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ON THE SOLVABILITY OF VECTOR FIELDS WITH REAL LINEAR COEFFICIENTS

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ABSTRACT. The following result is proved: for a vector field with real linear coefficients to be locally solvable in $\mathbb{R}^n$ it is necessary and sufficient that not all its orbits have a compact closure in the complement of the critical set of the vector field.

1. Introduction

In this paper the following notation is used: $\mathcal{C}_c^\infty(\Omega)$ for the space of test functions in an open subset Ω of $\mathbb{R}^n$ and $\mathcal{D}'(\Omega)$ for its dual, the space of distributions in Ω. We write $\partial_{x_j} = \frac{\partial}{\partial x_j}$ $(j = 1, ..., n)$; the upper sign $\top$ indicates *transpose* (of a matrix or of an operator in a distribution space). By a linear change of variables we always mean a *real* linear change of variables.

We use the following terminology:

Definition 1. *A smooth vector field L in an open subset Ω of $\mathbb{R}^n$ is said to be* **locally solvable** at a point $x^\circ \in \Omega$ if there exists an open neighborhood $U \subset \Omega$ of x° such that, to every $\varphi \in \mathcal{C}_c^\infty(U)$, there is $u \in \mathcal{D}'(\Omega)$ satisfying $Lu = \varphi$ in U.

Our purpose is to characterize those vector fields in $\mathbb{R}^n$ whose coefficients are real linear functionals, i.e., vector fields of the form

$$L = \sum_{j,k=1}^{n} a_{jk} x_j \partial_{x_k} \tag{1.1}$$

that are locally solvable at every point of $\mathbb{R}^n$ (this will turn out to be equivalent to local solvability at the origin). The characterization is the content of Theorem 1 below; it extends to higher dimensions the two-dimensional result in [Treves, 2009] (Theorem **10**). Related results in the context of hyperfunctions can be found in [Miwa, 1973].

We denote by $\mathfrak{m}(L)$ the complex matrix $(a_{jk})_{1\leq j,k\leq n}$; then $\operatorname{div} L = \operatorname{tr}\mathfrak{m}(L)$, the trace of the matrix $\mathfrak{m}(L)$. Nonsingular linear changes of the variables $x_1, ..., x_n$ are equivalent to conjugations $\mathfrak{m}(L) \to \Gamma^{-1}\mathfrak{m}(L)\Gamma$ by a matrix $\Gamma \in \mathbf{G}L(n, \mathbb{R})$.

The *critical points* of L are the points $x = (x_1, ..., x_n)$ in $\mathbb{R}^n$ such that $\mathfrak{m}(L)^\top \vec{x} = 0$. They form a vector subspace of $\mathbb{R}^n$, which will be denoted by $\boldsymbol{V}(L)$. We set

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$m = \operatorname{codim} \boldsymbol{V}(L)$; after a linear coordinate change we may assume that $\boldsymbol{V}(L) = \{x \in \mathbb{R}^n;\ x_1 = \cdots = x_m = 0\}$, in which case

$$L = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} x_j \partial_{x_k}. \tag{1.2}$$

The vector field (1.2) is invariant under translations in the variables $(x_{m+1}, ..., x_n)$, i.e., parallel to $\boldsymbol{V}(L)$; we have

$$L = \sum_{j=1}^{m} x_j L_j,\ L_j = \sum_{k=1}^{n} a_{jk} \partial_{x_k}. \tag{1.3}$$

Observe that the constant coefficients vector fields $L_1, ..., L_m$ must be linearly independent; if they were not, we would have $\operatorname{codim} \boldsymbol{V}(L) < m$.

When L has the expression (1.2) we can identify $\mathbb{R}^n/\boldsymbol{V}(L)$ to $\mathbb{R}^m$ and the push-forward of L under the quotient map $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\boldsymbol{V}(L)$ to the vector field in $\mathbb{R}^m$,

$$L^\circ = \sum_{j,k=1}^{m} a_{jk} x_j \frac{\partial}{\partial x_k}. \tag{1.4}$$

We point out that the spectrum of $\mathfrak{m}(L)$ is equal to that of $\mathfrak{m}(L^\circ)$ to which the eigenvalue zero has been adjoined, with the multiplicity $n-m$; in particular, $\operatorname{div} L = \operatorname{div} L^\circ$.

