PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 12, December 2009, Pages 4089–4097
S 0002-9939(09)10037-0
Article electronically published on July 10, 2009

HIGHER ORDER EMBEDDINGS OF CERTAIN BLOW-UPS OF $\mathbb{P}^2$

CINDY DE VOLDER AND HALSZKA TUTAJ-GASIŃSKA

(Communicated by Ted Chinburg)

Abstract. Let X_n be the blow-up of the projective plane along n general points of a smooth cubic plane curve and let $\mathcal{L}$ be the linear series of strict transforms of plane curves of degree d having multiplicity at least m_i at the i-th blown-up point. We prove that if $\mathcal{L}$ is k-very ample, then $\mathcal{L}$ is excellent and $\mathcal{L} \cdot (-K_n) \geq k+2$. Then we give a numerical criterion for the k-very ampleness of excellent classes with $\mathcal{L} \cdot (-K_n) \geq k+2$, which in many cases is a necessary and sufficient condition.

1. Introduction

Let X be a smooth projective variety with an ample line bundle L. The notions of k-very ampleness and k-jet ampleness of L were introduced in the 1980's by Beltrametti, Francia and Sommese, see [3, 4, 5]. We recall the definitions.

Definition 1.1. Let k be a nonnegative integer. We say that:

1. L is *k-very ample* on X if the restriction map

$$H^0(X,L) \to H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective for any 0-dimensional subscheme Z of length $k+1$. By *length of Z* we mean $l(Z) := h^0(Z, \mathcal{O}_Z)$.

2. L is *k-jet ample* on X if the restriction map

$$H^0(X,L) \to H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective for any 0-dimensional subscheme Z with the ideal $I_Z = \mathcal{M}_{P_1}^{\otimes k_1} \otimes \cdots \otimes \mathcal{M}_{P_r}^{\otimes k_r}$, with $k_1 + \cdots + k_r = k+1$ (where $\mathcal{M}_P$ denotes the maximal ideal of a point P).

Remark 1.2. Observe that the two notions are equivalent for $k = 0, 1$. In general k-jet ampleness implies k-very ampleness. If a line bundle is k-very ample, we say that it gives an *embedding of order k*.

The concepts of k-very ampleness and k-jet ampleness of line bundles have been widely studied; see for example [1, 2, 6, 7, 8, 9, 10, 11, 14]. In these papers the reader can find characterizations of k-very ample (or k-jet ample) line bundles on

Received by the editors May 17, 2008, and, in revised form, January 24, 2009, and April 30, 2009.
2000 *Mathematics Subject Classification.* Primary 14C20.
Key words and phrases. k-very ample, anticanonical rational surface.

different varieties. In the case of surfaces, Beltrametti, Francia and Sommese in [3, 4, 5] proved a criterion for deciding whether a given line bundle (of degree large enough) is k-very ample. We quote their result in this paper as Theorem 4.3. It is also worth mentioning that the line bundles of the degree smaller than required by this criterion were classified by Ballico and Sommese in [2].

When one studies k-very ampleness (or k-jet ampleness) of line bundles, a natural question arises. Let $X' \overset{\pi}{\to} X$ be a blow-up of X in pairwise distinct points $P_1, ..., P_n$, with exceptional divisors $E_1, ..., E_n$. Given a d-very ample (d-jet ample) line bundle on the variety X and nonnegative integers $a_1, ..., a_n$, what can we say about the k-very ampleness (k-jet ampleness) of the line bundle $\pi^*L - a_1E_1 - \cdots - a_nE_n$ on X'? This problem was studied by Beltrametti and Sommese in [7] or Ballico and Coppens in [1], and other authors.

Beltrametti and Sommese proved the following theorem (cf. Proposition 3.5 of [7]).

Theorem 1.3. *Let X be a complex projective variety and let L be a d-jet ample line bundle on X. Let $P_1, ..., P_n$ be pairwise distinct points in the regular part of X and let $a_1, ..., a_n$ be positive integers. Let $X' \overset{\pi}{\to} X$ be the blow-up of X in $P_1, ..., P_n$ with exceptional divisors $E_1, ..., E_n$. Let $k = \min\{d - (a_1 + \cdots + a_n), a_1, ..., a_n\}$. Then the line bundle $\pi^*L - a_1E_1 - \cdots - a_nE_n$ is k-jet ample (so also k-very ample) on X'.*

The conditions for k-very ampleness of line bundles on blow-ups were also studied by Ballico and Coppens in [1]. We quote a special case of their results as Theorem 4.9.

Szemberg and the second author in [14] investigated the k-very ampleness of line bundles on blow-ups of $\mathbb{P}^2$ in points in general position (i.e. general points).

