\mathcal{M}))} \\
&\leq \|a^*\|_{L_p(M_n(\mathcal{M}))},
\end{aligned}$$

the involution is an isometry on $L^p(M_n(\mathcal{M}))$, which can be identified with $S_p^n(L^p(\mathcal{M}))$. From [Pi2, Lemma 1.7], the involution is also an isometry on $M_n(L^p(\mathcal{M}))$. The Hölder inequality and [Te1, Proposition II.33] show that the positive cone $L^p(M_n(\mathcal{M}))_+$ is closed. Hence, the noncommutative L_p-space is a matrix ordered operator space.

Let $a \in S_p^n(L_p(\mathcal{M}))$ and $\frac{1}{p}+\frac{1}{q}=1$. From Lemma 2.5, it follows that

$$\begin{aligned}
\|a\|_{L_p(M_n(\mathcal{M}))} &= \sup\{|\mathrm{tr}_{\mathcal{M}}(b^*a)| : b \in L_q(M_n(\mathcal{M}))_1\} \\
&= \sup\{\mathrm{Re}\ \mathrm{tr}_{\mathcal{M}}(b^*a) : b \in L_q(M_n(\mathcal{M}))_1\} \\
&= \sup\{\frac{1}{2}(\mathrm{tr}_{\mathcal{M}}(b^*a)+\mathrm{tr}_{\mathcal{M}}(ba^*)) : b \in L_q(M_n(\mathcal{M}))_1\} \\
&= \sup\{\frac{1}{2}\mathrm{tr}_{M_2(\mathcal{M})}\begin{pmatrix} ba^* & |b^*|a \\ |b|a^* & b^*a \end{pmatrix} : b \in L_q(M_n(\mathcal{M}))_1\} \\
&= \sup\{\frac{1}{2}\mathrm{tr}_{M_2(\mathcal{M})}(\begin{pmatrix} |b^*| & b \\ b^* & |b| \end{pmatrix}\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}) : b \in L_q(M_n(\mathcal{M}))_1\} \\
&\le \sup\{\varphi(\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}) : \varphi \in L_p(M_{2n}(\mathcal{M}))^*_{1,+}\} \\
&= 2^{\frac{1}{p}}\nu^p_{L_p(\mathcal{M})}(a).
\end{aligned}$$

Thus, from Lemma 2.1, we obtain the desired result. □

By combining this with [W, Theorem 4.15], we obtain the following embedding theorem.

Corollary 2.7. *Noncommutative L_p-spaces can be embedded into $B(H)$ $2^{\frac{1}{p}}$-completely isomorphically and complete order isomorphically.*

For a C^*-algebra A, we define an involution on its dual space A^* as

$$\varphi^*(a) = \varphi(a^*)^*, \qquad \varphi \in A^*, a \in A.$$

Along the line of operator space duality, we identify $M_n(A^*)$ and $M_n(A)^*$ algebraically by using the parallel duality pairing

$$\langle [\varphi_{ij}], [a_{ij}] \rangle = \sum_{i,j} \varphi_{ij}(a_{ij}), \qquad \varphi_{ij} \in A^*, a_{ij} \in A.$$

We can now define the positive cone $M_n(A^*)_+$. However, the duality between the von Neumann algebra $M_n(\mathcal{M})$ and its noncommutative L_1-space $L^1(M_n(\mathcal{M}))$ is given by the trace duality pairing

$$\langle [a_{ij}], [b_{ij}] \rangle = \mathrm{tr}_{M_n} \otimes \mathrm{tr}_{\mathcal{M}}([a_{ij}][b_{ij}]) = \sum_{i,j} \mathrm{tr}_{\mathcal{M}}(a_{ij}b_{ji}).$$

Hence, we must be more careful.

For an operator space V, its opposite operator space V^{op} is defined by

$$\|[v_{ij}^{op}]\|_{M_n(V^{op})} = \|[v_{ji}]\|_{M_n(V)}.$$

Note that V^{op} and V are the same normed spaces. For a matrix ordered operator space X, we define its opposite matrix ordered operator space X^{op} as the opposite operator space with the matrix order

$$[x_{ij}^{op}] \ge 0 \Leftrightarrow [x_{ji}] \ge 0.$$

A functional $\varphi : M_n(X) \to \mathbb{C}$ can be written as

$$\varphi = [\varphi_{ij}], \qquad \varphi([x_{ij}]) = \sum_{i,j} \varphi_{ij}(x_{ij}).$$

Let $\tilde{\varphi} = [\varphi_{ji}]$. Then we have

$$\varphi([x_{ij}]) = \tilde{\varphi}([x_{ji}]).$$

A functional $\varphi : M_n(X) \to \mathbb{C}$ is positive and contractive if and only if $\tilde{\varphi} : M_n(X^{op}) \to \mathbb{C}$ is positive and contractive. From this, we see that X is an operator system if and only if X^{op} is an operator system.