We note that L defines a foliation of $\mathbb{R}^n$ whose leaves are either single points, precisely the critical points of L [i.e., the points of $\boldsymbol{V}(L)$] or "true" curves contained in $\mathbb{R}^n \backslash \boldsymbol{V}(L)$. More precisely, the latter are *immersed*, connected, one-dimensional analytic submanifolds without self-intersections whose tangent line is spanned by L at each one of their points (and are maximal for these properties). We shall refer to the leaves of the foliation defined by L as **L-orbits.**

We shall avail ourselves of the following:

Lemma 1. *The following properties of the vector field* (1.2) *are equivalent:*

(1) *every L-orbit in $\mathbb{R}^n \backslash \boldsymbol{V}(L)$ has a compact closure in $\mathbb{R}^n \backslash \boldsymbol{V}(L)$;*
(2) *every eigenvalue of $\mathfrak{m}(L^\circ)$ is purely imaginary, and the nilpotent part of $\mathfrak{m}(L^\circ)$ vanishes.*

Proof. Let $\chi = \xi + i\eta$ $(\xi, \eta \in \mathbb{R})$ be an eigenvalue of $\mathfrak{m}(L^\circ)$. If $\eta = 0$ there is a linear change of the variables $x_1, ..., x_m$ such that $a_{11} = \xi$ and $a_{1k} = 0$ if $k = 2, ...m$. In the new coordinates,

$$L = x_1 \left(\xi \frac{\partial}{\partial x_1} + \sum_{k=m+1}^{n} a_{1k} \frac{\partial}{\partial x_k} \right) + \sum_{j=2}^{m} \sum_{k=1}^{n} a_{jk} x_j \frac{\partial}{\partial x_k}. \tag{1.5}$$

The vector field (1.5) is tangent to the linear subspace $x_2 = \cdots = x_m = 0$. In the half-subspaces $x_1 \lessgtr 0$ the L-orbits coincide with those of the vector field

$$\xi \frac{\partial}{\partial x_1} + \sum_{k=m+1}^{n} a_{1k} \frac{\partial}{\partial x_k},$$

which are obviously not compact. This contradicts (1).

Now suppose $\xi\eta \neq 0$ and let $\vec{u} = \vec{v} + i\vec{w} \in \mathbb{C}^n$ be such that $\mathfrak{m}(L^\circ)\,\vec{u} = (\xi + i\eta)\,\vec{u}$. We get

$$\mathfrak{m}(L^\circ)\,\vec{v} = \xi\vec{v} - \eta\vec{w},\ \mathfrak{m}(L^\circ)\,\vec{w} = \eta\vec{v} + \xi\vec{w}.$$

Thus the span of $\vec{v}$, $\vec{w}$ is preserved by $\mathfrak{m}(L^\circ)$. We have necessarily $\vec{v} \wedge \vec{w} \neq 0$; otherwise $\mathfrak{m}(L^\circ)$ would have a real eigenvalue, a case already taken care of. A linear transformation enables us to assume that $\vec{v} = (1, 0, ..., 0)$ and $\vec{w} = (0, 1, 0, ..., 0)$ and

$$\mathfrak{m}(L^\circ) = \begin{pmatrix} \xi & -\eta & 0 \\ \eta & \xi & 0 \\ * & * & * \end{pmatrix}.$$

The change of variables

$$\tilde{x}_j = x_j - \frac{1}{\xi^2 + \eta^2} \left((\xi a_{1j} + \eta a_{2j}) x_1 + (-\eta a_{1j} + \xi a_{2j}) x_2 \right), \ j = m+1, ..., n,$$

transforms

$$\begin{aligned} L_1 &= \xi \partial_{x_1} - \eta \partial_{x_2} + \sum_{k=m+1}^{n} a_{1k} \partial_{x_k}, \\ L_2 &= \eta \partial_{x_1} + \xi \partial_{x_2} + \sum_{k=m+1}^{n} a_{2k} \partial_{x_k} \end{aligned}$$

into

$$L_1 = \xi \partial_{x_1} - \eta \partial_{x_2}, \ L_2 = \eta \partial_{x_1} + \xi \partial_{x_2}.$$

As a result of this and of (1.3) we have (after deleting the tildes)

$$L = x_1 (\xi \partial_{x_1} - \eta \partial_{x_2}) + x_2 (\eta \partial_{x_1} + \xi \partial_{x_2}) + \sum_{j=3}^{m} x_j L_j. \tag{1.6}$$

When L has the expression (1.6) L is tangent to the deleted affine subspace $\dot{\boldsymbol{V}}$ defined by $x_1^2 + x_2^2 \neq 0$, $x_3 = \cdots = x_m = 0$. If we use polar coordinates in this subspace we see that

$$L|_{\dot{\boldsymbol{V}}} = \xi r \partial_r - \eta \partial_\theta.$$

The orbits of L in $\dot{\boldsymbol{V}}$ being the spirals $r = r^\circ \exp\left(-\eta^{-1} \xi \theta\right)$, no orbit of L in $\dot{\boldsymbol{V}}$ has a compact closure, again contradicting (1).

