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RECURSIVE FORMULA FOR $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g$ IN $\overline{\mathcal{M}}_{g,1}$

D. ARCARA AND F. SATO

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Abstract. Mumford proved that $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g = 0$ in the Chow ring of $\overline{\mathcal{M}}_{g,1}$. We find an explicit recursive formula for $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g$ in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ as a combination of classes supported on boundary strata.

1. Introduction

Mumford proved in [4] that $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g = 0$ in the Chow ring of $\mathcal{M}_{g,1}$. Moreover, he showed that this class is supported on the boundary strata with a marked genus 0 component. Graber and Vakil proved in [3] that every codimension g class in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ is supported on the boundary strata with at least one genus 0 component.

We complement these results by finding an explicit recursive formula for $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g$ in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ as a combination of classes supported on boundary strata. It is clear from the formula being recursive that all the boundary strata have a genus 0 component in them, but it is not obvious from the formula that the marked point must be on a genus 0 component. We simplified the formula for $g < 5$ in Section 4 and checked that this is the case.

To make the statement of the Main Theorem easier to understand, let us introduce some notation. We shall denote the class $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g$ by $\Lambda_{g,g}$. More generally, whenever a moduli space $\overline{\mathcal{M}}_{g,n}$ has a natural choice of a special marked point (for example, it is the marked point that is glued via a gluing map), we shall define

$$\Lambda_{g,i} := \psi^i - \lambda_1\psi^{i-1} + \cdots + (-1)^g\lambda_g\psi^{i-g},$$

where the ψ class is at the special marked point, and we shall use the usual convention that $\psi^{-1} = 0$.

Theorem 1. *In the tautological ring of* $\overline{\mathcal{M}}_{g,1}$,

$$\Lambda_{g,g} = \sum_{h=1}^{g} \left(1 - \frac{h}{g}\right) \iota_{h*}(c_h),$$

where

$$c_h := \sum_{i=0}^{g-1} (-1)^{h+i} \Lambda_{h,i}\Lambda_{g-h,g-1-i}$$

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and ι_h is the natural boundary map

$$\iota_h \colon \overline{\mathcal{M}}_{h,2} \times \overline{\mathcal{M}}_{g-h,1} \longrightarrow \overline{\mathcal{M}}_{g,1}.$$

This formula is actually the first step of an algorithm which calculates each of the classes ψ^g, $\lambda_1\psi^{g-1}$, $\ldots$, λ_g in terms of classes supported on boundary strata. We want to single out the class $\Lambda_{g,g}$, though, because it is the only class we found so far in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ which has a nice recursive formula and can therefore be easily calculated.

Note that the formula is recursive because, once we obtain a formula for $\Lambda_{g,g}$, we can

- obtain a formula for $\Lambda_{g,g+1}$ (and then, similarly, for $\Lambda_{g,i}$ for $i > g$) by multiplying the formula obtained by ψ and simplifying;
- obtain a formula for $\Lambda_{g,i}$ $(i \geq g)$ on $\overline{\mathcal{M}}_{g,2}$ by pulling back the formula obtained on $\overline{\mathcal{M}}_{g,1}$ and simplifying using the pull-back formulas for ψ and λ classes.

2. Virtual localization

The main tool we use to prove our theorems is the virtual localization theorem by Graber and Pandharipande [2].

Theorem (Virtual localization theorem). *Suppose $f\colon X \to X'$ is a $\mathbb{C}^*$-equivariant map of proper Deligne–Mumford quotient stacks with a $\mathbb{C}^*$-equivariant perfect obstruction theory. If $i'\colon F' \hookrightarrow X'$ is a fixed substack and $c \in A^*_{\mathbb{C}^*}(X)$, let $f|_{F_i}\colon F_i \to F'$ be the restriction of f to each of the fixed substacks $F_i \subseteq f^{-1}(F')$. Then*

$$\sum_{F_i} f|_{F_i *} \frac{i^*_{F_i} c}{\epsilon_{\mathbb{C}^*}(F_i^{vir})} = \frac{i'^* f_* c}{\epsilon_{\mathbb{C}^*}(F'^{vir})},$$

where $i_{F_i}\colon F_i \to X$ and $\epsilon_{\mathbb{C}^*}(F^{vir})$ is the virtual equivariant Euler class of the "virtual" normal bundle F^{vir}.

