

A DIRECT PROOF OF THE THEOREM ON FORMAL FUNCTIONS

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ABSTRACT. We give a direct and elementary proof of the theorem on formal functions by studying the behaviour of the Godement resolution of a sheaf of modules under completion. This proof also works in some non-noetherian cases.

INTRODUCTION

Let $\pi: X \rightarrow \text{Spec } A$ be a proper scheme over a noetherian ring A . Let \mathcal{M} be a coherent \mathcal{O}_X -module and $Y \subset \text{Spec } A$ a closed subscheme. Let us denote by $^{\wedge}$ the completion along Y (respectively, along $\pi^{-1}(Y)$). The theorem on formal functions states that

$$H^i(X, \mathcal{M})^{\wedge} = H^i(X, \hat{\mathcal{M}}).$$

Two important corollaries of this theorem are Stein's factorization theorem and Zariski's Main Theorem ([H, III, 11.4, 11.5]).

Hartshorne [H] gives a proof of the theorem on formal functions for projective schemes (over a noetherian ring). Grothendieck [Gro] proves it for proper noetherian schemes. He first gives sufficient conditions for the commutation of the cohomology of complexes of A -modules with inverse limits ([Gro, 0, 13.2.3]); secondly, he gives a general theorem on the commutation of the cohomology of sheaves with inverse limits ([Gro, 0, 13.3.1]); finally, he laboriously checks that the theorem on formal functions is under the hypothesis of this general one ([Gro, 4.1.5]).

In this paper we give the "obvious direct proof" of the theorem on formal functions. Very briefly, we prove that the completion of the Godement resolution of a coherent sheaf is a flasque resolution of the completion of the coherent sheaf and that taking sections in the Godement complex commutes with completion. This, together with the finiteness theorem for the higher direct images of a proper morphism, yields the theorem. In fact, we prove that in the non-noetherian case the theorem on formal functions still holds under some finiteness hypothesis.

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1. COMPLETION OF GODEMENT RESOLUTION

Definition 1. Let X be a scheme, $\mathfrak{p} \subset \mathcal{O}_X$ a sheaf of ideals and \mathcal{M} an \mathcal{O}_X -module. The \mathfrak{p} -adic completion of \mathcal{M} , denoted by $\widehat{\mathcal{M}}$, is

$$\widehat{\mathcal{M}} := \varprojlim_n \mathcal{M}/\mathfrak{p}^n \mathcal{M}.$$

If $U = \text{Spec } A$ is an affine open subset and $I = \mathfrak{p}(U)$, one has a natural morphism

$$\Gamma(U, \mathcal{M}) \otimes_A A/I^n \rightarrow \Gamma(U, \mathcal{M}/\mathfrak{p}^n \mathcal{M})$$

and then a morphism

$$\Gamma(U, \mathcal{M})^\wedge \rightarrow \Gamma(U, \widehat{\mathcal{M}}),$$

where $\Gamma(U, \mathcal{M})^\wedge$ is the I -adic completion of $\Gamma(U, \mathcal{M})$.

Definition 2. We say that \mathcal{M} is affinely \mathfrak{p} -acyclic if for any affine open subset U and any natural number n , the sheaves \mathcal{M} and $\mathcal{M}/\mathfrak{p}^n \mathcal{M}$ are acyclic on U and the morphism $\Gamma(U, \mathcal{M}) \otimes_A A/I^n \rightarrow \Gamma(U, \mathcal{M}/\mathfrak{p}^n \mathcal{M})$ is an isomorphism. In particular, $\Gamma(U, \mathcal{M})^\wedge \rightarrow \Gamma(U, \widehat{\mathcal{M}})$ is an isomorphism.

Every quasi-coherent module is affinely \mathfrak{p} -acyclic (with \mathfrak{p} quasi-coherent).

Notation. For any sheaf F , let us denote

$$0 \rightarrow F \rightarrow C^0 F \rightarrow C^1 F \rightarrow \dots \rightarrow C^n F \rightarrow \dots$$

as its Godement resolution (see [Go, II, 4.3]). We shall also denote $C^* F = \bigoplus_{i \geq 0} C^i F$ and $F_i = \text{Ker}(C^i F \rightarrow C^{i+1} F)$. One has that $C^0 F_i = C^i F$. The functors $F \rightsquigarrow C^i F$ and $F \rightsquigarrow F_i$ are exact.