Now suppose that $\xi = 0$, $\eta \neq 0$, but that the nilpotent part of $\mathfrak{m}(L^\circ)$ does not vanish. If $i\eta$ $(0 \neq \eta \in \mathbb{R})$ is an eigenvalue of $\mathfrak{m}(L^\circ)$, the same is true of $-i\eta$. There is a linear change of the variables $x_1, ..., x_m$ which puts $\mathfrak{m}(L^\circ)$ into the form

$$\overbrace{\begin{pmatrix} 0 & \eta & 0 & 0 & 0 & 0 \\ -\eta & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \eta & 0 & 0 \\ 0 & 1 & -\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}}^{2p}.$$

The expression of L in the new coordinates will be

$$\begin{aligned} L &= \eta (x_1 \partial_{x_2} - x_2 \partial_{x_1} + x_3 \partial_{x_4} - x_4 \partial_{x_3}) \\ &\quad + x_3 \partial_{x_1} + x_4 \partial_{x_2} + \sum_{j=5}^{m} \sum_{k=m+1}^{n} a_{jk} x_j \partial_{x_k}. \end{aligned} \tag{1.7}$$

When L has the expression (1.7), L is tangent to the deleted affine subspace $\overset{\bullet}{\boldsymbol{V}}$ defined by $x_1^2+x_2^2+x_3^2+x_4^2\neq 0$, $x_5=\cdots=x_n=0$. We see that

$$L|_{\overset{\bullet}{\boldsymbol{V}}}=\eta\left(x_1\partial_{x_2}-x_2\partial_{x_1}+x_3\partial_{x_4}-x_4\partial_{x_3}\right)+x_3\partial_{x_1}+x_4\partial_{x_2}.$$

It is checked straightaway that L is tangent to the curve in $\overset{\bullet}{\boldsymbol{V}}$ defined by the parametric equations

$$x_1=\eta^{-1}t\cos t,\ x_2=\eta^{-1}t\sin t,\ x_3=\cos t,\ x_4=\sin t,\ t\in\mathbb{R},$$

whose closure in $\overset{\bullet}{\boldsymbol{V}}$ is noncompact, contradicting (1). We have thus proved that the negation of (2) implies that of (1).

Now suppose (2) is true. There is a linear change of the variables $x_1,...,x_m$ that puts L° into the form

$$L^\circ=\sum_{j=1}^{p}\eta_j\left(x_{2j-1}\partial_{x_{2j}}-x_{2j}\partial_{x_{2j-1}}\right),\tag{1.8}$$

and therefore L into the form

$$L=\sum_{j=1}^{p}\eta_j\left(x_{2j-1}L_{2j}-x_{2j}L_{2j-1}\right),$$

where

$$L_k=\partial_{x_k}+\sum_{\ell=2p+1}^{n}b_{k\ell}\partial_\ell,\ k=1,...,2p.$$

We carry out the change of variables

$$\begin{aligned}\tilde{x}_k &= x_k,\ k=1,...,2p,\\ \tilde{x}_\ell &= x_\ell-\sum_{k=1}^{2p}b_{k\ell}x_k,\ \ell=2p+1,...,n.\end{aligned}$$

In the new coordinates, $L_k=\partial_{\tilde{x}_k}$ which, after deletion of the tildes, puts L itself into the form (1.8). If we write $x_{2j-1}=r_j\cos\theta_j$, $x_{2j}=r_j\sin\theta_j$, we get

$$L=\sum_{j=1}^{p}\eta_j\partial_{\theta_j}.\tag{1.9}$$

We see that the L-orbit through a point $x^\circ\in\mathbb{R}^n\backslash\boldsymbol{V}(L)$ is a geodesic of the torus

$$\widetilde{\mathbb{T}}^{2p}(x^\circ)=\Big\{x\in\mathbb{R}^n\backslash\boldsymbol{V}(L);\ x_j^2+x_{p+j}^2=\left(x_j^\circ\right)^2+\left(x_{p+j}^\circ\right)^2,\ j=1,...,p,\ x_k=x_k^\circ,\ k=2p+1,...,n\Big\}.\tag{1.10}$$

As a consequence, the closure of every L-orbit is compact. □

Corollary 1. *If* $\operatorname{div}L\neq 0$*, then there is an* L*-orbit in* $\mathbb{R}^n\backslash\boldsymbol{V}(L)$ *whose closure is not a compact subset of* $\mathbb{R}^n\backslash\boldsymbol{V}(L)$*.*

Proof. If $\operatorname{tr}\mathfrak{m}(L)\neq 0$ there is an eigenvalue of $\mathfrak{m}(L)$ whose real part does not vanish. □

Remark 1. Property (2) in Lemma 1 is equivalent to the following property:

- The one-parameter subgroup $\mathbb{R}\ni t\longrightarrow\exp t\mathfrak{m}(L)\in\mathbf{G}L(n,\mathbb{R})$ has a compact closure.