In this paper we study k-very ampleness of line bundles on a special blow-up of $\mathbb{P}^2$; namely we consider the blow-up X_n of $\mathbb{P}^2$ along n general points of a smooth cubic curve in $\mathbb{P}^2$. This implies, in particular, that the anticanonical divisor of X_n is effective, smooth, irreducible and reduced. The idea and the techniques used in the paper were inspired by the works of Harbourne [10, 11]. Harbourne studied 0- and 1-very ampleness of line bundles on X_n, and he solved the problem completely, so in our paper we consider the case $k \geq 2$. The main result of the paper, Theorem 4.1, gives a characterization of k-very ample line bundles on X_n.

At the end of the paper we compare our result with the results of Beltrametti and Sommese and of Ballico and Coppens (cf. Theorem 1.3 and Theorem 4.9).

2. Notation

We assume that the base field is algebraically closed and of characteristic 0.

Let D be a smooth cubic curve in $\mathbb{P}^2$ and let $P_1, \ldots, P_n$ be general points of D. Let $\pi : X_n \longrightarrow \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ in $P_1, \ldots, P_n$. By H we denote the pull-back of a line, by E_i, $i = 1, \ldots, n$, we denote the exceptional divisor corresponding to P_i. Then the ordered set $\mathcal{E} := \{H, E_1, \ldots, E_n\}$ generates $\operatorname{Pic} X_n$, and it is called an *exceptional configuration* of X_n. Note that in our situation every base of $\operatorname{Pic} X_n$ will be an exceptional configuration in the sense of [10, 11], which is not true if the points P_i are not in general position on D. By K_n we denote the canonical class on X_n and by D_n the strict transform of D on X_n, so $D_n \in -K_n = |3H - E_1 - \cdots - E_n|$.

Let $\mathcal{L}$ be a line bundle on X_n. Then $\mathbb{P}(\mathcal{L}) = |dH - m_1E_1 - \cdots - m_nE_n|$ for some integers $d, m_1, \ldots, m_n$. In what follows, we will use $\mathcal{L}$ to denote the line bundle as well as its corresponding complete linear system on X_n.

Given an exceptional configuration $\mathcal{E} := \{H, E_1, \ldots, E_n\}$, one can define the $\mathcal{E}$-*simple roots* $r_0 := H - E_1 - E_2 - E_3$ and $r_i := E_i - E_{i+1}$ for $i = 0, \ldots, n-1$; and the *reflections* $s_i(\mathcal{F}) := \mathcal{F} + (\mathcal{F} \cdot r_i)r_i$, $i = 0, \ldots, n-1$, $\mathcal{F} \in \operatorname{Pic} X_n$. The reflections generate a subgroup $W \subset \mathrm{GL}(\operatorname{Pic} V)$ which preserves the intersection product and fixes K_n. Moreover, it is known that the elements r_i, $i = 0, \ldots, n-1$ form the simple roots of a root system having Weyl group W (see [13, 12]). Note that $s_0(\mathcal{E}) = \{2H - E_1 - E_2 - E_3, H - E_2 - E_3, H - E_1 - E_3, H - E_1 - E_2, E_3, \ldots, E_n\}$ and $s_i(\mathcal{E}) = \{H, E_1, \ldots, E_{i-1}, E_{i+1}, E_i, E_{i+2}, \ldots, E_n\}$, $i = 1, \ldots, n-1$. So, if $\mathcal{L} = |dH - m_1E_1 - \cdots - m_nE_n| \in \operatorname{Pic} X_r$ and $\mathcal{E}_i = \{H^{(i)}, E_1^{(i)}, \ldots, E_n^{(i)}\} := s_i(\mathcal{E})$ for $i = 0, \ldots, n-1$, then

$$\begin{aligned} s_0(\mathcal{L}) = {} & |(2d - m_1 - m_2 - m_3)H^{(0)} - (d - m_2 - m_3)E_1^{(0)} - (d - m_1 - m_3)E_2^{(0)}, \\ & - (d - m_1 - m_2)E_3^{(0)} - m_4E_4^{(0)} - \cdots - m_nE_n^{(0)}|, \\ s_i(\mathcal{L}) = {} & |dH^{(i)} - m_1E_1^{(i)} - \cdots - m_{i-1}E_{i-1}^{(i)} - m_{i+1}E_i^{(i)} - m_iE_{i+1}^{(i)} - m_{i+2}E_{i+2}^{(i)} \\ & - \cdots - m_nE_n^{(i)}|, \ i = 1, \ldots, n-1. \end{aligned}$$

Thus s_0 is in fact the Cremona transformation with respect to the first three multiplicities, and s_i interchanges m_i and m_{i+1}. Moreover, the exceptional curves $E_i^{(i)}$ also correspond to general points of D.

