RECURSIVE FORMULA FOR $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ IN $\overline{M}_{g,1}$

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Abstract. Mumford proved that $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g = 0$ in the Chow ring of $\overline{M}_{g,1}$. We find an explicit recursive formula for $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ in the tautological ring of $\overline{M}_{g,1}$ as a combination of classes supported on boundary strata.

1. Introduction

Mumford proved in [4] that $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g = 0$ in the Chow ring of $\overline{M}_{g,1}$. Moreover, he showed that this class is supported on the boundary strata with a marked genus 0 component. Graber and Vakil proved in [3] that every codimension $g$ class in the tautological ring of $\overline{M}_{g,1}$ is supported on the boundary strata with at least one genus 0 component.

We complement these results by finding an explicit recursive formula for $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ in the tautological ring of $\overline{M}_{g,1}$ as a combination of classes supported on boundary strata. It is clear from the formula being recursive that all the boundary strata have a genus 0 component in them, but it is not obvious from the formula that the marked point must be on a genus 0 component. We simplified the formula for $g < 5$ in Section 4 and checked that this is the case.

To make the statement of the Main Theorem easier to understand, let us introduce some notation. We shall denote the class $\psi^g - \lambda_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ by $\Lambda_{g,g}$. More generally, whenever a moduli space $\overline{M}_{g,n}$ has a natural choice of a special marked point (for example, it is the marked point that is glued via a gluing map), we shall define

$$\Lambda_{g,i} := \psi^i - \lambda_1 \psi^{i-1} + \cdots + (-1)^g \lambda_g \psi^{i-g},$$

where the $\psi$ class is at the special marked point, and we shall use the usual convention that $\psi^{-1} = 0$.

Theorem 1. In the tautological ring of $\overline{M}_{g,1}$,

$$\Lambda_{g,g} = \sum_{h=1}^{g} \left( 1 - \frac{h}{g} \right) \iota_{h}*(c_{h}),$$

where

$$c_{h} := \sum_{i=0}^{g-1} (-1)^{h+i} \Lambda_{h,i} \Lambda_{g-h,g-1-i}.$$
and $\iota_h$ is the natural boundary map
\[ \iota_h : \overline{M}_{h,2} \times \overline{M}_{g-h,1} \to \overline{M}_{g,1}. \]

This formula is actually the first step of an algorithm which calculates each of the classes $\psi^g$, $\lambda_1 \psi^{g-1}$, \ldots, $\lambda_g$ in terms of classes supported on boundary strata. We want to single out the class $\Lambda_{g,g}$, though, because it is the only class we found so far in the tautological ring of $\overline{M}_{g,1}$ which has a nice recursive formula and can therefore be easily calculated.

Note that the formula is recursive because, once we obtain a formula for $\Lambda_{g,g}$, we can

- obtain a formula for $\Lambda_{g,g+1}$ (and then, similarly, for $\Lambda_{g,i}$ for $i > g$) by multiplying the formula obtained by $\psi$ and simplifying;
- obtain a formula for $\Lambda_{g,i}$ ($i \geq g$) on $\overline{M}_{g,2}$ by pulling back the formula obtained on $\overline{M}_{g,1}$ and simplifying using the pull-back formulas for $\psi$ and $\lambda$ classes.

2. VIRTUAL LOCALIZATION

The main tool we use to prove our theorems is the virtual localization theorem by Graber and Pandharipande [2].

**Theorem** (Virtual localization theorem). Suppose $f : X \to X'$ is a $\mathbb{C}^*$-equivariant map of proper Deligne–Mumford quotient stacks with a $\mathbb{C}^*$-equivariant perfect obstruction theory. If $i' : F' \to X'$ is a fixed substack and $c \in A^*_c(X)$, let $f|_{F_i} : F_i \to F'$ be the restriction of $f$ to each of the fixed substacks $F_i \subseteq f^{-1}(F')$. Then
\[ \sum_{F_i} f|_{F_i}* \frac{i'_* c}{\epsilon_{C^*}(F'_{\text{vir}})} = i^* f_* c \frac{[1]_{\text{vir}}}{\epsilon_{C^*}(F'_{\text{vir}})}, \]

where $i_{F_i} : F_i \to X$ and $\epsilon_{C^*}(F'_{\text{vir}})$ is the virtual equivariant Euler class of the “virtual” normal bundle $F'_{\text{vir}}$.