Corollary 2.8. *The dual space of a C^*-algebra is an operator system.*

Proof. The bidual A^{**} of a C^*-algebra A can be identified with the enveloping von Neumann algebra of A. The opposite matrix ordered operator space $(A^*)^{op}$ is completely isometric and completely order isomorphic to $L_1(A^{**})$ by [Te1, Theorem II.7]. Hence, the dual space A^* is an operator system. □

Finally, we show that the constant $2^{\frac{1}{p}}$ in the inequality

$$\|a\|_{L_p(M_n(\mathcal{M}))} \leq 2^{\frac{1}{p}} \nu^p_{L_p(\mathcal{M})}(a)$$

is the best one. However, this does not imply that $2^{\frac{1}{p}}$ is the best constant in Theorem 2.6. Unexpectedly, it is achieved by the 2-dimensional commutative case. We consider an element $(1,1)$ in ℓ^2_p. It is sufficient to show that

$$\nu^p_{\ell^2_p}((1,1)) \leq 1.$$

For $\frac{1}{p} + \frac{1}{q} = 1$, we let

$$\begin{pmatrix} a_1 & \cdot & b_1 & \cdot \\ \cdot & a_2 & \cdot & b_2 \\ \bar{b}_1 & \cdot & d_1 & \cdot \\ \cdot & \bar{b}_2 & \cdot & d_2 \end{pmatrix} \in (S^4_q)_{1,+}.$$

We have

$$\begin{aligned} 0 &\leq \mathrm{tr}(\begin{pmatrix} a_1 & \cdot & b_1 & \cdot \\ \cdot & a_2 & \cdot & b_2 \\ \bar{b}_1 & \cdot & d_1 & \cdot \\ \cdot & \bar{b}_2 & \cdot & d_2 \end{pmatrix} \begin{pmatrix} 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \\ -1 & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & 1 \end{pmatrix}) \\ &= a_1 + a_2 + d_1 + d_2 - b_1 - b_2 - \bar{b}_1 - \bar{b}_2. \end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname{tr}(\begin{pmatrix} a_1 & \cdot & b_1 & \cdot \\ \cdot & a_2 & \cdot & b_2 \\ \bar{b}_1 & \cdot & d_1 & \cdot \\ \cdot & \bar{b}_2 & \cdot & d_2 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}) &= b_1 + b_2 + \bar{b}_1 + \bar{b}_2 \\
&\leq \frac{1}{2}(a_1 + a_2 + d_1 + d_2 + b_1 + b_2 + \bar{b}_1 + \bar{b}_2) \\
&= \frac{1}{2}\operatorname{tr}(\begin{pmatrix} a_1 & \cdot & b_1 & \cdot \\ \cdot & a_2 & \cdot & b_2 \\ \bar{b}_1 & \cdot & d_1 & \cdot \\ \cdot & \bar{b}_2 & \cdot & d_2 \end{pmatrix} \begin{pmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix}) \\
&\leq \frac{1}{2}\| \begin{pmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix} \|_{S_p^4} \\
&= \frac{1}{2}\| \begin{pmatrix} 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix} \|_{S_p^4} \\
&= 2^{\frac{1}{p}}.
\end{aligned}$$

In the same manner, we can also show that

$$\operatorname{tr}(\begin{pmatrix} a_1 & \cdot & b_1 & \cdot \\ \cdot & a_2 & \cdot & b_2 \\ \bar{b}_1 & \cdot & d_1 & \cdot \\ \cdot & \bar{b}_2 & \cdot & d_2 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}) \geq -2^{\frac{1}{p}}.$$

We obtain the inequality

$$\nu^p_{\ell^2_p}((1,1)) \leq 1.$$

References

[CE] M.-D. Choi and E. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1997), 156–209. MR0430809 (55:3814)

[HJX] U. Haagerup M. Junge, and Q. Xu, *A reduction method for noncommutative L_p-spaces and applications* , Trans. Amer. Math. Soc., to appear.

[JRX] M. Junge, Z-J. Ruan and Q. Xu, *Rigid $\mathcal{OL}_p$ structures of non-commutative L_p-spaces associated with hyperfinite von Neumann algebras*, Math. Scand. **96** (2005), 63–95. MR2142873 (2006b:46086)

[K] H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L^p-spaces*, J. Funct. Anal. **56** (1984), 29–78 MR735704 (86a:46085)

[Pa] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics, 78, Cambridge University Press, Cambridge, UK, 2002. MR1976867 (2004c:46118)

[Pi1] G. Pisier, *The operator Hilbert space OH, complex interpolation and tensor norms*, Mem. Amer. Math. Soc. **122**, No. 585, 1996. MR1342022 (97a:46024)

[Pi2] G. Pisier, *Non-commutative vector valued L_p-spaces and completely p-summing maps*, Astérisque **247**, 1998. MR1648908 (2000a:46108)

[S] I. E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. (2) **57** (1953), 401–457. MR0054864 (14:991f)

[Ta] M. Takesaki, *Theory of Operator Algebras. II*, Encyclopaedia of Mathematical Sciences, Vol. 125, Springer-Verlag, 2002. MR1873025 (2002m:46083)

[Te1] M. Terp, *L^p spaces associated with von Neumann algebras*, Math. Institute, Copenhagen University, 1981.

[Te2] M. Terp, *Interpolation spaces between a von Neumann algebra and its predual*, J. Operator Theory **8** (1982), 327-360. MR677418 (85b:46075)

[W] W. Werner, *Subspaces of $L(H)$ that are $*$-invariant*, J. Funct. Anal. **193** (2002), 207–223. MR1929500 (2003h:46086)

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