The following terminology was introduced in 1985 by Harbourne; see [10]. Let $\mathcal{E} = \{H, E_1, \ldots, E_n\}$ be an exceptional configuration. A line bundle $\mathcal{L} \in \operatorname{Pic} X_n$ is called $\mathcal{E}$-*standard* if $\mathcal{L} = |dH - m_1E_1 - \cdots - m_nE_n|$ with $d \geq m_1 + m_2 + m_3$ and $d \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. A bundle $\mathcal{L} \in \operatorname{Pic} X_n$ is called *almost $\mathcal{E}$-excellent* (resp. *$\mathcal{E}$-excellent*), if $\mathcal{L}$ is $\mathcal{E}$-standard and $\mathcal{L} \cdot K_n \leq 0$ (resp. $\mathcal{L} \cdot K_n < 0$). We say that a line bundle is *standard* (resp. *(almost) excellent*) if there exists an exceptional configuration $\mathcal{E}$ of X_n such that $\mathcal{L}$ is $\mathcal{E}$-standard (resp. (almost) $\mathcal{E}$-excellent).

To conclude this section we list some results of [10, 11] which we use in this paper.

Lemma ([10, Lemma 1.4]). *Let $\mathcal{E} := \{H, E_1, \ldots, E_n\}$ be an exceptional configuration on X_n. The $\mathcal{E}$-standard divisor classes are precisely the nonnegative sums of the classes H, $H - E_1$, $2H - E_1 - E_2$ and $-K_i := 3H - E_1 - \cdots - E_i$, $3 \leq i \leq n$.*

Theorem ([10, Theorem 3.1 (a)]). *Let $\mathcal{L}$ be an almost $\mathcal{E}$-excellent class for some exceptional configuration $\mathcal{E}$. If $\mathcal{L} \cdot K_n = 0$, then $-K_n$ is a fixed component of $\mathcal{L}$.*

Corollary ([10, Corollary 3.2]). *A class $\mathcal{L}$ is almost excellent iff $\mathcal{L} \cdot \mathcal{C} \geq 0$ for every irreducible class $\mathcal{C}$.*

Lemma ([11, Lemma 1.2]). *Let $\mathcal{L}$ be an element of* $\operatorname{Pic} X_n$ *and let $r_0, \ldots, r_{n-1}$ be the $\mathcal{E}$-simple roots of an exceptional configuration $\mathcal{E} = \{H, E_1, \ldots, E_n\}$ on X_n.*

(1) *If $\mathcal{L}$ is a nonzero standard class, then $\mathcal{L}$ equals $\mathcal{I} + a_0r_0 + \cdots + a_{n-1}r_{n-1}$, where $\mathcal{I}$ is a nonzero $\mathcal{E}$-standard class and $a_i \geq 0$, $i = 0, \ldots, n-1$.*

(2) *If $\mathcal{L}$ is the class of an exceptional divisor, then either $\mathcal{L} = E_n + a_0r_0 + \cdots + a_{n-1}r_{n-1}$, $a_i \geq 0$, $i = 0, \ldots, n-1$, or $\mathcal{L} = H - E_1 - E_2$ and* $\operatorname{rk}\operatorname{Pic} X_n = 3$.

We will need also the following fact from [4].

Lemma ([4, Corollary 1.4])**.** *Let $\mathcal{L}$ be a k-very ample line bundle on a curve S of genus $g(C) > 0$. Let $d = \deg \mathcal{L}$. Then $d \geq k + 2$.*

3. Reducing to $\mathcal{E}$-standard classes

In this section we prove that if $\mathcal{L}$ is k-very ample on X_n, then it must be excellent.

Proposition 3.1. *Let $\mathcal{E} = \{H, E_1, \ldots, E_n\}$ and $\mathcal{L} = |dH - m_1E_1 - \cdots - m_nE_n| \in \operatorname{Pic} X_n$. If $\mathcal{L}$ is k-very ample, then $\mathcal{L}$ is excellent and $\mathcal{L} \cdot (-K_n) \geq k + 2$.*

Proof. We will give a very simple algorithm, based on the algorithm stated in [10], to see that the statement is true.

Step 1: Rearrange m_i so that $m_1 \geq \cdots \geq m_n$ and go to step 2.

Step 2: Check whether $d \geq m_1 + m_2 + k$, $m_1 \geq \cdots \geq m_n \geq k$ and $3d \geq m_1 + \cdots + m_n + k + 2$ (which is equivalent to $\mathcal{L} \cdot (-K_n) \geq k + 2$).

If the three conditions in Step 2 are satisfied, then go to step 3; else $\mathcal{L}$ cannot be k-very ample. Indeed, it is then clear that $\mathcal{L}$ cannot separate the 0-dimensional subschemes of length $k + 1$, lying on either the strict transform of a line, or on an exceptional divisor, or on D_n (cf. [4, Corollary 1.4]).

Step 3: Check whether $\mathcal{L}$ is now in standard form; i.e. check if $d \geq m_1+m_2+m_3$. If this condition is satisfied, then we are done; otherwise apply the Cremona transformation on $\mathcal{L}$ with respect to the first 3 multiplicities and go to step 1.