**Remark.** The conditions in the theorem are satisfied for the Kontsevich–Manin spaces $\overline{M}_{g,n}(\mathbb{P}^1, d)$ of stable maps, and $\epsilon_{C^*}(F'_{\text{vir}})$ can be explicitly computed in terms of $\psi$ and $\lambda$-classes [2] (see also [1]).

We define a $\mathbb{C}^*$-action on $\mathbb{P}^1$ by $a \cdot [x : y] = [x : ay]$ for $a \in \mathbb{C}^*$ and $[x : y] \in \mathbb{P}^1$. There are two fixed points, 0 and $\infty$, and the torus acts with weight 1 on the tangent space at 0 and $-1$ on the tangent space at $\infty$. This $\mathbb{C}^*$-action induces $\mathbb{C}^*$-actions on $\overline{M}_{g,n}(\mathbb{P}^1, d)$, and we shall consider the trivial $\mathbb{C}^*$-action on $\overline{M}_{g,n}$.

3. PROOF OF THEOREM [1]

We use virtual localization on the function $f : \overline{M}_{g,3}(\mathbb{P}^1, 1) \to \overline{M}_{g,3} \times (\mathbb{P}^1)^3$ defined by sending a map $g : (C, p_1, p_2, p_3) \to \mathbb{P}^1$ to the point
\[ (\{[C, p_1, p_2, p_3] \to \mathbb{P}^1) \to [C_{\text{stab}}, p_1, p_2, p_3], g(p_1), g(p_2), g(p_3)). \]

Consider the fixed locus
\[ F' := \overline{M}_{g,3} \times \{0\} \times \{\infty\} \times \{\infty\} \subseteq \overline{M}_{g,3} \times (\mathbb{P}^1)^3 \]
and apply the virtual localization theorem with $c = [1]_{\text{vir}}$ to obtain
\[ \sum_{F_i} (f|_{F_i})* \frac{[1]_{\text{vir}}}{\epsilon_{C^*}(F'_{\text{vir}})} = i^* f_* [1]_{\text{vir}} \frac{1}{t(-t)(-t)}. \]
There are $g + 1$ fixed loci mapping to $F'$. One fixed locus has a marked point mapping to 0 and a curve in $\overline{M}_{g,2}$ mapping to $\infty$. We shall denote it by $F_0$. Then there are $g$ fixed loci which have a curve in $\overline{M}_{h,2}$ mapping to 0 and a curve in $\overline{M}_{g-h,3}$ mapping to $\infty$ (with $1 \leq h \leq g$). We shall denote these fixed loci by $F_h$. Note that $F_0 \simeq \overline{M}_{g,3}$ and $F_h \simeq \overline{M}_{h,2} \times \overline{M}_{g-h,3}$ ($1 \leq h \leq g$).

Since $\psi^*f_*[1]_{\text{vir}}$ is a polynomial in $t$, the sum of the contributions from the coefficient of $t^{-4}$ on each fixed locus is 0. Call this contribution $a_{-4}$. We shall calculate it one fixed locus at the time, denoting by $\psi_0$ and $\psi_\infty$ the $\psi$ classes at the points where the curves are attached.

- For $F_0$, we obtain
  \[
  \frac{[1]_{\text{vir}}}{\epsilon_{C^*}(F_0^\text{vir})} = \frac{1}{t} \cdot \frac{(-1)^g t^g + \lambda_1 t^{g-1} + \cdots + \lambda_g}{-t(-t - \psi_\infty)},
  \]
  and the coefficient of $t^{-4}$ is $-\Lambda_{g,g+1}$ (the natural $\psi$ class here is $\psi_1$).

- For $F_h$ ($1 \leq h \leq g$), we obtain
  \[
  \frac{[1]_{\text{vir}}}{\epsilon_{C^*}(F_h^\text{vir})} = \frac{t^h - \lambda_1^0 t^{h-1} + \cdots + (-1)^h \lambda_h^0}{t(t - \psi_0)} \cdot \frac{(-1)^{g-h} t^{g-h} + \lambda_1^0 t^{g-h-1} + \cdots + \lambda_g^{g-h}}{-t(-t - \psi_\infty)},
  \]
  and the coefficient of $t^{-4}$ is $[1]_{\text{vir}}$.

  \[
  \epsilon'_h := \sum_{i=0}^{g} (-1)^{h+i} \Lambda_{h,i} \Lambda_{g-h,g-i}.
  \]

This is a class of codimension $g$ in $\overline{M}_{h,2} \times \overline{M}_{g-h,3}$ which maps to the codimension $g + 1$ class $\iota_{h,*}(\epsilon'_h)$ in $\overline{M}_{g,3}$ under $(f|_{P_h})_*$.

