PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 12, December 2009, Pages 4109–4116
S 0002-9939(09)10009-6
Article electronically published on July 20, 2009

A KUROSH-TYPE THEOREM FOR TYPE III FACTORS

JASON ASHER

(Communicated by Marius Junge)

ABSTRACT. We prove a generalization of N. Ozawa's Kurosh-type theorem to the setting of free products of semiexact II_1 factors with respect to arbitrary (non-tracial) faithful normal states. We are thus able to distinguish certain resulting type III factors. For example, if $M = LF_n \otimes LF_m$ and $\{\varphi_i\}$ is any sequence of faithful normal states on M, then the l-various $(M,\varphi_1)*...*(M,\varphi_l)$ are all mutually non-isomorphic.

1. INTRODUCTION

In [Oz], Ozawa obtained analogues of the Kurosh subgroup theorem (and its consequences) in the setting of free products of semiexact II_1 factors with respect to the canonical tracial states. In particular he was able to prove a certain unique-factorization theorem that distinguishes, for example, the n-various $L(F_\infty) * (L(F_\infty) \otimes R)^{*n}$. The paper was a continuation of the joint work [OP] with Popa that proved various unique-factorization results for tensor products of II_1 factors.

These papers shared a particular combination of ideas from [Oz2] and [Po3]. The first was a C^*-algebraic method for detecting injectivity of a von Neumann algebra, adopted from its original context of proving solidity of finite von Neumann algebras with the Akemann-Ostrand property. This method was used in concert with Popa's intertwining-by-bimodules technique that (roughly speaking) allows one to conclude unitary conjugacy results from a spatial condition. More precisely, if A and B are diffuse von Neumann subalgebras of a finite von Neumann algebra M, the presence of an A-B subbimodule of $L^2(M)$ with finite right B-dimension implies that a 'corner' of A can be conjugated into B by a partial isometry in M. The technique was actually shown by Popa to apply to the case of (M,φ) not necessarily finite (but with discrete decomposition), though one needs the additional assumption that A and B are subalgebras of the centralizer M^φ.

In this paper we will demonstrate how the above results can be extended to the case of free products of semiexact II_1 factors with respect to arbitrary (non-tracial) states. The primary difficulty this setting presents is that the resulting factors will be necessarily of type III. Following the same basic outline of the proof from [Oz], we will therefore utilize a generalization of Popa's intertwining technique that allows the ambient algebra M to be completely arbitrary and makes no assumption

Received by the editors November 13, 2008, and, in revised form, March 8, 2009.
2000 *Mathematics Subject Classification.* Primary 46L10, 46L09.
Research supported in part by NSF grant DMS-0555680 and NSF VIGRE grant DMS-0701302.

about the relative position of the subalgebras A and B in M. One of the fruits of our labors will be the result that for M_i semiexact non-prime non-injective II_1 factors with faithful normal states φ_i, the reduced free product $*_{i=1}^m (M_i, \varphi_i)$ of the (M_i, φ_i) is uniquely written in such a way as the free product of such factors, up to permutation of indices and stable isomorphism.

The study of primeness in the type III setting has been undertaken by several authors, and we would be remiss not to mention them here. In [Sh], Shlyakhtenko proved that for $0 < \lambda < 1$ the type III_λ free Araki-Woods factors are prime. Vaes and Vergnioux gave examples of prime type III factors that arose from quantum groups in [VV]. Gao and Junge also made use of Ozawa's techniques for type III to prove in [GJ] that any conditioned non-injective von Neumann subalgebra of the non-tracial reduced free product of injective von Neumann algebras is prime. Finally, in [CH], Chifan and Houdayer constructed prime factors of type III by using amalgamated free products.

2. Reduced free product

We will first review the reduced free product construction [Vo] in our particular case, retaining the notation from [Oz]. For $i \in \{1, \ldots, m\}$, let M_i be semiexact II_1 factors with faithful normal states φ_i, and let $H_i = L^2(M_i, \varphi_i)$. Denote by ξ_i the M_i-cyclic-and-separating state vector $\hat{1} \in H_i$, and let J_i be the modular conjugation on H_i. Finally, let $H_i^0 = H_i \ominus \mathbb{C}\xi_i$ and construct a new Hilbert space H as follows:

$$H = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{i_1, \ldots, i_n \in \{1, \ldots, m\} \\ i_1 \neq i_2 \neq \cdots \neq i_n}} H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0.$$