Since the actions applied on $\mathcal{L}$ form a sequel of some s_i, $i = 0, \ldots, n-1$, we have written $\mathcal{L}$ with respect to some exceptional configuration $\overline{\mathcal{E}}$. So, we have proved that either $\mathcal{L}$ is not k-very ample or $\mathcal{L}$ is standard and $\mathcal{L} \cdot (-K_n) \geq k + 2$. □

4. The k-very ampleness of $\mathcal{E}$-standard classes

In this section we formulate and prove the main result of the paper, namely the criterion for a line bundle on X_n to be k-very ample.

From Proposition 3.1 we know that either $\mathcal{L}$ is not k-very ample or $\mathcal{L}$ is excellent (and $\mathcal{L} \cdot (-K_n) \geq k+2$). Thus, it is enough to formulate the criterion for excellent line bundles. To simplify notation, we will assume that $\mathcal{L}$ is $\mathcal{E}$-standard, with $\mathcal{E} = \{H, E_1, \ldots, E_n\}$.

Theorem 4.1. *If $\mathcal{L} = |dH - m_1E_1 - \cdots - m_nE_n| \in \operatorname{Pic} X_n$, $n \geq 3$ is $\mathcal{E}$-standard, $m_n \geq k \geq 2$, $3d \geq m_1 + \cdots + m_n + k + 2$ and $3d \geq m_1 + \cdots + m_n + n - 9$, then $\mathcal{L}$ is k-very ample.*

Remark 4.2. The conditions of Theorem 4.1 say exactly that:

(1) $d \geq m_1 + m_2 + m_3$ (because $\mathcal{L}$ is in a standard form),
(2) $m_1 \geq \cdots \geq m_n \geq k$,
(3) $3d \geq m_1 + \cdots + m_n + k + 2$,
(4) $3d \geq m_1 + \cdots + m_n + n - 9$.

Obviously, when treating a particular case, (3) will imply (4) or vice versa, and since all but (4) are necessary conditions, we will obtain a necessary and sufficient criterion for all cases where (3) implies (4) (which occurs whenever $k+2 \geq n-9$).

To prove this result we will use the following Reider-type criterion.

Theorem 4.3 ([4, Theorem 2.1])**.** *Let $\mathcal{M}$ be a nef line bundle on a smooth surface S, such that $\mathcal{M}^2 \geq 4k + 5$, for some $k \geq 0$. Then either $\mathcal{L} := \mathcal{M} + K_S$ is k-very* *ample or there exists an effective divisor C such that $\mathcal{M}-2C$ is $\mathbb{Q}$-effective and C contains some 0-dimensional subscheme where the k-very ampleness fails and*

$$\mathcal{M}\cdot C-k-1\leq C^2<\frac{\mathcal{M}\cdot C}{2}<k+1.$$

In order to make the proof of Theorem 4.1 clearer, we first prove some auxiliary results.

Lemma 4.4. *There do not exist any nodal curves on X_n. (A nodal curve is an irreducible reduced curve C with $C^2=-2$, $C\cdot K_n=0$ and $p_a(C)=0$.)*

Proof. Assume that $\mathcal{C}\in \operatorname{Pic} X_n$ is irreducible and reduced (i.e. there exists an effective irreducible reduced divisor $C\in\mathcal{C}$), $\mathcal{C}^2=-2$ and $\mathcal{C}\cdot K_n=0$. Since $\mathcal{C}$ is irreducible, reduced and not exceptional (an exceptional curve is an irreducible reduced curve C with $C^2=C\cdot K_n=-1$), there exists some exceptional configuration $\mathcal{E}'$ such that $\mathcal{C}$ is $\mathcal{E}'$-standard, i.e. $\mathcal{C}$ is standard. But we also know that $\mathcal{C}\cdot K_n=0$, so it follows from [10, Theorem 3.1 (a)] that D_n is a fixed component of $\mathcal{C}$, which contradicts the fact that $\mathcal{C}$ is irreducible, unless $\mathcal{C}=-K_n$. This in turn contradicts $\mathcal{C}^2=-2$ and $\mathcal{C}\cdot K_n=0$. □

Lemma 4.5. *If $\mathcal{C}\in \operatorname{Pic} X_n$ is an exceptional class (i.e. there exists an exceptional curve $C\in\mathcal{C}$), then $\mathcal{M}\cdot\mathcal{C}\geq k+1$ with $\mathcal{M}:=\mathcal{L}-K_n$ and $\mathcal{L}$ as in Theorem 4.1.*

Proof. Let $\mathcal{C}\in \operatorname{Pic} X_n$ be an exceptional class. Because of [11, Lemma 1.2 (2)], we know that $\mathcal{C}=E_n+a_0r_0+\cdots+a_{n-1}r_{n-1}$ with $a_i\geq 0$ for all $i=0,\dots,n-1$. Obviously $\mathcal{L}\cdot r_0=d-m_1-m_2-m_3\geq 0$, $\mathcal{L}\cdot r_i=m_i-m_{i+1}\geq 0$ for $i=1,\dots,n-1$ and $\mathcal{L}\cdot E_n=m_n\geq k$. So $\mathcal{M}\cdot\mathcal{C}=\mathcal{L}\cdot\mathcal{C}-K_n\cdot\mathcal{C}\geq k+1$. □