Remark. The conditions in the theorem are satisfied for the Kontsevich–Manin spaces $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^m, d)$ of stable maps, and $\epsilon_{\mathbb{C}^*}(F^{\mathrm{vir}})$ can be explicitly computed in terms of ψ and λ-classes [2] (see also [1]).

We define a $\mathbb{C}^*$-action on $\mathbb{P}^1$ by $a \cdot [x : y] = [x : ay]$ for $a \in \mathbb{C}^*$ and $[x : y] \in \mathbb{P}^1$. There are two fixed points, 0 and ∞, and the torus acts with weight 1 on the tangent space at 0 and -1 on the tangent space at ∞. This $\mathbb{C}^*$-action induces $\mathbb{C}^*$-actions on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$, and we shall consider the trivial $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{g,n}$.

3. Proof of Theorem 1

We use virtual localization on the function $f\colon \overline{\mathcal{M}}_{g,3}(\mathbb{P}^1, 1) \longrightarrow \overline{\mathcal{M}}_{g,3} \times (\mathbb{P}^1)^3$ defined by sending a map $g\colon (C, p_1, p_2, p_3) \to \mathbb{P}^1$ to the point

$$((C_{\mathrm{stab}}, p_1, p_2, p_3), g(p_1), g(p_2), g(p_3)).$$

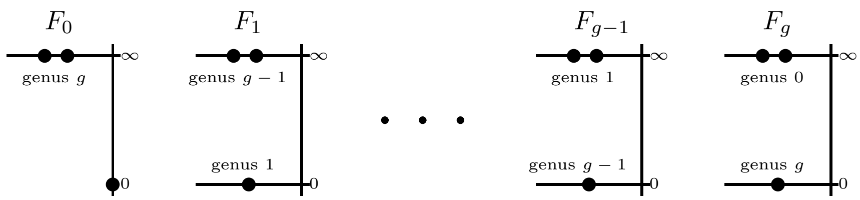
Consider the fixed locus

$$F' := \overline{\mathcal{M}}_{g,3} \times \{0\} \times \{\infty\} \times \{\infty\} \hookrightarrow \overline{\mathcal{M}}_{g,3} \times (\mathbb{P}^1)^3$$

and apply the virtual localization theorem with $c = [1]^{\mathrm{vir}}$ to obtain

$$\sum_{F_i} (f|_{F_i})_* \frac{[1]^{\mathrm{vir}}}{\epsilon_{\mathbb{C}^*}(F_i^{\mathrm{vir}})} = \frac{i'^* f_* [1]^{\mathrm{vir}}}{t(-t)(-t)}.$$

There are $g+1$ fixed loci mapping to F'. One fixed locus has a marked point mapping to 0 and a curve in $\overline{\mathcal{M}}_{g,3}$ mapping to ∞. We shall denote it by F_0. Then there are g fixed loci which have a curve in $\overline{\mathcal{M}}_{h,2}$ mapping to 0 and a curve in $\overline{\mathcal{M}}_{g-h,3}$ mapping to ∞ (with $1 \leq h \leq g$). We shall denote these fixed loci by F_h. Note that $F_0 \simeq \overline{\mathcal{M}}_{g,3}$ and $F_h \simeq \overline{\mathcal{M}}_{h,2} \times \overline{\mathcal{M}}_{g-h,3}$ $(1 \leq h \leq g)$.



Since $i'^*f_*[1]^{\text{vir}}$ is a polynomial in t, the sum of the contributions from the coefficient of t^{-4} on each fixed locus is 0. Call this contribution a_{-4}. We shall calculate it one fixed locus at the time, denoting by ψ_0 and ψ_∞ the ψ classes at the points where the curves are attached.