Next, let $H(i)$ be the subspace of H given by the direct sum of $\mathbb{C}\xi$ and those terms in the direct sum above with $i_1 \neq i$. For each i there is a unitary isomorphism

$$U_i : H \to H_i \otimes H(i)$$

which identifies $H(i)$ with $\mathbb{C}\xi \otimes H(i)$ and $H(i)^\perp$ with $H_i^0 \otimes H(i)$. Define a representation $\lambda_i : \mathbb{B}(H_i) \to \mathbb{B}(H)$ by

$$\lambda_i(a) = U_i^*(a \otimes 1_{H(i)})U_i.$$

The reduced free product $(M, \varphi) = *_{i=1}^m (M_i, \varphi_i)$ is then the von Neumann algebra in $\mathbb{B}(H)$ generated by $\lambda_i(M_i)$ for $i \in \{1, \ldots, m\}$, with faithful normal state φ given by

$$\varphi(x) = \langle x\xi, \xi \rangle_H .$$

Note that ξ is an M-cyclic-and-separating state vector, and so $H = L^2(M, \varphi)$ with the associated modular conjugation J given by

$$J(\eta_{i_1} \otimes \cdots \otimes \eta_{i_n}) = J_{i_n}\eta_{i_n} \otimes \cdots \otimes J_{i_1}\eta_{i_1},$$

where $\eta_{i_1} \otimes \cdots \otimes \eta_{i_n} \in H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0$ as in the direct sum decomposition above. Furthermore, one can thus see that $JH(i)$ is the direct sum of $\mathbb{C}\xi$ and those terms in the direct sum above with $i_n \neq i$.

For each i there is thus another unitary isomorphism

$$V_i : H \to JH(i) \otimes H_i$$

which identifies $JH(i)$ with $JH(i) \otimes \mathbb{C}\xi$ and $JH(i)^\perp$ with $JH(i) \otimes H_i^0$. Then, precisely as in [Oz], Lemma 3.1, we have the following.

Lemma 2.1. *For $a \in \mathbb{B}(H_i)$, $j \neq i$, and with P_ξ the projection of H onto $\mathbb{C}\xi$ we have*

$$\begin{aligned}\lambda_i(a) &= JV_i^*(1_{JH(i)} \otimes J_i a J_i)V_i J \\ &= V_i^*(P_\xi \otimes a + \lambda_i(a)_{|_{JH(i)\ominus\mathbb{C}\xi}} \otimes 1_{H_i})V_i \\ &= V_j^*(\lambda_i(a)_{|_{JH(i)}} \otimes 1_{H_i})V_j.\end{aligned}$$

This shows that V_i is a right M_i-module isomorphism of $L^2(M,\varphi)$ with $K_i \otimes H_i$, where $K_i = JH(i)$. Furthermore, if we let $D_i = V_i^*(\mathbb{K}(K_i) \otimes_{min} \mathbb{B}(H_i))V_i$, then this lemma also shows that $\lambda_j(a)D_i \subseteq D_i$ for $i, j \in \{1, \dots, m\}$.

Let $B_i \subseteq M_i$ be a σ-weakly dense exact C^*-algebra, and let B be the C^*-algebra in $\mathbb{B}(H)$ generated by $\lambda_i(B_i)$ for $i \in \{1, \dots, m\}$. Then for $C = JBJ$ we have the following lemma, as in [Oz], Prop. 3.2.

Lemma 2.2. *Suppose that $\Psi : \mathbb{B}(H) \to \mathbb{B}(H)$ is a C-bimodule unital completely positive map. If $D_i \subset \ker\Psi$ for $i \in \{1, \dots, m\}$, then the unital completely positive map*

$$\begin{aligned}\tilde{\Psi} : B \otimes C &\to \mathbb{B}(H) \\ \sum_{k=1}^n a_k \otimes x_k &\mapsto \Psi\left(\sum_{k=1}^n a_k x_k\right)\end{aligned}$$

is continuous with respect to the minimal tensor norm.

Finally, we will need the following control of partial isometries that normalize diffuse subalgebras in free products; see [Po2] and [Oz], Lemma 2.3. For completeness, we include the proof.