Lemma 4.6. *Let $\mathcal{M}:=\mathcal{L}-K_n$ with $\mathcal{L}$ as in Theorem 4.1, and let $\mathcal{D}$ be a nonzero standard class which does not contain an exceptional curve in its base locus. Then $\mathcal{M}\cdot\mathcal{D}\geq 2k+2$ unless $\mathcal{D}=c_n\mathcal{S}_n$, with $n\geq 8$, $c_n\geq 1$ and $\mathcal{S}_n=-K_n$.*

Proof. Because of [11, Lemma 1.2 (1)], we know that $\mathcal{D}=w\mathcal{I}=\mathcal{I}+a_0r_0+\cdots+a_{n-1}r_{n-1}$, where $\mathcal{I}$ is a nonzero $\mathcal{E}$-standard class, $w\in W$ and $a_i\geq 0$ for $i=0,\dots,n-1$. Since $\mathcal{I}$ is $\mathcal{E}$-standard and by [10, Lemma 1.4], there exist nonnegative integers c_i, $i=0,\dots,n$, such that

$$\mathcal{I}=c_0\mathcal{S}_0+c_1\mathcal{S}_1+\cdots+c_n\mathcal{S}_n,$$

with $\mathcal{S}_0:=H$, $\mathcal{S}_1:=H-E_1$, $\mathcal{S}_2:=2H-E_1-E_2$ and $\mathcal{S}_i:=3H-E_1-\cdots-E_i$ for $i=3,\dots,n$.

Moreover, $\mathcal{M}\cdot r_0=d-m_1-m_2-m_3\geq 0$ and $\mathcal{M}\cdot r_i=m_i-m_{i+1}\geq 0$, for $i=1,\dots,n-1$. So $\mathcal{M}\cdot\mathcal{D}\geq\mathcal{M}\cdot\mathcal{I}$. On the other hand,

$$\begin{aligned}
\mathcal{M}\cdot\mathcal{S}_0&=d+3\geq 3k+3,\\
\mathcal{M}\cdot\mathcal{S}_1&=d+2-m_1\geq 2k+2,\\
\mathcal{M}\cdot\mathcal{S}_2&=2d+4-m_1-m_2\geq 4k+4,\\
\mathcal{M}\cdot\mathcal{S}_i&=\mathcal{M}\cdot(\mathcal{S}_n+E_{i+1}+\cdots+E_n)\\
&\geq m_{i+1}+\cdots+m_n+n-i\geq 2k+2 \qquad \text{if } 3\leq i\leq n-2,\\
\mathcal{M}\cdot\mathcal{S}_{n-1}&=\mathcal{M}\cdot(\mathcal{S}_n+E_n)\geq m_n+1\geq k+1,\\
\mathcal{M}\cdot\mathcal{S}_n&=3d-m_1-\cdots-m_n-n+9\geq 0.
\end{aligned}$$

So obviously $\mathcal{M} \cdot \mathcal{I} \geq 2k+2$, unless

$$\mathcal{I} = \mathcal{S}_{n-1} + c_n\mathcal{S}_n,\ c_n \geq 0 \ \text{ or } \ \mathcal{I} = c_n\mathcal{S}_n,\ c_n \geq 1.$$

To this end, assume first that $n \leq 9$. Then

$$\begin{aligned}\mathcal{M} \cdot (\mathcal{S}_{n-1} + c_n\mathcal{S}_n) &\geq \mathcal{M} \cdot \mathcal{S}_{n-1} = \mathcal{L} \cdot \mathcal{S}_{n-1} - c_nK_n\mathcal{S}_n \\ &\geq \mathcal{L} \cdot \mathcal{S}_n + \mathcal{L} \cdot E_n \geq k+2+m_n \geq 2k+2.\end{aligned}$$

On the other hand, if $n \geq 10$, then

$$(\mathcal{S}_{n-1} + c_n\mathcal{S}_n).K_n \geq 0,$$

so $D_n \subset \mathrm{Bs}(\mathcal{S}_{n-1} + c_n\mathcal{S}_n)$, and this argument holds c_n+1 times. More precisely, the class $\mathcal{I} = \mathcal{S}_{n-1} + c_n\mathcal{S}_n$ consists of just one effective divisor $(c_n+1)D_n + E_n$. But this contradicts the fact that $\mathcal{D}$, and thus also $\mathcal{I}$, does not contain an exceptional divisor in its base locus. Thus $\mathcal{M} \cdot \mathcal{I} \geq 2k+2$, unless $\mathcal{I} = c_n\mathcal{S}_n$. Because $w\mathcal{S}_n = \mathcal{S}_n$ for all $w \in W$, from $\mathcal{I} = c_n\mathcal{S}_n$ it follows that $\mathcal{D} = w\mathcal{I} = c_n\mathcal{S}_n$. Moreover, if $n \leq 7$, then $\mathcal{M} \cdot (c_n\mathcal{S}_n) \geq c_n\mathcal{L} \cdot \mathcal{S}_n + 2 \geq c_n2k+2$. □