2. Statement and proof of theorem

2.1. **Statement of theorem and first observations.**

Theorem 1. *The following properties of the vector field* (1.1) *are equivalent:*

- **(a)** *L is not locally solvable at any point of $\boldsymbol{V}(L)$.*
- **(b)** *The closure of every L-orbit in $\mathbb{R}^n \backslash \boldsymbol{V}(L)$ is a compact subset of $\mathbb{R}^n \backslash \boldsymbol{V}(L)$.*
- **(c)** *There are coordinates $x_1, ..., x_n$ in $\mathbb{R}^n$ such that the L-orbit of an arbitrary point $x^\circ \in \mathbb{R}^n \backslash \boldsymbol{V}(L)$ is a geodesic of the torus* (1.10).

Remark 2. Owing to the invariance of L under translations in the $\boldsymbol{V}(L)$ directions, Property **(a)** could have as well been stated as

- **(a′)** L is not locally solvable at the origin.

Remark 3. Property **(c)** requires $m = \dim \mathbb{R}^n / \boldsymbol{V}(L) = 2p$ $(0 \neq p \in \mathbb{Z}_+)$. Lemma 1 states that **(b)** is equivalent to the property that the nonzero eigenvalues of $\mathfrak{m}(L)$ are purely imaginary numbers $\pm i\eta_j$ $(j = 1, ..., p)$ and the nilpotent part of $\mathfrak{m}(L^\circ)$ vanishes. The real numbers η_j determine which geodesic in the torus (1.10) is the L-orbit through x°.

That **(c)** implies **(b)** is trivial; the converse is part of Lemma 1. Proving that **(a)** and **(b)** are equivalent will be a matter of settling a few special cases. The corresponding results will be stated as independent propositions. The proofs of these propositions will make repeated use of the next two classical lemmas. In what follows $(\cdot, \cdot)$ and $\|\cdot\|$ stand for the inner product and the norm in $L^2(\Omega)$, respectively.

Lemma 2. *If X is a vector field with constant coefficients, not all vanishing, and if h is a complex-valued analytic function in an open subset Ω of $\mathbb{R}^n$, then the vector field $h(x)X$ is locally solvable in Ω.*

Proof. An arbitrary distribution $f \in \mathcal{D}'(\Omega)$ can be divided by the analytic function h ([Lojasiewicz, 1965]). Then it suffices to solve the equation $Xu = h^{-1}f$. □

Lemma 3. *Let Ω be an open subset of $\mathbb{R}^n$. Suppose there is a linear partial differential operator with constant coefficients $P(\partial)$ such that, for some $C > 0$ and all $\varphi \in \mathcal{C}_c^\infty(\Omega)$,*

$$\|\varphi\| \leq C \left\| P(\partial) L^\top \varphi \right\|. \tag{2.1}$$

Then $L\mathcal{D}'(\Omega) \supset L^2(\Omega)$.

Proof. The estimate (2.1) entails that the linear map $P(\partial) L^\top \mathcal{C}_c^\infty(\Omega) \ni P(\partial) L^\top \varphi \longrightarrow \varphi \in \mathcal{C}_c^\infty(\Omega)$ extends as a bounded linear operator $E : L^2(\Omega) \hookleftarrow$ having the property that $EP(\partial) L^\top \varphi = \varphi$ for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$. We derive that $LP(-\partial) E^\top f = f$ for every $f \in L^2(\Omega)$. □

We begin the proof proper of Theorem 1 with the following two simple observations.

The case $m = 1$ is a direct consequence of Lemma 2 since, in this case, $L = x_1 L_1$ with L_1 having constant coefficients not all vanishing. The L-orbits are the same as the orbits of X (which are straight lines) possibly "interrupted" by the hyperplane $x_1 = 0$.

In the remainder of the proof we limit our attention to the cases $2 \leq m \leq n$.

Proposition 1. *Let L be the vector field* (1.2)*. If* $\operatorname{div} L \neq 0$*, then there is a bounded linear operator E on $L^2(\mathbb{R}^n)$ such that $LEf = f$ for all $f \in L^2(\mathbb{R}^n)$.*

Proof. We have, whatever $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$2\operatorname{Re}\left(L^\top \varphi, \varphi\right)_{L^2} = -(\operatorname{div} L)\|\varphi\|_{L^2}.$$

The claim follows then from the Cauchy-Schwarz inequality and Lemma 3. □

In the sequel we restrict our attention to divergence-free vector fields (1.1), in which case $L^\top = -L$.