Lemma 3. Let X be a scheme, \mathfrak{p} a sheaf of ideals of finite type on X and \mathcal{M} an \mathcal{O}_X -module. Denote $I = \Gamma(X, \mathfrak{p})$ and assume that \mathfrak{p} is generated by a finite number of global sections (this holds for example when X is affine). For any open subset $V \subseteq X$ one has

$$\Gamma(V, C^0(\mathfrak{p}\mathcal{M})) = I \cdot \Gamma(V, C^0 \mathcal{M}).$$

In particular, the natural morphism $\mathfrak{p}C^0 \mathcal{M} \rightarrow C^0(\mathfrak{p}\mathcal{M})$ is an isomorphism.

Proof. If J is a finitely generated ideal of a ring A and M_i is a collection of A -modules, then $J \cdot \prod M_i = \prod (J \cdot M_i)$. Now, by hypothesis, \mathfrak{p} is generated by a finite number of global sections f_1, \dots, f_r . Let $J = (f_1, \dots, f_r)$. Then

$$\Gamma(V, C^0(\mathfrak{p}\mathcal{M})) = \prod_{x \in V} \mathfrak{p}_x \cdot \mathcal{M}_x = \prod_{x \in V} J \cdot \mathcal{M}_x = J \cdot \prod_{x \in V} \mathcal{M}_x = J \cdot \Gamma(V, C^0 \mathcal{M}).$$

Since $I \cdot \prod_{x \in V} \mathcal{M}_x$ is contained in $\Gamma(V, C^0(\mathfrak{p}\mathcal{M}))$ one concludes the lemma. In particular, if V is affine, then $\Gamma(V, C^0(\mathfrak{p}\mathcal{M})) = I_V \cdot \Gamma(V, C^0 \mathcal{M})$, with $I_V = \Gamma(V, \mathfrak{p})$. It follows that $\mathfrak{p}C^0 \mathcal{M} \rightarrow C^0(\mathfrak{p}\mathcal{M})$ is an isomorphism. \square

Proposition 4. Let X be a scheme and let \mathfrak{p} be an ideal of finite type. For any \mathcal{O}_X -module \mathcal{M} one has:

- (1) $\mathfrak{p}C^i \mathcal{M} = C^i(\mathfrak{p}\mathcal{M})$ and $(C^i \mathcal{M})/\mathfrak{p}(C^i \mathcal{M}) = C^i(\mathcal{M}/\mathfrak{p}\mathcal{M})$, for any i .
- (2) $\mathfrak{p}\mathcal{M}_i = (\mathfrak{p}\mathcal{M})_i$ and $(\mathcal{M}/\mathfrak{p}\mathcal{M})_i = \mathcal{M}_i/\mathfrak{p}\mathcal{M}_i$, for any i .
- (3) $C^i \mathcal{M}$ is affinely \mathfrak{p} -acyclic.

- (4) $\widehat{C^i \mathcal{M}}$ is flasque. Moreover, if \mathfrak{p} is generated by a finite number of global sections, then

$$\Gamma(X, \widehat{C^i \mathcal{M}}) = \Gamma(X, C^i \mathcal{M})^\wedge.$$

Proof. (1) and (2). We may assume that X is affine. Hence $\mathfrak{p}C^0 \mathcal{M} = C^0(\mathfrak{p}\mathcal{M})$ by the previous lemma and $(C^0 \mathcal{M})/\mathfrak{p}C^0 \mathcal{M} = C^0 \mathcal{M}/C^0(\mathfrak{p}\mathcal{M}) = C^0(\mathcal{M}/\mathfrak{p}\mathcal{M})$. From the exact sequence

$$\mathcal{M}/\mathfrak{p}\mathcal{M} \rightarrow C^0 \mathcal{M}/\mathfrak{p}C^0 \mathcal{M} \rightarrow \mathcal{M}_1/\mathfrak{p}\mathcal{M}_1 \rightarrow 0$$