Lemma 2.3. *Let $(M,\varphi) = \ast_{i=1}^l (M_i, \varphi_i)$ be the reduced free product of II_1 factors M_i with faithful normal states φ_i, and let $Q \subseteq M_j$ be a diffuse von Neumann subalgebra. If $v \in M$ is a non-zero partial isometry with $vv^* \in Q' \cap M$ and such that $v^*Qv \subseteq M_i$, then $i = j$ and $v \in M_j$.*

Proof. Let $\zeta \in L^2(M_i, \varphi_i) \subseteq L^2(M, \varphi)$ be such that $\langle \cdot \zeta, \zeta \rangle$ gives the canonical trace on M_i. Note that then $x\zeta = J_\varphi x^* J_\varphi \zeta = \zeta x$ for all $x \in M_i$. If we let $\eta = v\zeta$, then $\eta \neq 0$ and

$$q\eta = vv^*qv\zeta = v\zeta(v^*qv) = \eta(v^*qv)$$

for all $q \in Q$.

Suppose, by contradiction, that $i \neq j$. Then $L^2(M,\varphi)$ as an M_j-M_i bimodule is an infinite multiple of the 'coarse' bimodule $L^2(M_j, \varphi_j) \otimes L^2(M_i, \varphi_i)$. Considering the projection of η onto any coarse component, we get a Hilbert-Schmidt operator T on $L^2(M, \varphi)$ such that

$$qT = T(v^*qv)$$

for all $q \in Q$. As Q is diffuse, we must have that $Tv^* = 0$, so that $T = Tv^*v = 0$. Thus, $\eta = 0$, a contradiction.

So, we must have that $i = j$. Now, note that $L^2(M,\varphi)$ as an M_j-M_j bimodule is isomorphic to the direct sum of $L^2(M_j, \varphi_j)$ and an infinite multiple of the coarse bimodule $L^2(M_j, \varphi_j) \otimes L^2(M_j, \varphi_j)$. By the above argument, we must have that $\eta \in L^2(M_j, \varphi_j)$. Thus, considering these L^2-vectors as unbounded operators affiliated with M_j, we get that $v = \eta\zeta^{-1} \in M_j$. $\square$

3. Intertwining-by-bimodules

We present here a generalization of the intertwining-by-bimodules technique of Popa; see Theorem 2.1 of [Po3]. The technical advantage is that we allow the ambient algebra to be arbitrary, and we make no assumption about the location of the subalgebras.

The following lemma is standard; for a reference on this and other aspects of the modular theory of Tomita and Takesaki, see [Ta].

Lemma 3.1. *Let M be a von Neumann algebra in a standard representation (M, H, J, P), where J is the modular conjugation, and P is the self-dual cone. Then any vector $\eta \in H$ has a unique polar decomposition as $\eta = u|\eta|$ for $|\eta| \in P$ and a partial isometry $u \in M$ such that the source of u is $[M'|\eta|]$, and the target of u is $[M'\eta]$. Furthermore if η satisfies an equation of the form $x\eta = Jx^*J\eta$ for all x in a (possibly non-unital) von Neumann subalgebra A of M, then $|\eta|$ satisfies the same equation, and u satisfies $xu = ux$ for all $x \in A$.*

Proof. Note that (M', H, J, P) is a standard representation for M', and consider the positive normal functional φ_η on M' given by

$$\varphi_\eta(z) = \langle z\eta, \eta\rangle\,.$$

Then there is a unique vector $|\eta| \in B$ such that $\varphi_\eta(z) = \langle z|\eta|, |\eta|\rangle$. Defining u on $M'|\eta|$ by $uz|\eta| = z\eta$ is well-defined (and when extended by continuity gives a partial isometry) because $||z\eta||^2 = \varphi_\eta(z^*z) = ||z|\eta|||^2$. Then u is clearly in M, and furthermore, uniqueness is now obvious.

Suppose further that η satisfies $x\eta = Jx^*J\eta$ for all $x \in A$. By considering $A + \mathbb{C}(1-p)$ for p the unit of A, we may assume that A is unital. Then for any unitary $w \in A$ one has that

$$(wuw^*)(wJwJ|\eta|) = wJwJ\eta = \eta.$$

Also, $\zeta = wJwJ|\eta| \in B$, and wuw^* is a partial isometry with source $[M'\zeta]$ and target $[M'\eta]$. Thus, by uniqueness, we obtain the result. $\square$

Proposition 3.2. *Let M be a von Neumann algebra with faithful normal state φ, and let $A, B \subseteq M$ be two factors of type II_1. Suppose that there is a non-zero A-B bimodule $H \subset L^2(M, \varphi)$ such that the von Neumann dimension of H as a right B-module satisfies $\dim_B H_B < \infty$. Then there are non-zero projections $q \in A$ and $p \in B$, an injective *-homomorphism $\theta : qAq \to pBp$, and a non-zero partial isometry $v \in M$ such that $v^*v \le p$, $vv^* \le q$, and $xv = v\theta(x)$ for all $x \in qAq$.*