2.2. **The matrix $\mathfrak{m}(L^\circ)$ has at least one real eigenvalue ξ.** In this case we may assume that L has the form (1.5).

Proposition 2. *Let L be the vector field* (1.5)*. If* $\operatorname{div} L = 0$ *and if* $\mathfrak{m}(L^\circ)$ *has at least one real eigenvalue ξ (possibly equal to zero), then L is locally solvable in $\mathbb{R}^n$.*

Proof. We may assume that L is the vector field (1.5).

Case $\xi \neq 0$. We use the fact that $\partial_{x_1} L = (L+\xi)\partial_{x_1}$. Since $\operatorname{div} L = 0$ we have, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\operatorname{Re}\left((L+\xi)\varphi, \varphi\right)_{L^2} = \xi\|\varphi\|_{L^2},$$

implying

$$\operatorname{Re}(\partial_{x_1}^2 L\varphi, \varphi)_{L^2} = \operatorname{Re}\left(\partial_{x_1}(L+\xi)\partial_{x_1}\varphi, \varphi\right)_{L^2} = -\xi\|\partial_{x_1}\varphi\|_{L^2}.$$

To every number $T > 0$ there is $C_T > 0$ such that $\|\varphi\|_{L^2} \leq C_T\|\partial_{x_1}\varphi\|_{L^2}$ for every φ whose support is contained in the slab $(-T,T)\times\mathbb{R}^{n-1}$. The claim then follows from the Cauchy-Schwarz inequality and Lemma 3.

Case $\xi = 0$. In this case we have $L = x_1\sum_{k=m+1}^n a_{1k}\frac{\partial}{\partial x_k} + \sum_{j=2}^m\sum_{k=1}^n a_{jk}x_j\frac{\partial}{\partial x_k}$ according to (1.5). There is a linear change of the variables $x_{m+1}, \dots., x_n$ that transforms L into

$$x_1\partial_{x_{m+1}} + \sum_{j=2}^m x_j\left(\sum_{1\leq k\leq n} a_{jk}\partial_{x_k}\right). \tag{2.2}$$

Since $\operatorname{div} L = 0$ we see that

$$2\operatorname{Re}\left(L\varphi, \partial_{x_1}\partial_{x_{m+1}}\varphi\right)_{L^2} = -\left\|\partial_{x_{m+1}}\varphi\right\|_{L^2}^2$$

for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. We define $\Omega_T = \{x \in \mathbb{R}^n;\ |x_{m+1}| < T\}$ $(T > 0)$. There is $C_T > 0$ such that

$$\|\varphi\|_{L^2} \leq C_T\left\|\partial_{x_{m+1}}\varphi\right\|_{L^2}$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega_T)$. That the vector field (2.2) is locally solvable in $\mathbb{R}^n$ then ensues directly from the Cauchy-Schwarz inequality and Lemma 3. □

2.3. **The matrix $\mathfrak{m}(L^\circ)$ has no real eigenvalues.**

2.3.1. *Case I: The real part of some eigenvalue of $\mathfrak{m}(L^\circ)$ is different from zero.*

Proposition 3. *Let L be the vector field* (1.2). *If $\operatorname{div} L = 0$ and if $\mathfrak{m}(L^\circ)$ has at least one eigenvalue whose real part is different from zero, then L is locally solvable in $\mathbb{R}^n$.*

Proof. We may assume that L has the expression (1.6); it follows that, for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\partial_{x_1} L\varphi &= L(\partial_{x_1}\varphi) + \xi\partial_{x_1}\varphi - \eta\partial_{x_2}\varphi,\\ \partial_{x_2} L\varphi &= L(\partial_{x_2}\varphi) + \eta\partial_{x_1}\varphi + \xi\partial_{x_2}\varphi,\end{aligned}$$

whence

$$(\partial_{x_1} + i\partial_{x_2}) L\varphi = L(\partial_{x_1}\varphi + i\partial_{x_2}\varphi) + (\xi + i\eta)(\partial_{x_1}\varphi + i\partial_{x_2}\varphi).$$

Set $\Phi = \partial_{x_1}\varphi + i\partial_{x_2}\varphi$; we have

$$(\partial_{x_1} + i\partial_{x_2}) L\varphi = L\Phi + (\xi + i\eta)\Phi$$

and

$$\operatorname{Re}(L\Phi + (\xi + i\eta)\Phi, \Phi)_{L^2} = \xi \|\Phi\|_{L^2}.$$

But this can be rewritten as

$$\operatorname{Re}\left((\partial_{x_1} + i\partial_{x_2})^2 L\varphi, \varphi\right)_{L^2} = \xi \|\partial_{x_1}\varphi + i\partial_{x_2}\varphi\|_{L^2}.$$

Let $\Omega_R = \left\{x \in \mathbb{R}^n;\ x_1^2 + x_2^2 < R\right\}$, $R > 0$. We have, for some $C_R > 0$ and all $\varphi \in \mathcal{C}_c^\infty(\Omega_R)$,

$$\|\varphi\|_{L^2} \le C_R \|\partial_{x_1}\varphi + i\partial_{x_2}\varphi\|_{L^2}.$$

The Cauchy-Schwarz inequality and Lemma 3 entail the claim. □

2.3.2. *Case II: Every eigenvalue of $\mathfrak{m}(L^\circ)$ is purely imaginary.* We continue to assume that $\operatorname{div} L = 0$.