and the isomorphism $C^0 \mathcal{M}/\mathfrak{p}C^0 \mathcal{M} = C^0(\mathcal{M}/\mathfrak{p}\mathcal{M})$, it follows that $\mathcal{M}_1/\mathfrak{p}\mathcal{M}_1 = (\mathcal{M}/\mathfrak{p}\mathcal{M})_1$. By the exactness of the functor $M \rightsquigarrow M_1$ one concludes that $\mathfrak{p}\mathcal{M}_1 = (\mathfrak{p}\mathcal{M})_1$. Consequently $\mathfrak{p}C^1 \mathcal{M} = \mathfrak{p}C^0(\mathcal{M}_1) = C^0(\mathfrak{p}\mathcal{M}_1) = C^0((\mathfrak{p}\mathcal{M})_1) = C^1(\mathfrak{p}\mathcal{M})$, and analogously $C^1 \mathcal{M}/\mathfrak{p}C^1 \mathcal{M} = C^1(\mathcal{M}/\mathfrak{p}\mathcal{M})$. Repeating this argument one concludes (1) and (2).

3. Since $C^i \mathcal{M} = C^0 \mathcal{M}_i$, we may assume that $i = 0$. Denote $\mathcal{N} = C^0 \mathcal{M}$. By (1), $\mathcal{N}/\mathfrak{p}^n \mathcal{N}$ is acyclic on any open subset. From the long exact sequence of cohomology associated to $0 \rightarrow \mathfrak{p}^n \mathcal{N} \rightarrow \mathcal{N} \rightarrow \mathcal{N}/\mathfrak{p}^n \mathcal{N} \rightarrow 0$ and the acyclicity of $\mathfrak{p}^n \mathcal{N}$ (by (1)), one obtains that

$$\Gamma(U, \mathcal{N}/\mathfrak{p}^n \mathcal{N}) = \Gamma(U, \mathcal{N})/\Gamma(U, \mathfrak{p}^n \mathcal{N}).$$

Moreover, if U is affine, $\Gamma(U, \mathfrak{p}^n \mathcal{N}) = \mathfrak{p}^n(U)\Gamma(U, \mathcal{N})$, by Lemma 3. We have concluded (3).

(4) Again, we may assume that $i = 0$. Let us prove that the completion of $\mathcal{N} = C^0 \mathcal{M}$ is flasque. It suffices to prove that its restriction to any affine open subset is flasque, so we may assume that X is affine. Let us denote $I = \mathfrak{p}(X)$. For any open subset V , one has as in the proof of (3)

$$\Gamma(V, \widehat{\mathcal{N}}) = \lim_{\leftarrow n} \Gamma(V, \mathcal{N}/\mathfrak{p}^n \mathcal{N}) = \lim_{\leftarrow n} \Gamma(V, \mathcal{N})/\Gamma(V, \mathfrak{p}^n \mathcal{N}),$$

and by Lemma 3, $\Gamma(V, \mathfrak{p}^n \mathcal{N}) = I^n \Gamma(V, \mathcal{N})$. In conclusion, $\Gamma(V, \widehat{\mathcal{N}}) = \Gamma(V, \mathcal{N})^\wedge$. One concludes that $\widehat{\mathcal{N}}$ is flasque because \mathcal{N} is flasque and that the I -adic completion preserves surjections. The same arguments prove the second part of the statement. \square

Lemma 3 and Proposition 4 will be used very often in the sequel without referencing.