- For F_0, we obtain
$$\frac{[1]^{\text{vir}}}{\epsilon_{\mathbb{C}^*}(F_0^{\text{vir}})} = \frac{1}{t} \cdot (-1)^g \frac{t^g + \lambda_1 t^{g-1} + \cdots + \lambda_g}{-t(-t-\psi_\infty)},$$
and the coefficient of t^{-4} is $-\Lambda_{g,g+1}$ (the natural ψ class here is ψ_1).
- For F_h $(1 \leq h \leq g)$, we obtain
$$\frac{[1]^{\text{vir}}}{\epsilon_{\mathbb{C}^*}(F_h^{\text{vir}})} = \frac{t^h - \lambda_1^0 t^{h-1} + \cdots + (-1)^h\lambda_h^0}{t(t-\psi_0)} \cdot (-1)^{g-h} \frac{t^{g-h} + \lambda_1^\infty t^{g-h-1} + \cdots + \lambda_{g-h}^\infty}{-t(-t-\psi_\infty)},$$
and the coefficient of t^{-4} is[1]
$$c'_h := \sum_{i=0}^{g} (-1)^{h+i} \Lambda_{h,i}\Lambda_{g-h,g-i}.$$
This is a class of codimension g in $\overline{\mathcal{M}}_{h,2} \times \overline{\mathcal{M}}_{g-h,3}$ which maps to the codimension $g+1$ class $\iota_{h*}(c'_h)$ in $\overline{\mathcal{M}}_{g,3}$ under $(f|_{F_h})_*$.

To summarize, we obtain that
$$-\Lambda_{g,g+1} + \sum_{h=1}^{g} \iota_{h*}(c'_h) = 0$$
in $\overline{\mathcal{M}}_{g,3}$. We now multiply by ψ_3 and push forward to $\overline{\mathcal{M}}_{g,2}$.

- If $h=0$, we obtain $-2g\Lambda_{g,g+1}$ in $\overline{\mathcal{M}}_{g,2}$.
- If $1 \leq h < g$, note that, since the third marked point is on the curve at ∞, we are really multiplying by ψ_3 in $\overline{\mathcal{M}}_{g-h,3}$ and pushing forward to $\overline{\mathcal{M}}_{g-h,2}$. We therefore obtain, by dilaton, the class $2(g-h)\iota_{h*}(c'_h)$, which is a class of codimension $g+1$ in $\overline{\mathcal{M}}_{g,2}$.
- If $h=g$, then $\psi_3 = 0$ because it is a descendent at a marked point of a genus 0 curve with 3 markings (the curve mapping to ∞).

We now push forward via the map that forgets the second marked point.

- If $h=0$, we obtain, by string, $-2g\Lambda_{g,g}$.

[1]Note that c'_h is the summation (with the appropriate sign) of all possible products of codimension g of a class on the curve mapping to 0 with a class on the curve mapping to ∞.

- If $1 \leq h < g$, we obtain, by string, the class $2(g-h)\iota_{h*}(c_h)$, where c_h is just c'_h with every power of ψ_∞ lowered by 1, i.e.,

$$c_h := \sum_{i=0}^{g-1} (-1)^{h+i} \Lambda_{h,i} \Lambda_{g-h,g-1-i}.$$

Putting it all together, we obtain that

$$-2g\Lambda_{g,g} + \sum_{h=1}^{g} 2(g-h)\iota_{h*}(c_h) = 0,$$

from which we can derive the formula of Theorem 1. □

Remarks. (I) By taking the coefficient of t^{-3-j} with $j > 1$, it is possible to find a similar formula for $\Lambda_{g,g+j-1}$ in terms of classes supported on boundary strata. As mentioned at the end of the introduction, we can also calculate it by starting with the formula for $\Lambda_{g,g}$ and multiplying by powers of ψ.

(II) In [3], Graber and Vakil proved that a codimension g class in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ can be written as a sum of classes supported on boundary strata with at least one genus 0 component. By induction on g, it is easy to see that this is the case for our c_h classes.