Proof. We can pick non-zero projections $q \in A$ and $p \in B$ such that $K = qJ_\varphi pJ_\varphi H$ is a non-zero qAq-pBp bimodule with $\dim_{pBp} K_{pBp} = 1$. Then, we can find a right Hilbert pBp-module isomorphism $\alpha : L^2(pBp, \tau_{pBp}) \to H$, and define $\theta : qAq \to pBp$ by

$$\theta(x)\zeta = \alpha^{-1}(x\alpha(\zeta)) \text{ for all } \zeta \in L^2(pBp, \tau_{pBp}).$$

Note that θ is a non-zero, and hence injective, *-homomorphism.

Furthermore, if we let $\xi = \alpha(\hat{p})$, then for $x \in qAq$,

$$x\xi = \alpha(\theta(x)\hat{p}) = \alpha(\hat{p}\theta(x)) = J_\varphi\theta(x)^*J_\varphi\xi.$$

Consider now $M_2 \otimes M$ with the faithful normal state $\tilde{\varphi} = \mathrm{Tr} \otimes \varphi$, and define for $x \in qAq$,

$$\tilde{x} = \begin{bmatrix} x & 0 \\ 0 & \theta(x) \end{bmatrix}.$$

Also, let $\tilde{\xi} \in L^2(M_2 \otimes M, \tilde{\varphi})$ be defined as

$$\tilde{\xi} = \begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix}$$

and note that we then have the equation

$$\tilde{x}\tilde{\xi} = J_{\tilde{\varphi}} \tilde{x}^* J_{\tilde{\varphi}} \tilde{\xi}$$

for all $x \in qAq$.

By the lemma above, if we let $\tilde{\xi} = u|\tilde{\xi}|$ be the polar decomposition of $\tilde{\xi}$, then u in $M_2 \otimes M$ is a partial isometry satisfying

$$\tilde{x}u = u\tilde{x}$$

for all $x \in qAq$. It is easy to see that u has the form

$$u = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}$$

for a partial isometry $w \in M$ that satisfies $xw = w\theta(x)$ for all $x \in qAq$. In fact w is the polar part of ξ, and so letting $v = qw = qw\theta(q) = w\theta(q)$, we obtain the desired result. □

4. Kurosh-type theorem

We will now present the main technical result, a generalization of Theorem 3.3 in [Oz]. Note that much of the argument remains the same, but we include it for narrative coherence. Also, in everything that follows, all von Neumann algebras are assumed to be separable.

Theorem 4.1. *Let M_i, for $i \in \{1, \dots, l\}$, be semiexact II_1 factors with faithful normal states φ_i, and let $(M, \varphi) = \ast_{i=1}^{l} (M_i, \varphi_i)$ be their reduced free product. Suppose that $Q \subseteq M$ is an injective II_1 factor with a normal conditional expectation $E_Q : M \to Q$. If $Q' \cap M$ is non-injective, then there exists an index i and a maximal partial isometry $u \in M$ with $uu^* \in Q' \cap M$ and such that $u^*Qu \subseteq M_i$.*

Proof. Recall all of the notation from the reduced free product construction above. Let

$$\Psi_Q : \mathbb{B}(L^2(M, \varphi)) \to Q' \cap \mathbb{B}(L^2(M, \varphi))$$

be a proper conditional expectation, i.e. such that for all $x \in \mathbb{B}(L^2(M, \varphi))$ we have

$$\Psi_Q(x) \in \overline{\mathrm{co}}^{\omega}\{uxu^* : u \in U(Q)\}.$$

Note that Ψ_Q is a $Q' \cap \mathbb{B}(L^2(M, \varphi))$-bimodule map.

As Q is in the range of a normal conditional expectation from M, we can find a faithful normal state ν on M which centralizes Q. Then $\Psi_{Q_{|M}}$ is ν-preserving, and so $\Psi_{Q_{|_M}}$ is the unique ν-preserving normal conditional expectation $E_{Q'\cap M} : M \to Q' \cap M$. Lemma 5 from [Oz2] thus implies that, as $Q' \cap M$ is noninjective, the map

$$\tilde{\Psi}_Q : B \otimes C \to \mathbb{B}(L^2(M,\varphi))$$
$$\sum_{k=1}^{n} a_k \otimes x_k \mapsto \Psi_Q\left(\sum_{k=1}^{n} a_k x_k\right)$$

is not continuous with respect to the minimal tensor norm. In turn, Lemma 2.2 above implies that there must be an index i such that $D_i \not\subset \ker(\Psi_Q)$.