Proposition 4. *If all the eigenvalues of $\mathfrak{m}(L^\circ)$ are purely imaginary and if the nilpotent part of $\mathfrak{m}(L^\circ)$ is not equal to zero, then L is locally solvable at every point of $\boldsymbol{V}(L)$.*

Proof. In this case we may assume that L has the expression (1.7). We derive

$$\begin{aligned}\partial_{x_1} L\varphi &= L\partial_{x_1}\varphi + \eta\partial_{x_2}\varphi,\\ \partial_{x_2} L\varphi &= L\partial_{x_2}\varphi - \eta\partial_{x_1}\varphi,\\ \partial_{x_3} L\varphi &= L\partial_{x_3}\varphi + \eta\partial_{x_4}\varphi + \partial_{x_1}\varphi,\\ \partial_{x_4} L\varphi &= L\partial_{x_4}\varphi - \eta\partial_{x_3}\varphi + \partial_{x_2}\varphi.\end{aligned}$$

Integration by parts implies

$$\begin{aligned}(\partial_{x_3} L\varphi, \partial_{x_1}\varphi) &= (L\partial_{x_3}\varphi, \partial_{x_1}\varphi) + \eta(\partial_{x_4}\varphi, \partial_{x_1}\varphi) + \|\partial_{x_1}\varphi\|^2\\ &= -(\partial_{x_3}\varphi, L\partial_{x_1}\varphi) + \eta(\partial_{x_4}\varphi, \partial_{x_1}\varphi) + \|\partial_{x_1}\varphi\|^2\\ &= -(\partial_{x_3}\varphi, \partial_{x_1} L\varphi) + \eta(\partial_{x_3}\varphi, \partial_{x_2}\varphi) + \eta(\partial_{x_4}\varphi, \partial_{x_1}\varphi) + \|\partial_{x_1}\varphi\|^2,\end{aligned}$$

whence

$$2\operatorname{Re}(\partial_{x_3} L\varphi, \partial_{x_1}\varphi) = \|\partial_{x_1}\varphi\|^2 + \eta\left((\partial_{x_3}\varphi, \partial_{x_2}\varphi) + (\partial_{x_4}\varphi, \partial_{x_1}\varphi)\right).$$

Likewise,

$$\begin{aligned}(\partial_{x_4}L\varphi,\partial_{x_2}\varphi) &= (L\partial_{x_4}\varphi,\partial_{x_2}\varphi)-\eta\left(\partial_{x_3}\varphi,\partial_{x_2}\varphi\right)+\left\|\partial_{x_2}\varphi\right\|^2\\ &= -\left(\partial_{x_4}\varphi,L\partial_{x_2}\varphi\right)-\eta\left(\partial_{x_3}\varphi,\partial_{x_2}\varphi\right)+\left\|\partial_{x_2}\varphi\right\|^2\\ &= -\left(\partial_{x_4}\varphi,\partial_{x_2}L\varphi\right)-\eta\left(\partial_{x_4}\varphi,\partial_{x_1}\varphi\right)-\eta\left(\partial_{x_3}\varphi,\partial_{x_2}\varphi\right)+\left\|\partial_{x_2}\varphi\right\|^2,\end{aligned}$$

whence

$$2\,\mathrm{Re}\left(\partial_{x_4}L\varphi,\partial_{x_2}\varphi\right)=\left\|\partial_{x_2}\varphi\right\|^2-\eta\left(\left(\partial_{x_3}\varphi,\partial_{x_2}\varphi\right)+\left(\partial_{x_4}\varphi,\partial_{x_1}\varphi\right)\right).$$

We conclude that

$$2\,\mathrm{Re}\left(\left(\partial_{x_3}L\varphi,\partial_{x_1}\varphi\right)+\left(\partial_{x_4}L\varphi,\partial_{x_2}\varphi\right)\right)=\left\|\partial_{x_1}\varphi\right\|^2+\left\|\partial_{x_2}\varphi\right\|^2 \tag{2.3}$$

for all $\varphi\in\mathcal{C}_c^\infty\left(\mathbb{R}^2\right)$. In view of (2.3) the Cauchy-Schwarz inequality and Lemma 3 entail the claim. □

The proof of Theorem 1 is completed with the following result.