To summarize, we obtain that
\[
-\Lambda_{g,g+1} \sum_{h=1}^{g} \iota_{h,*}(\epsilon'_h) = 0
\]
in $\overline{M}_{g,3}$. We now multiply by $\psi_3$ and push forward to $\overline{M}_{g,2}$.

- If $h = 0$, we obtain $-2g\Lambda_{g,g+1}$ in $\overline{M}_{g,2}$.
- If $1 \leq h < g$, note that, since the third marked point is on the curve at $\infty$, we are really multiplying by $\psi_3$ in $\overline{M}_{g-h,3}$ and pushing forward to $\overline{M}_{g-h,2}$. We therefore obtain, by dilaton, the class $2(g-h)\iota_{h,*}(\epsilon'_h)$, which is a class of codimension $g + 1$ in $\overline{M}_{g,2}$.
- If $h = g$, then $\psi_3 = 0$ because it is a descendent at a marked point of a genus 0 curve with 3 markings (the curve mapping to $\infty$).

We now push forward via the map that forgets the second marked point.

- If $h = 0$, we obtain, by string, $-2g\Lambda_{g,g}$.

\footnote{Note that $\epsilon'_h$ is the summation (with the appropriate sign) of all possible products of co-dimension $g$ of a class on the curve mapping to 0 with a class on the curve mapping to $\infty$.}
• If $1 \leq h < g$, we obtain, by string, the class $2(g-h)\iota_h(c_h)$, where $c_h$ is just $c'_h$ with every power of $\psi_\infty$ lowered by 1, i.e.,

$$c_h := \sum_{i=0}^{g-1} (-1)^{h+i}\Lambda_{h,i}A_{g-h,g-1-i}.$$ 

Putting it all together, we obtain that

$$-2g\Lambda_{g,g} + \sum_{h=1}^{g} 2(g-h)\iota_h(c_h) = 0,$$

from which we can derive the formula of Theorem 1. \[\square\]

Remarks. (I) By taking the coefficient of $t^{-3-j}$ with $j > 1$, it is possible to find a similar formula for $\Lambda_{g,g+j-1}$ in terms of classes supported on boundary strata. As mentioned at the end of the introduction, we can also calculate it by starting with the formula for $\Lambda_{g,g}$ and multiplying by powers of $\psi$.

(II) In [3], Graber and Vakil proved that a codimension $g$ class in the tautological ring of $M_{g,1}$ can be written as a sum of classes supported on boundary strata with at least one genus 0 component. By induction on $g$, it is easy to see that this is the case for our $c_h$ classes.

(III) Using the same function $f$ as above but with the fixed locus $M_{g,2} \times \{0\} \times \{\infty\}$ instead of $M_{g,2} \times \{0\} \times \{\infty\}^2$, it is possible to obtain the following tautological relation on $M_{g,1}$:

$$\sum_{h=1}^{g-1} (2h)\iota_h(c_h) + (2g)\pi_*(\psi_{g+1} - \lambda_1\psi_g + \cdots + (-1)^g\lambda_g\psi_2) = 0.$$ 

4. Explicit formulas for low genus

The formula of Theorem 1 can be simplified recursively, and we have calculated the answer for low values of $g$. Note that these formulas were already known for $g = 1$ and $g = 2$, but they were unknown for higher $g$'s.

Genus 1: In $M_{1,1}$,

$$\psi - \lambda_1 = 0.$$ 

Genus 2: In $M_{2,1}$,

$$\psi^2 - \lambda_1\psi + \lambda_2 = 0.$$ 

Genus 3: In $M_{3,1}$,

$$\psi^3 - \lambda_1\psi^2 + \lambda_2\psi - \lambda_3 = 0.$$ 

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Genus 4: In $\overline{M}_{4,1}$.

We also have calculated the formula for $\Lambda_{5,5}$ in $\overline{M}_{5,1}$. We do not write it here because it was calculated via a (possibly incorrect) computer program and because it is rather long. Note that non-integer coefficients do appear in genus 5.

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