Proposition 5. *Let \mathfrak{p} be of finite type and \mathcal{M} be affinely \mathfrak{p} -acyclic. Then $\widehat{C \cdot \mathcal{M}}$ is a flasque resolution of $\widehat{\mathcal{M}}$.*

Proof. We already know that $\widehat{C \cdot \mathcal{M}}$ is flasque. Let us prove now that \mathcal{M}_1 is affinely \mathfrak{p} -acyclic. From the exact sequence

$$0 \rightarrow \mathcal{M}/\mathfrak{p}^n \mathcal{M} \rightarrow C^0(\mathcal{M}/\mathfrak{p}^n \mathcal{M}) \rightarrow \mathcal{M}_1/\mathfrak{p}^n \mathcal{M}_1 \rightarrow 0$$

one has that $\mathcal{M}_1/\mathfrak{p}^n \mathcal{M}_1$ is acyclic on any affine open subset. Moreover, taking sections on an affine open subset $U = \text{Spec } A$, one obtains the exact sequence (let us denote $I = \mathfrak{p}(U)$)

$$0 \rightarrow \Gamma(U, \mathcal{M}) \otimes_A A/I^n \rightarrow \Gamma(U, C^0 \mathcal{M}) \otimes_A A/I^n \rightarrow \Gamma(U, \mathcal{M}_1/\mathfrak{p}^n \mathcal{M}_1) \rightarrow 0$$

and then $\Gamma(U, \mathcal{M}_1) \otimes_A A/I^n = \Gamma(U, \mathcal{M}_1/\mathfrak{p}^n \mathcal{M}_1)$; i.e. \mathcal{M}_1 is affinely \mathfrak{p} -acyclic.

Now, taking the inverse limit in the above exact sequence (and taking into account that the I -adic completion preserves surjections) one obtains the exact sequence

$$(*) \quad 0 \rightarrow \Gamma(U, \widehat{\mathcal{M}}) \rightarrow \Gamma(U, \widehat{C^0\mathcal{M}}) \rightarrow \Gamma(U, \widehat{\mathcal{M}}_1) \rightarrow 0.$$

Therefore the sequence $0 \rightarrow \widehat{\mathcal{M}} \rightarrow \widehat{C^0\mathcal{M}} \rightarrow \widehat{\mathcal{M}}_1 \rightarrow 0$ is exact. The conclusion follows easily. \square

Remark 6. From the exact sequence $(*)$ one obtains that $H^1(U, \widehat{\mathcal{M}}) = 0$ for any affine open subset U and any affinely \mathfrak{p} -acyclic module \mathcal{M} . Since \mathcal{M}_1 is also affinely \mathfrak{p} -acyclic, one has $H^2(U, \widehat{\mathcal{M}}) = 0$. Recurrently, $H^i(U, \widehat{\mathcal{M}}) = 0$ for any $i > 0$. Moreover, $\widehat{\mathcal{M}}/\mathfrak{p}^n\widehat{\mathcal{M}} = \mathcal{M}/\mathfrak{p}^n\mathcal{M}$. In fact, $\widehat{\mathcal{M}}/\mathfrak{p}^n\widehat{\mathcal{M}}$ is the sheaf associated to the presheaf $\Gamma(U, \widehat{\mathcal{M}})/I^n\Gamma(U, \widehat{\mathcal{M}})$, with $I = \mathfrak{p}(U)$. Since $\Gamma(U, \widehat{\mathcal{M}}) = \Gamma(U, \mathcal{M})^\wedge$, this presheaf is $\Gamma(U, \mathcal{M})^\wedge/I^n\Gamma(U, \mathcal{M})^\wedge = \Gamma(U, \mathcal{M})/I^n\Gamma(U, \mathcal{M})$ (because I is finitely generated), whose associated sheaf is $\mathcal{M}/\mathfrak{p}^n\mathcal{M}$. Finally

$$\begin{aligned} \Gamma(U, \mathcal{M}/\mathfrak{p}^n\mathcal{M}) &= \Gamma(U, \mathcal{M})/I^n\Gamma(U, \mathcal{M}) = \Gamma(U, \mathcal{M})^\wedge/I^n\Gamma(U, \mathcal{M})^\wedge \\ &= \Gamma(U, \widehat{\mathcal{M}})/I^n\Gamma(U, \widehat{\mathcal{M}}). \end{aligned}$$

In conclusion, if \mathfrak{p} is a finite type ideal and \mathcal{M} is affinely \mathfrak{p} -acyclic, then $\widehat{\mathcal{M}}$ is also affinely \mathfrak{p} -acyclic. This result will not be used in the sequel.