By continuity of Ψ_Q, there must be a finite rank projection $f \in \mathbb{B}(K_i)$ such that $x = \Psi_Q(f \otimes 1) \neq 0$, where we have identified $L^2(M,\varphi)$ with $K_i \otimes H_i$ via V_i. Note that x commutes with the right M_i-action, and so $x \in (\mathbb{B}(K_i)\bar{\otimes} M_i) \cap Q'$. Furthermore, $\mathrm{Tr} \otimes \tau_{M_i}(x) \leq \mathrm{Tr}(f) < \infty$, and so by taking a suitable spectral projection e of x we get that $eL^2(M,\varphi)$ is a non-zero Q-M_i bimodule with $\dim_{M_i} eL^2(M,\varphi) < \infty$.

Thus, by Proposition 3.2, there are non-zero projections $q \in Q$ and $p \in M_i$, a *-homomorphism $\theta : qQq \to pM_ip$, and a non-zero partial isometry $v \in M$ such that $v^*v \leq p$, $vv^* \leq q$ and for all $a \in qQq$ we have

$$av = v\theta(a).$$

Note that this implies that $vv^* \in (qQq)' \cap qMq = q(Q' \cap M)q$ and so we can write $vv^* = qq'$ for $q' \in Q' \cap M$. Furthermore, this also implies that $v^*v \in \theta(qQq)' \cap pMp$, and so by Lemma 2.3 we have that $v^*v \in pM_ip$. Thus for all $a \in Q$,

$$v^*av = v^*v\theta(qaq) \in M_i.$$

Also, note that all of this remains true after trimming q and q' (and v, θ, in turn). So, by restricting q (and hence v and θ) we may assume that $\tau_Q(q) = \frac{1}{n}$ for some integer n. Then, by restricting q' (and hence v), we may assume that $\tau_{M_i}(v^*v) = \frac{1}{mn}$ for some integer m. We can then find n partial isometries $u_i \in Q$ and $w_i \in M_i$, respectively, such that $\sum_i u_iu_i^* = 1$, $u_i^*u_i = q$, $\sum_i w_i^*w_i \leq 1$, and $w_iw_i^* = v^*v$. Then, letting $w = \sum_i u_ivw_i$ we get that for $a \in Q$,

$$w^*aw = \sum_i \sum_j w_i^*v^*u_i^*au_jvw_j \in M_i$$

while

$$ww^* = \sum_i u_iq'u_i^* = q' \in Q' \cap M.$$

Note that $\tau_{M_i}(w^*w) = \frac{1}{m}$. If $Q' \cap M$ is infinite, then by repeating an argument such as the one above, we can enlarge w to a partial isometry u with $u^*u = 1$, $uu^* \in Q' \cap M$ and such that $u^*Qu \subseteq M_i$. Furthermore, if $Q' \cap M$ is type III (or if q' is an infinite projection and $Q' \cap M$ is type II_∞) we can easily dilate u to a unitary.

Otherwise, if $Q' \cap M$ is type II_1 we have two cases. First, if $\tau_{Q'\cap M}(q') \leq \frac{1}{m}$, then we can proceed precisely as in the infinite situation and obtain u with the same properties. If on the other hand $\tau_{Q'\cap M}(q') > \frac{1}{m}$, we can restrict w and assume that $\tau_{Q'\cap M}(ww^*) = \frac{1}{m} \geq \tau_{M_i}(w^*w)$. Then, proceeding as above we can enlarge w to a partial isometry u with $uu^* = 1$ and such that $u^*Qu \subseteq M_i$. □

The following lemma follows easily from some well-known results of Popa; see A.1.1 and A.1.2 of [Po]. The proof in the case that the ambient algebra M is finite can be found as Proposition 13 of [OP], and the general case follows by the exact same argument where one replaces trace-preserving conditional expectations with those that preserve a state φ on M that centralizes N.