Lemma 4.7. *Define $\mathcal{M} := \mathcal{L} - K_n$, with $\mathcal{L}$ as in Theorem* 4.1. *Then $\mathcal{M}^2 \geq 4k+5$.*

Proof. If $n \leq 8$, then $\mathcal{L} = c_0\mathcal{S}_0 + c_1\mathcal{S}_1 + \cdots + c_n\mathcal{S}_n$, with $\mathcal{S}_i$ as in the proof of Lemma 4.6, $c_i \geq 0$ for $i = 0, \ldots, n$ and $c_n = m_n \geq k$ (see [10, Lemma 1.4]). Since $\mathcal{S}_i \cdot \mathcal{S}_j \geq 0$ for all $i, j \in \{0, \ldots, n\}$ and $\mathcal{S}_n^2 = 9-n \geq 1$, we have that $\mathcal{L}^2 \geq c_n^2\mathcal{S}_n^2 \geq k^2$. Moreover $\mathcal{L} \cdot (-K_n) = 3d - m_1 - \cdots - m_n \geq k+2$ and $K_n^2 = 9-n \geq 1$. So

$$\mathcal{M}^2 = \mathcal{L}^2 - 2\mathcal{L} \cdot K_n + K_n^2 \geq k^2 + 2k + 5 \geq 4k+5.$$

In case $n \geq 9$, we know that $\mathcal{M}^2 = (\mathcal{L}^2 - \mathcal{L} \cdot K_n) + (K_n^2 - \mathcal{D} \cdot K_n)$ and $K_n^2 - \mathcal{D} \cdot K_n = 3d - m_1 - \cdots - m_n + 9 - n \geq 0$. So

$$\mathcal{M}^2 \geq \mathcal{L}^2 - \mathcal{L} \cdot K_n = \mathcal{L}^2 + 3d - m_1 - \cdots - m_n \geq \mathcal{L}^2 + k + 2.$$

Define $\mathcal{A} := |aH - m_1E_1 - \cdots - m_nE_n|$, with a being the minimal integer such that $a \geq m_1 + m_2 + m_3$ and $3a \geq m_1 + \cdots + m_n$. If $a \neq m_1 + m_2 + m_3$, then a is equal to $(m_1 + \cdots + m_n + \alpha)/3$ with $\alpha \in \{0, 1, 2\}$. So in particular $3a \leq m_1 + \cdots + m_n + 2$. On the other hand, $3d \geq m_1 + \cdots + m_n + k + 2 > m_1 + \cdots + m_n + 2 \geq 3a$, and thus $d > a$. However, if $a = m_1 + m_2 + m_3$, then it could happen that $d = a$. We now have to consider the following two cases:

(1) $d \geq a+1$,
(2) $d = a = m_1 + m_2 + m_3$.

(1) Assume $d \geq a+1$. Since $\mathcal{A}$ is in standard form and $\mathcal{A} \cdot K_n \leq 0$ (because of the choice of a), we know that $\mathcal{A}$ is nef (see [10, Corollary 3.2]). Moreover $\mathcal{A}$ is effective (because it is standard), so we must have $\mathcal{A}^2 \geq 0$. Thus

$$\mathcal{L}^2 = d^2 - \sum_{i=1}^{n} m_i^2 = (d-a)(d+a) + \mathcal{A}^2 \geq d+a.$$

On the other hand, $m_i \geq k$ for all $i = 1, \ldots, n$, $n \geq 9$, $3d \geq m_1 + \cdots + m_n + k + 2$ and $3a \geq m_1 + \cdots + m_n$, so

$$d + a \geq \frac{(n+1)k + 2 + nk}{3} \geq 6k,$$

which implies that

$$\mathcal{M}^2 \geq \mathcal{L}^2 + k + 2 \geq 7k + 2 \geq 4k + 5.$$

(2) Assume $d = a = m_1 + m_2 + m_3$. Define $\mathcal{A}' := |(a-1)H - (m_1-1)E_1 - m_2E_2 - \cdots - m_nE_n|$. Then $\mathcal{A}'$ is obviously standard unless $m_4 > m_1 - 1$, which occurs if and only if $m := m_1 = m_2 = m_3 = m_4$ (and then $d = a = 3m$).