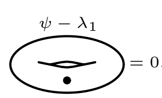
(III) Using the same function f as above but with the fixed locus $\overline{\mathcal{M}}_{g,2} \times \{0\}^2 \times \{\infty\}$ instead of $\overline{\mathcal{M}}_{g,2} \times \{0\} \times \{\infty\}^2$, it is possible to obtain the following tautological relation on $\overline{\mathcal{M}}_{g,1}$:

$$\sum_{h=1}^{g-1} (2h)\iota_{h*}(c_h) + (2g)\pi_* \left(\psi_2^{g+1} - \lambda_1 \psi_2^g + \cdots + (-1)^g \lambda_g \psi_2 \right) = 0.$$

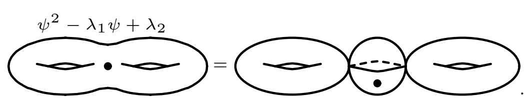
4. Explicit formulas for low genus

The formula of Theorem 1 can be simplified recursively, and we have calculated the answer for low values of g. Note that these formulas were already known for $g = 1$ and $g = 2$, but they were unknown for higher g's.

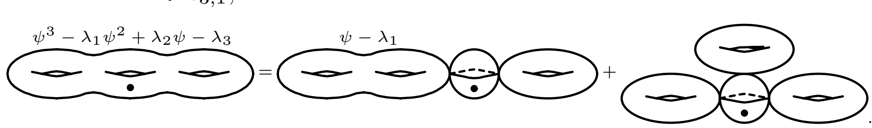
Genus 1: In $\overline{\mathcal{M}}_{1,1}$,



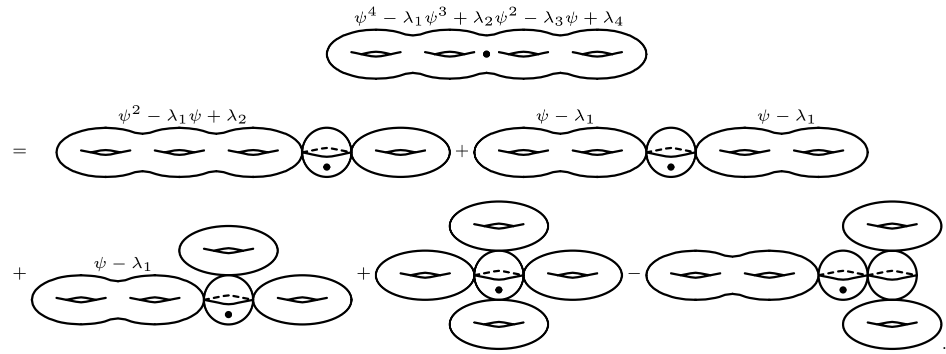
Genus 2: In $\overline{\mathcal{M}}_{2,1}$,



Genus 3: In $\overline{\mathcal{M}}_{3,1}$,



Genus 4: In $\overline{\mathcal{M}}_{4,1}$,



We also have calculated the formula for $\Lambda_{5,5}$ in $\overline{\mathcal{M}}_{5,1}$. We do not write it here because it was calculated via a (possibly incorrect) computer program and because it is rather long. Note that non-integer coefficients do appear in genus 5.

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References

[1] C. Faber, R. Pandharipande, Relative maps and tautological classes, *J. Eur. Math. Soc. (JEMS)* 7 (2005), no. 1, 13–49. MR2120989 (2005m:14046)

[2] T. Graber, R. Pandharipande, Localization of virtual classes, *Invent. Math.* 135 (1999), no. 2, pp. 487–518. MR1666787 (2000h:14005)

[3] T. Graber, R. Vakil, Relative virtual localization and vanishing of tautological classes on moduli spaces of curves, *Duke Math. J.* 130 (2005), no. 1, 1–37. MR2176546 (2006j:14035)

[4] D. Mumford, Towards an Enumerative Geometry of the Moduli Space of Curves, in *Arithmetic and geometry, Vol. II*, Progr. Math. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328. MR717614 (85j:14046)

Department of Mathematics, St. Vincent College, 300 Fraser Purchase Road, Latrobe, Pennsylvania 15650-2690

E-mail address: daniele.arcara@email.stvincent.edu

Department of Mathematics, Nagoya University Furocho, Chikusaku, Nagoya 464-8602, Japan

E-mail address: fumi@math.utah.edu