Lemma 4.2. *Let $N \subseteq M$ be an inclusion of factors with N type II_1 and in the range of a normal conditional expectation $E_N : M \to N$. Then there is an injective II_1 factor $Q \subseteq N$ such that $Q' \cap M = N' \cap M$.*

Corollary 4.3. *Let M_i, for $i \in \{1, \dots, l\}$, be semiexact II_1 factors with faithful normal states φ_i, and let $(M, \varphi) = \ast_{i=1}^{l}(M_i, \varphi_i)$ be their reduced free product. Suppose that $N \subseteq M$ is a non-prime non-injective II_1 factor with a normal conditional expectation $E_N : M \to N$. Then there is an index i and a maximal partial isometry u such that $u^* N u \subseteq M_i$.*

Proof. Write $N = N_1 \otimes N_2$ for II_1 factors N_i with N_2 non-injective. By Lemma 4.2, there is an injective II_1 factor $Q \subseteq N_1$ such that $Q' \cap M = N_1' \cap M$. Note that $N_2 \subseteq Q' \cap M$, and thus $Q' \cap M$ is non-injective. Furthermore, Q is in the range of a normal conditional expectation from M. Thus, by Theorem 4.1, there is an index i and a maximal partial isometry u such that $uu^* \in Q' \cap M = N_1' \cap M$, and $u^* Q u \subseteq M_i$.

Now, note that if $x \in N_2$ and $q \in Q$, then

$$(u^*xu)(u^*qu) = u^*xqu = u^*qxu = (u^*qu)(u^*xu)$$

and so by Lemma 2.3, $u^*xu \in M_i$.

Furthermore, if $y \in N_1$ and $x \in N_2$, then as $uu^* \in N_1' \cap M$,

$$(u^*yu)(u^*xu) = u^*yxu = u^*xyu = (u^*xu)(u^*yu)$$

and so by Lemma 2.3 again, $u^*yu \in M_i$.

Thus, $u^*N_1N_2u = u^*N_1uu^*N_2u \subseteq M_i$, and so $u^*Nu \subseteq M_i$. □

Corollary 4.4. *Let $\{M_i\}_{i=1}^m$, $\{N_j\}_{j=1}^n$ be semiexact non-prime non-injective II_1 factors with faithful normal states φ_i, ψ_j, and let M, φ, N, ψ be their respective reduced free products. If $M \simeq N$, then $m = n$ and, after reordering indices, M_i is isomorphic to an amplification of N_i.*

Proof. Identify M and N via the implied isomorphism. By the proof of Corollary 4.3, for each index i there is an injective II_1 factor $Q_i \subseteq M_i$, an index $\alpha(i)$, and a maximal partial isometry $u_i \in M$ such that $u_i u_i^* \in Q_i' \cap M$ and $u_i^* M_i u_i \subseteq N_{\alpha(i)}$. Similarly for each index j there is an injective II_1 factor $P_j \subseteq N_j$, an index $\beta(j)$, and a maximal partial isometry $v_j \in M$ such that $v_j v_j^* \in P_j' \cap M$ and $v_j^* N_j v_j \subseteq M_{\beta(j)}$.

Consider the projections $p_i = u_i^* u_i$ and $q_i = v_{\alpha(i)} v_{\alpha(i)}{}^*$ in $N_{\alpha(i)}$ (Lemma 2.3 implies that $q_i \in N_{\alpha(i)}$). First, by restricting the target projection of u_i in $Q_i' \cap M$, we may assume that $\tau_{N_{\alpha(i)}}(p_i) \leq \tau_{N_{\alpha(i)}}(q_i)$. Furthermore, by factoriality of $N_{\alpha(i)}$ we can rotate v_i by a partial isometry to assure that $p_i \leq q_i$. Then, if we let $w_i = u_i v_{\alpha(i)}$, w_i is a partial isometry such that $w_i^* Q_i w_i \subseteq M_{\beta(\alpha(i))}$, and $w_i w_i^* = u_i u_i^* \in Q_i' \cap M$. Thus, by Lemma 2.3, $\beta(\alpha(i)) = i$ and $w_i \in M_i$. So, u_i implements an isomorphism of $(u_i u_i^*) A (u_i u_i^*)$ with $pN_\alpha(i)p$.

A completely symmetric argument shows that $\alpha(\beta(j)) = j$. □

Note that this last corollary implies that, for example, if $d_i \in M = L(\mathbb{F}_n) \otimes L(\mathbb{F}_m)$ are positive and $\varphi_i(\cdot) \sim \tau_M(d_i \cdot)$, then $\ast_{i=1}^{l}(M, \varphi_i)$ are non-isomorphic for different l.

Acknowledgement

The author would like to thank Dimitri Shlyakhtenko for initially suggesting this investigation and for his invaluable subsequent advice and direction.

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Department of Mathematics, University of California, Los Angeles, California 90095
E-mail address: asherj@math.ucla.edu