First, let us assume that $\mathcal{A}'$ is standard. Since $\mathcal{A}' \cdot K_n = -3a + 3 + m_1 + \cdots + m_n - 1 \leq -k < 0$, we see that $\mathcal{A}'$ is excellent and therefore also nef (see [10, Corollary 3.2]). As $\mathcal{A}'$ is effective and nef, it follows that $(\mathcal{A}')^2 = a^2 - m_1^2 - \cdots - m_n^2 - 2a + 2m_1 \geq 0$. This implies that

$$\mathcal{L}^2 = a^2 - \sum_{i=1}^{n} m_i^2 \geq 2a - 2m_1 = 2m_2 + 2m_3 \geq 4k$$

and thus $\mathcal{M}^2 \geq \mathcal{L}^2 + k + 2 \geq 5k + 2$. So we certainly have that $\mathcal{M}^2 \geq 4k + 5$ unless $k = 2 = m_2$. Now, if $m_2 = k = 2$ and $a = m_1 + m_2 + m_3$, then (using Remark 4.2(2)) we obtain

$$3a = 3m_1 + 12 \geq m_1 + \cdots + m_n + k + 2 = m_1 + 2n + 2.$$

This implies that $m_1 \geq n - 5$ (with $n \geq 9$) and thus also

$$\begin{aligned} \mathcal{M}^2 &\geq (m_1+4)^2 - m_1^2 - 4(n-1) + 3(m_1+4) - m_1 - 2(n-1) \\ &\geq 10m_1 - 6n + 34 \geq 4n - 16 \geq 13 = 4k + 5. \end{aligned}$$

If $m_4 > m_1 - 1$ denote $m := m_1 = \cdots = m_4$, so $d = a = 3m$. Thus $\mathcal{L}^2 = 5m^2 - m_5^2 - \cdots - m_n^2$ and $5m \geq m_5 + \cdots + m_n + k + 2$ (because of Remark 4.2 (3)). Multiplying this last inequality by m, and using the fact that $m \geq m_i$ for all $i = 5, \ldots, n$, we obtain $5m^2 - m_5^2 - \cdots - m_n^2 \geq m(k+2)$. So $\mathcal{M}^2 \geq \mathcal{L}^2 + k + 2 \geq (m+1)(k+2)$, which implies $\mathcal{M}^2 \geq 4k + 5$ unless $m = k = 2$. But if $m = k = 2$, then $m_i = 2$ for all $i = 1, \ldots, n$ ($n \geq 9$) and $d = 3m = 6$, which contradicts the fact that $3d \geq m_1 + \cdots + m_n + k + 2$. □

Proof of Theorem 4.1. Let $\mathcal{M} := \mathcal{L} - K_n$. Obviously $\mathcal{M}$ is $\mathcal{E}$-standard because $\mathcal{L}$ is $\mathcal{E}$-standard. Moreover $\mathcal{M} \cdot K_n = -3d + m_1 + \cdots + m_n + n - 9 \leq 0$, so $\mathcal{M}$ is nef (see [10, Corollary 3.2]).

Because of Lemma 4.7 we know that $\mathcal{M}^2 \geq 4k + 5$. So, if $\mathcal{L}$ is not k-very ample, then, according to Theorem 4.3, there must exist an effective divisor C where the k-very ampleness fails and such that

$$\mathcal{M} \cdot C - k - 1 \leq C^2 < \frac{\mathcal{M} \cdot C}{2} < k + 1. \tag{4.1}$$

Assume that such a curve C exists. Then we can write $C = \sum_{i=1}^{x} n_i C_i$, with C_i distinct irreducible curves and $n_i > 0$. Because of Lemma 4.4 we know that the only irreducible curves with negative self-intersection are exceptional curves and D_n (if $n \geq 10$). Also $\mathcal{M} \cdot C_i \geq 0$ for all i (because $\mathcal{M}$ is nef). Let $I := \{i : C_i \text{ is exceptional}\}$. Then, using Lemma 4.5, we see that $\mathcal{M} \cdot C \geq \sum_{i \in I} n_i(k+1)$. But in order to have (4.1), we certainly need $\mathcal{M} \cdot C < 2k + 2$, so either $I = \emptyset$ or I is a singleton, say $I = \{x\}$, and $n_x = 1$. Now define

$$C' := \begin{cases} C & \text{if } I = \emptyset, \\ \sum_{i=1}^{x-1} n_i C_i & \text{if } I = \{x\}. \end{cases}$$

Let $\mathcal{C}'$ denote the class of C'. Then $\mathcal{C}'$ is numerically effective and thus standard (see e.g. [10, Corollary 3.2]). Since C' does not contain an exceptional curve in its decomposition, no exceptional curve can be contained in the base locus of $\mathcal{C}'$. So, Lemma 4.6 and the fact that we need $2k+1 \geq \mathcal{M} \cdot C \geq \mathcal{M} \cdot \mathcal{C}'$ imply that $\mathcal{C}'$ can only be zero or equal to $c_n \mathcal{S}_n$, $c_n \geq 1$. Thus, C must satisfy one of the following conditions:

(1) C is irreducible and exceptional;
(2) $C = C' + C_x$ with C_x exceptional and $C' \in c_n \mathcal{S}_n$, $c_n \geq 1$, $n \geq 8$;
(3) $C \in c_n \mathcal{S}_n$, $c_n \geq 1$, $n \geq 8$.