Proposition 5. *If all the eigenvalues of* $\mathfrak{m}\left(L^\circ\right)$ *are purely imaginary and the nilpotent part of* $\mathfrak{m}\left(L^\circ\right)$ *is equal to zero, then* L *is not locally solvable at any point of* $\boldsymbol{V}(L)$.

Proof. In this case we may assume that $L=\sum_{j=1}^{p}\eta_j\partial_{\theta_j}$ [see (1.9)], where $\theta_j=\arg\left(x_{2j-1}+\sqrt{-1}x_{2j}\right)$. Let $\mathfrak{B}_R=\left\{x\in\mathbb{R}^n;\ \sum_{j=1}^{n}x_j^2<R^2\right\}$, $R>0$. If $u\in\mathcal{D}'\left(\mathfrak{B}_R\right)$ and $F\in\mathcal{C}_c^\infty\left(\mathfrak{B}_R\right)$ are such that $Lu=F$ in $\mathfrak{B}_R$, then necessarily

$$\int_0^{2\pi}\cdots\int_0^{2\pi}F\left(r_1\cos\theta_1,r_1\sin\theta_1,...,r_p\cos\theta_p,r_p\sin\theta_p,\tilde{x}_{m+1},...,\tilde{x}_n\right)d\theta_1\cdots d\theta_p=0$$

for all $r_j\geq 0$ and all $\left(\tilde{x}_{m+1},...,\tilde{x}_n\right)\in\mathbb{R}^{n-m}$: F cannot be an arbitrary test function in $\mathfrak{B}_R$. □

3. Vector fields with complex linear coefficients defining a rank-one foliation

In this section we allow the coefficients of the vector field (1.1) to be complex. We reason under the hypothesis that L *is locally solvable in the open set* $\Omega=\mathbb{R}^n\backslash\boldsymbol{V}(L)$. We begin by recalling what this entails. Let $\mathfrak{g}(L)$ denote the Lie algebra generated by the real vector fields $X=\mathrm{Re}\,L$, $Y=\mathrm{Im}\,L$. The *Nagano foliation* of $\mathbb{R}^n\backslash\boldsymbol{V}(L)$ defined by $\mathfrak{g}(L)$ consists of *connected* analytic submanifolds whose tangent space at any one of their points is equal to the "freezing" of $\mathfrak{g}(L)$ at that point and which are maximal for these properties. A classical result of the local solvability theory of smooth complex vector fields without critical points (see [Treves, 1992], Section VIII.7) states that for L to be locally solvable in $\mathbb{R}^n\backslash\boldsymbol{V}(L)$ it is necessary and sufficient that the following requirement be satisfied:

(P) *The leaves of the Nagano foliation of* $\mathbb{R}^n\backslash\boldsymbol{V}(L)$ *defined by the Lie algebra* $\mathfrak{g}(L)$ *have dimension* 1 *or* 2. *On each two-dimensional leaf the sign of* $\frac{1}{2i}L\wedge\overline{L}$ *does not change.*

Here we shall assume that there are no two-dimensional Nagano leaves and we shall not concern ourselves with the sign of $\frac{1}{2i}L\wedge\overline{L}$. To say that there are only one-dimensional Nagano leaves is equivalent to saying that $L\wedge\overline{L}$ vanishes identically. The extension of Theorem 1 is not quite automatic since the vanishing of $L\wedge\overline{L}$ at every point does not imply that $\zeta^{-1}L$ is a real vector field for some $\zeta \in \mathbb{C}\backslash\{0\}$. We take a closer look at this question.

In the notation (1.3), $L \wedge \overline{L} \equiv 0$ is equivalent to the equations

$$\operatorname{Im}\left(L_j \wedge \overline{L}_k\right) = 0 \tag{3.1}$$

$(1 \leq j \leq k \leq m)$. For each $j = 1, ..., m$, we can select a real vector field $L_j^{\mathbb{R}} \neq 0$ such that $L_j = \zeta_j L_j^{\mathbb{R}}$ for some $\zeta_j \in \mathbb{C}\backslash\{0\}$. With this, (3.1) reads

$$\operatorname{Im}\left(\zeta_j \bar{\zeta}_k\right) L_j^{\mathbb{R}} \wedge L_k^{\mathbb{R}} = 0,$$