2. FINITENESS HYPOTHESIS

Definition 7. An \mathcal{O}_X -module \mathcal{M} is said to be pseudo-coherent if for each $n \in \mathbb{N}$ there exists locally on X an exact sequence $L_n \rightarrow \dots \rightarrow L_0 \rightarrow \mathcal{M} \rightarrow 0$, where L_i are free modules of finite type. A morphism $f: X \rightarrow Y$ is said to be pseudo-coherent if every $x \in X$ has an open neighborhood U such that the restriction $f|_U$ factors as $U \xrightarrow{i} Z \xrightarrow{p} Y$, where i is a closed immersion such that $i_*\mathcal{O}_U$ is pseudo-coherent on Z , and p is smooth ([I, p. 228, Def. 1.2]).

Let $f: X \rightarrow Y$ be a pseudo-coherent morphism of schemes, \mathcal{M} a pseudo-coherent \mathcal{O}_X -module and \mathfrak{p} a sheaf of ideals on Y of finite type. Let us denote $D_{\mathfrak{p}}\mathcal{O}_Y = \bigoplus_{n \in \mathbb{N}} \mathfrak{p}^n$, which has an obvious structure of sheaf of rings. Moreover, $\bigoplus_{n \in \mathbb{N}} R^i f_*(\mathfrak{p}^n \mathcal{M})$ has a natural structure of $D_{\mathfrak{p}}\mathcal{O}_Y$ -module. We shall assume that

$$(1) \quad \bigoplus_{n \in \mathbb{N}} R^i f_*(\mathfrak{p}^n \mathcal{M}) \text{ is a } D_{\mathfrak{p}}\mathcal{O}_Y\text{-module of finite type.}$$

Examples 8. Let $f: X \rightarrow Y$, \mathcal{M} and \mathfrak{p} be as above. Let us see some cases where the hypothesis (1) holds:

(a) Noetherian case: Assume that $f: X \rightarrow Y$ is a proper morphism between noetherian schemes. Then f is pseudo-coherent, \mathcal{M} is coherent and \mathfrak{p} is a coherent sheaf of ideals. By [Gro, 3.3.2], (1) holds. In fact, this statement is a consequence of the Finiteness Theorem: let $Y' = \text{Spec } D_{\mathfrak{p}}\mathcal{O}_Y$, $X' = \text{Spec } D_{\mathfrak{p}}\mathcal{O}_X$, and $\mathcal{M}' = D_{\mathfrak{p}}\mathcal{M} = \bigoplus_{n=0}^{\infty} \mathfrak{p}^n \mathcal{M}$, with the obvious $\mathcal{O}_{X'}$ -module structure and $f': X' \rightarrow Y'$ the natural morphism. \mathcal{M}' is a coherent $\mathcal{O}_{X'}$ -module and f' is proper. By the Finiteness Theorem $R^i f'_*\mathcal{M}'$ is a coherent $\mathcal{O}_{Y'}$ -module. Since $\pi: Y' \rightarrow Y$ and $\pi': X' \rightarrow X$ are affine morphisms, one concludes that $\pi_*(R^i f'_*\mathcal{M}') = R^i f_*(\pi'_*\mathcal{M}') = \bigoplus_{n \in \mathbb{N}} R^i f_*(\mathfrak{p}^n \mathcal{M})$ is a $\pi_*\mathcal{O}_{Y'} = D_{\mathfrak{p}}\mathcal{O}_Y$ -module of finite type.

(b) Assume that f is proper and flat and that \mathcal{M} is flat over Y . With the same notation as in the preceding example, one has that f' is a proper pseudo-coherent map (see [I, p. 233, Cor 1.10]) and $\mathcal{M}' \simeq \mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{M}$ is a pseudo-coherent module on X' (see [I, p. 111, Cor 2.16.1]). By Kiehl's Finiteness Theorem ([K, p. 315, Thm. 2.9']) $R^i f'_* \mathcal{M}'$ is a pseudo-coherent $\mathcal{O}_{Y'}$ -module. One concludes as before that (1) holds.