(1) Assume C is irreducible and exceptional. Then $\mathcal{M} \cdot C \geq k+1$ (see Lemma 4.5) and $C^2 = -1$, so $\mathcal{M} \cdot C - k - 1 \geq 0 > C^2$, which contradicts inequality (4.1).

(2) Assume $C = C' + C_x$ with C_x exceptional and $C' \in c_n \mathcal{S}_n$, $c_n \geq 1$ and $n \geq 8$. If $8 \leq n \leq 10$, then $\mathcal{M} \cdot \mathcal{S}_n = \mathcal{L} \cdot \mathcal{S}_n - K_n \cdot \mathcal{S}_n \geq k+1$ (use arguments as in the proof of Lemma 4.6). So $\mathcal{M} \cdot C \geq k+1+c_n(k+1) \geq 2k+2$, which contradicts inequality (4.1).

If $n \geq 11$, then

$$C^2 = C_x^2 + 2c_n \mathcal{S}_n C_x + c_n^2 \mathcal{S}_n^2 = -1 + 2c_n + (9-n)c_n^2 < 0.$$

On the other hand, $\mathcal{M} \cdot C = \mathcal{M} \cdot C_x + c_n \mathcal{M} \cdot \mathcal{S}_n \geq k+1$, which contradicts $\mathcal{M} \cdot C - k - 1 \leq C^2$.

(3) Assume $C \in c_n \mathcal{S}_n$, $c_n \geq 1$ and $n \geq 8$. If $8 \leq n \leq 10$, then $\mathcal{M} \cdot C = c_n \mathcal{M} \cdot \mathcal{S}_n \geq c_n(k+1)$, which contradicts inequality (4.1) unless $c_n = 1$. But if $c_n = 1$, then $\mathcal{M} \cdot C = \mathcal{M} \cdot \mathcal{S}_n \geq k+2+9-n$ and $C^2 = 9-n$, which contradicts $\mathcal{M} \cdot C \leq C^2$.

If $n \geq 11$, then $C^2 = c_n^2(9-n)$ and $\mathcal{M} \cdot C = \mathcal{M} \cdot \mathcal{S}_n + (c_n - 1)\mathcal{M} \cdot \mathcal{S}_n \geq k+2+9-n$. Because of inequality (4.1), we must have $\mathcal{M} \cdot C - k - 1 \leq C^2$. So we certainly need

$$1 + (9-n) \leq \mathcal{M} \cdot C - k - 1 \leq C^2 = c_n^2(9-n).$$

But this would imply $1 \leq (c_n^2 - 1)(9-n) \leq 0$, which obviously gives a contradiction. □

Remark 4.8. As mentioned in the introduction, a characterization of the k-very ampleness of classes $\mathcal{L}$ on X_n can also be obtained by applying the results of Ballico and Coppens; see [1, Proposition 2.2]. In particular, this proposition implies the following:

Theorem 4.9. *A standard class $\mathcal{L} = |dH - m_1 E_1 - \cdots - m_n E_n| \in \operatorname{Pic} X_n$, with $d = t + k$, $k, t, n \geq 1$ and $m_n \geq k$ is k-very ample on X_n if $m_i + m_j \leq t$ for all $i \neq j$ and $h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t) \otimes \mathcal{I}_Z) = 0$ with $Z = m_1 P_1 + \cdots + m_n P_n$.*

The condition $h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t) \otimes \mathcal{I}_Z) = 0$ can be checked in any given particular case using the results from [10], but it is not possible to write down numerical conditions (as in Remark 4.2). The main reason for this is that the class $\mathcal{L}_t := |tH - m_1 E_1 - \cdots - m_n E_n|$ does not have to be standard. One can however fairly easily see that if $3t < m_1 + \cdots + m_n$, then $h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t) \otimes \mathcal{I}_Z) = h^1(X_n, \mathcal{L}_t) > 0$. So in order to have some idea as to how the result of this paper compares to the result of Ballico and Coppens, one can compare the condition (BC): $3t \geq m_1 + \cdots + m_n$ to the condition (DT): $3d \geq m_1 + \cdots + m_n + n - 9$. (Note that the other conditions of Theorem 4.1 are necessary conditions.) So one can easily see that (DT) is a weaker condition than (BC) if n is not "too big" with respect to k, more precisely if $n \leq 3k + 9$.

Observe also that if $d \geq m_1 + \cdots + m_n + k$, then from Theorem 1.3 we obtain the k-very ampleness of $\mathcal{L}$ on X_n. The conditions of Theorem 4.1 are weaker (we assume $3d \geq m_1 + \cdots + m_n + k + 2$ and $3d \geq m_1 + \cdots + m_n + n - 9$), but of course the situation treated in Theorem 4.1 is a very special case of the one treated in Theorem 1.3.

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Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, S22, B-9000 Ghent, Belgium

E-mail address: `cindy.devolder@ugent.be`

Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, PL-30348 Kraków, Poland – and – Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, PL-00956 Warszawa, Poland

E-mail address: `htutaj@im.uj.edu.pl`