implying either $\operatorname{Im}\left(\zeta_j \bar{\zeta}_k\right) = 0$ or $L_j^{\mathbb{R}} \wedge L_k^{\mathbb{R}} = 0$. We can relabel all the vector fields L_j having the property that $L_j^{\mathbb{R}} \wedge L_1^{\mathbb{R}} = 0$ so that they become $L_1, ..., L_{\nu_1}$. Next we relabel all the vector fields L_j such that $L_j^{\mathbb{R}} \wedge L_{\nu_1+1}^{\mathbb{R}} = 0$ so that they become $L_{\nu_1+1}, ..., L_{\nu_2}$, etc. We end up with a set of vector fields $X_1, ..., X_r$ $(1 \leq r \leq m)$ with real *constant* coefficients such that $X_\alpha \wedge X_\beta \neq 0$ if $\alpha \neq \beta$ and such that to each $j \in [1, ..., m]$ there is $\alpha \in [1, ..., r]$ and $\zeta_j' \in \mathbb{C}\backslash\{0\}$ such that $L_j = \zeta_j' X_\alpha$. Moreover, if $L_k = \zeta_k' X_\beta$ with $\alpha \neq \beta$, then $\operatorname{Im}\left(\zeta_j' \bar{\zeta}_k'\right) = 0$, meaning that $\zeta_j' = c_{j,k}\zeta_k'$ for some $c_{j,k} \in \mathbb{R}\backslash\{0\}$. In particular, this implies $\zeta_k' = c_{k,1}\zeta_j'$ for all $j = 1, ..., \nu_1$ and all $k > \nu_1$. Moreover, when $r > 1$, what we have just said implies $\zeta_j' = c_{j,1}\zeta_1'$ for all $j \in [2, ..., \nu_1]$. We find ourselves in one of the following two situations:

(1) $r = 1$; *there is a single vector field X with real constant coefficients such that $L_j = \zeta_j X$ for each $j = 1, ..., m$, with $\zeta_j \in \mathbb{C}\backslash\{0\}$.* In this case,

$$L = \left(\sum_{j=1}^{m} \zeta_j x_j\right) X. \tag{3.2}$$

(2) $r > 1$; *there are numbers $\zeta \in \mathbb{C}\backslash\{0\}$ and $c_{\alpha,j} \in \mathbb{R}\backslash\{0\}$ such that*

$$L = \zeta \sum_{\alpha=1}^{r} \left(\sum_{j=\nu_{\alpha-1}+1}^{\nu_\alpha} c_{\alpha,j} x_j\right) X_\alpha. \tag{3.3}$$

When (3.3) holds Theorem 1 applies to $\zeta^{-1}L$ and therefore extends to L.

Since $\operatorname{codim} \boldsymbol{V}(L) = m$, when (3.2) holds we have

$$\sum_{k=1}^{m} (\operatorname{Re}\zeta_k) x_k = \sum_{k=1}^{m} (\operatorname{Im}\zeta_k) x_k = 0 \Longrightarrow x_1 = \cdots = x_m = 0.$$

This is possible only if $m \leq 2$. Moreover, if $m = 2$, it requires $\operatorname{Im}\left(\zeta_1 \bar{\zeta}_2\right) \neq 0$; in turn the latter allows one to carry out a real linear change of the variables x_1, x_2 such that $L = \zeta(x_1 + ix_2)X$, $0 \neq \zeta \in \mathbb{C}$. If $m = 1$, we must have $L = \zeta x_1 X$. The only one of these two cases not covered by Theorem 1 (after division by ζ) is the case of $L = (x_1 + ix_2)X$, whose local solvability is a consequence of Lemma 2.

If $L = (x_1 + ix_2)X$ (and thus $m = 2$), not every eigenvalue of $\mathfrak{m}(L^\circ)$ [see (1.4)] can be purely imaginary (and different from zero). Indeed, in an arbitrary basis of $\mathbb{R}^2$ we will have

$$\mathfrak{m}(L^\circ) = \begin{pmatrix} a_{11} & a_{12} \\ ia_{11} & ia_{12} \end{pmatrix}, \; a_{11}, a_{12} \in \mathbb{R},$$

whose eigenvalues are $a_{11} + ia_{12}$ and 0. The orbits of $L = (x_1 + ix_2)X$ in $\mathbb{R}^n \backslash \boldsymbol{V}(L)$ are equal to the connected components of the intersections of $\mathbb{R}^n \backslash \boldsymbol{V}(L)$ with the orbits of X in $\mathbb{R}^n$. The latter being straight lines, the closures of the orbits of L in $\mathbb{R}^n \backslash \boldsymbol{V}(L)$ are not compact (X has real constant coefficients, not all equal to zero). We can state

Theorem 2. *Let the vector field* (1.1) *have complex coefficients. If $L \wedge \overline{L}$ vanishes identically, then the properties* (a), (b), (c) *in Theorem* 1 *are equivalent.*

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