Theorem 9 (on formal functions). *Let $f: X \rightarrow Y$ be a morphism of schemes, \mathfrak{p} a sheaf of ideals of finite type on Y and $\mathfrak{p}\mathcal{O}_X$ the ideal induced in X . Let \mathcal{M} be an affinely \mathfrak{p} -acyclic \mathcal{O}_X -module satisfying (1). The natural morphisms (where completions are made by \mathfrak{p} and $\mathfrak{p}\mathcal{O}_X$ respectively)*

$$R^i f_* \widehat{\mathcal{M}} \rightarrow R^i f_*(\widehat{\mathcal{M}})$$

are isomorphisms. If $Y = \text{Spec } A$, then

$$H^i(X, \mathcal{M})^\wedge = H^i(X, \widehat{\mathcal{M}}).$$

Proof. The question is local on Y , so we may assume that $Y = \text{Spec } A$ is affine. It suffices to show that $H^i(X, \mathcal{M})^\wedge = H^i(X, \widehat{\mathcal{M}})$. It is clear that $\mathfrak{p}\mathcal{O}_X$ is generated by the global sections $I = \Gamma(Y, \mathfrak{p})$.

Let $C^\bullet \mathcal{M}$ be the Godement resolution of \mathcal{M} . Let us denote by d_i the differential of the complex $\Gamma(X, C^\bullet \mathcal{M})$ on degree i and d_i^n the differential of the complex $\Gamma(X, \mathfrak{p}^n C^\bullet \mathcal{M}) = I^n \Gamma(X, C^\bullet \mathcal{M})$ on degree i . Observe that $\text{Im } d_i^n = I^n \text{Im } d_i$ and $\text{Ker } d_i = \Gamma(X, \mathcal{M}_i)$ because \mathcal{M}_i is the kernel of $C^i \mathcal{M} \rightarrow C^{i+1} \mathcal{M}$ (recall that $C^i \mathcal{M} = C^0 \mathcal{M}_i$). Moreover, $(I^n \text{Ker } d_i) \cap \text{Im } d_{i-1} = I^n \text{Im } d_{i-1}$, as follows from the inclusion $I^n \text{Im } d_{i-1} \subseteq (I^n \text{Ker } d_i) \cap \text{Im } d_{i-1}$ and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n \Gamma(X, \mathcal{M}_i) \cap \text{Im } d_{i-1} & \longrightarrow & I^n \Gamma(X, \mathcal{M}_i) & \longrightarrow & I^n H^i(X, \mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^n \text{Im } d_{i-1} = \text{Im } d_{i-1}^n & \longrightarrow & \Gamma(X, \mathfrak{p}^n \mathcal{M}_i) & \longrightarrow & H^i(X, \mathfrak{p}^n \mathcal{M}) & \longrightarrow & 0. \end{array}$$

Hence,

$$H^i(X, \mathcal{M})^\wedge = (\text{Ker } d_i / \text{Im } d_{i-1})^\wedge = \widehat{\text{Ker } d_i} / \widehat{\text{Im } d_{i-1}}.$$

$\widehat{C^\bullet \mathcal{M}}$ is a flasque resolution of $\widehat{\mathcal{M}}$ (by Proposition 5) and $\Gamma(X, \widehat{C^\bullet \mathcal{M}}) = \Gamma(X, C^\bullet \mathcal{M})^\wedge$ (by Proposition 4(4)). Completing the exact sequences

$$0 \rightarrow \text{Ker } d_i \rightarrow \Gamma(X, C^i \mathcal{M}) \rightarrow \text{Im } d_i \rightarrow 0$$

we obtain the exact sequences

$$0 \rightarrow \widehat{\text{Ker } d_i} \rightarrow \Gamma(X, \widehat{C^i \mathcal{M}}) \rightarrow \widehat{\text{Im } d_i} \rightarrow 0$$

because, as we shall see below, the I -adic topology of $\Gamma(X, C^i \mathcal{M})$ induces the I -adic topology on $\text{Ker } d_i$. Then $H^i(\Gamma(X, \widehat{C^\bullet \mathcal{M}})) = \widehat{\text{Ker } d_i} / \widehat{\text{Im } d_{i-1}}$ and the natural map $H^i(X, \mathcal{M})^\wedge = [H^i \Gamma(X, C^\bullet \mathcal{M})]^\wedge \rightarrow H^i(\Gamma(X, C^\bullet \mathcal{M})^\wedge) = H^i(\Gamma(X, \widehat{C^\bullet \mathcal{M}})) = H^i(X, \widehat{\mathcal{M}})$ is an isomorphism.

Let us prove that the I -adic topology of $\Gamma(X, C^i \mathcal{M})$ induces the I -adic topology on $\text{Ker } d_i = \Gamma(X, \mathcal{M}_i)$. Intersecting the equality $I^n \Gamma(X, C^0 \mathcal{M}_i) = \Gamma(X, C^0(\mathfrak{p}^n \mathcal{M}_i))$ with $\Gamma(X, \mathcal{M}_i)$, one obtains that the induced topology on $\Gamma(X, \mathcal{M}_i)$ is given by the filtration $\{\Gamma(X, \mathfrak{p}^n \mathcal{M}_i)\}$. Hence it suffices to show that this filtration is I -stable.

Since $\mathfrak{p}^n \mathcal{M}_i = (\mathfrak{p}^n \mathcal{M})_i$ (see Proposition 4(2)), it is enough to prove that the filtration $\{\Gamma(X, (\mathfrak{p}^n \mathcal{M})_i)\}$ is I -stable. This is equivalent to showing that $\bigoplus_{n=0}^{\infty} \Gamma(X, (\mathfrak{p}^n \mathcal{M})_i)$ is a $D_I A$ -module generated by a finite number of homogeneous components, where $D_I A = \bigoplus_{n=0}^{\infty} I^n$. By the exact sequence

$$\bigoplus_{n=0}^{\infty} \Gamma(X, C^{i-1}(\mathfrak{p}^n \mathcal{M})) \rightarrow \bigoplus_{n=0}^{\infty} \Gamma(X, (\mathfrak{p}^n \mathcal{M})_i) \rightarrow \bigoplus_{n=0}^{\infty} H^i(X, \mathfrak{p}^n \mathcal{M}) \rightarrow 0$$

it suffices to see the statement for the first and third members. The statement for the first one is obvious because $\Gamma(X, C^{i-1}(\mathfrak{p}^n \mathcal{M})) = I^n \Gamma(X, C^{i-1} \mathcal{M})$. For the third one, it is true by hypothesis (1). \square

Remark 10. Reading the above proof carefully, it is not difficult to see that one has already showed that $H^i(X, \mathcal{M})^\wedge = \varprojlim_n H^i(X, \mathcal{M}/\mathfrak{p}^n \mathcal{M})$.

Corollary 11. *Assume now that $f: X \rightarrow Y$ is a pseudo-coherent projective morphism of schemes; that is, locally on Y , f factors as $X \xrightarrow{i} \mathbb{P}_Y^r \xrightarrow{\pi} Y$, where i is a pseudo-coherent closed immersion and π is the natural projection. Let \mathcal{M} be a pseudo-coherent \mathcal{O}_X -module, flat over Y , and \mathfrak{p} a sheaf of ideals on Y of finite type. Then the theorem on formal functions holds:*

$$R^i \widehat{f_* \mathcal{M}} \xrightarrow{\sim} R^i f_*(\widehat{\mathcal{M}}).$$

Proof. The question is local on Y , so we can assume that f factors as $X \xrightarrow{i} \mathbb{P}_Y^r \xrightarrow{\pi} Y$. The theorem holds obviously for i , i.e., $\widehat{i_* \mathcal{M}} = i_* \widehat{\mathcal{M}}$, and it holds also for π (by Theorem 9). Then

$$R^i \widehat{f_* \mathcal{M}} = R^i \widehat{\pi_*(i_* \mathcal{M})} = R^i \pi_*(\widehat{i_* \mathcal{M}}) = R^i \pi_*(i_* \widehat{\mathcal{M}}) = R^i f_*(\widehat{\mathcal{M}}). \quad \square$$

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