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ON TRACES OF SOBOLEV FUNCTIONS ON THE BOUNDARY OF EXTENSION DOMAINS

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Abstract. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded $W^{1,p}$-extension domain and that μ is an upper d-Ahlfors measure on $\partial\Omega$ with $p \in (1, N)$ and $d \in (N-p, N)$. Then there exist continuous trace operators from $W^{1,p}(\Omega)$ into $L^q(\partial\Omega, d\mu)$ and into $B^p_\beta(\partial\Omega, d\mu)$ for every $q \in [1, dp/(N-p)]$ and every $\beta \in (0, 1-(N-d)/p]$.

1. The main result

The following theorem is the main result in this note.

Theorem 1.1. *Let $p \in [1, \infty]$ be fixed and let $\Omega \subset \mathbb{R}^N$ be a bounded $W^{1,p}$-extension domain and μ be an upper d-Ahlfors measure on a closed subset $S \subset \overline{\Omega}$ with $d \in (N-p, N) \cap (0, N)$.*

1. *Let $q = s = \infty$ if $p = N = 1$.*
2. *Let $q, s \in [1, \infty)$ if $p = N \geq 2$.*
3. *Let $q = s = \infty$ if $p > N$.*
4. *Let $q = dp/(N-p) > p$ and $s \in [1, q)$ if $1 < p < N$.*

Then there exists a mapping $T : W^{1,p}(\Omega) \to L^q(S, d\mu)$ satisfying

- *$T : W^{1,p}(\Omega) \to L^q(S, d\mu)$ is linear and continuous,*
- *$T : W^{1,p}(\Omega) \to L^s(S, d\mu)$ is compact,*
- *$Tu = u|_S$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.*

Of particular interest is the case when S equals the boundary of Ω.

Proof. Case (3) is obvious since $W^{1,p}(\Omega)$ is compactly embedded into $C(\overline{\Omega})$ by the Theorem of Rellich-Kondrachov. For case (1) one has to note that Ω is an open and bounded interval in $\mathbb{R}$. Case (2) follows from Corollary 7.4, and case (4) follows from Corollary 7.3. □

2. Preliminaries

In this article we will distinguish between pointwise defined functions and equivalence classes of functions. For example, the L^p-spaces consist of equivalence classes of p-integrable functions which coincide up to a set of measure zero. Another example are the Sobolev spaces $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open set. These spaces

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consist of (equivalence classes of) functions $u \in L^p(\Omega)$ such that the distributional derivatives $D_j u$ $(j = 1, \dots, N)$ belong again to $L^p(\Omega)$. Equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{L^p(\Omega)} ,$$

it is a Banach space. It is well-known that for every $f \in W^{1,p}(\Omega)$ there exists a pointwise defined function $u \in f$ which is Cap_p-quasi continuous on Ω. Here $\mathrm{Cap}_p : \mathcal{P}(\mathbb{R}^N) \to [0, \infty]$ denotes the classical p-capacity given by

$$\mathrm{Cap}_p(A) := \inf \left\{ \|u\|^p_{W^{1,p}(\mathbb{R}^N)} : u \geq 1 \text{ a.e. on a neighborhood of } A \right\} .$$

Since such a Cap_p-quasi continuous representative is unique up to a Cap_p-polar set, we can redefine the Sobolev space $W^{1,p}(\Omega)$ as follows:

$$\mathcal{W}^{1,p}(\Omega) := \left\{ [u] : u \in f \in W^{1,p}(\Omega) \text{ is } \mathrm{Cap}_p \text{ --quasi continuous} \right\} ,$$

where the above equivalence class $[u]$ consists of all Cap_p-quasi continuous functions v such that $v = u$ almost everywhere on Ω.

3. Nullspaces

In order to define general trace operators we have to introduce nullspaces and equivalence classes of functions related to such nullspaces.

Definition 3.1. Let M be an arbitrary set. Then we call a set $\mathcal{N} \subset \mathcal{P}(M)$ a *nullspace* on M if $\mathcal{N}$ satisfies the following two conditions:

(N1) $\emptyset \in \mathcal{N}$;

(N2) $A_n \in \mathcal{N}$ for $n \in \mathbb{N}$ implies $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{N}$.

On the vector space $\mathcal{F}(M)$ of all functions from M into $\mathbb{R}$ we define the equivalence relation $\sim_{\mathcal{N}}$ related to a nullspace $\mathcal{N}$ on M by

$$f \sim_{\mathcal{N}} g :\Leftrightarrow \exists N \in \mathcal{N} \text{ such that } f = g \text{ on } M \setminus N. \tag{3.1}$$

Then the space $F(M, \mathcal{N}) := \mathcal{F}(M)/ \sim_{\mathcal{N}}$ consists of equivalence classes of functions from M into $\mathbb{R}$ which coincide outside a nullset $N \in \mathcal{N}$. The name 'nullset' indicates that the set N is in some sense 'small', such as the nullsets for a measure or the polar sets for a capacity.

4. The trace

In this section we will define what we mean by a trace operator $\mathrm{Tr} : X \to Y$ for certain function spaces X and Y. Note that we do not require any topology on the range space Y.

Definition 4.1. Let B be a topological space, A a subset of B and let $\mathcal{N}_A$ and $\mathcal{N}_B$ be nullspaces on A and B, respectively. Let X be a subspace of $F(B, \mathcal{N}_B)$ equipped with a norm $\|\cdot\|_X$ such that $C(B) \cap X$ is dense in X and let Y be a subspace of $F(A, \mathcal{N}_A)$. Then we say that Y is a *trace space* of X if the following holds:

> *For every f in X there exists an element $f_A \in Y$ such that for every sequence $f_n \in C(B) \cap X$ which converges to f in X and for every $u_A \in f_A$ there exists a subsequence f_{n_k} and a set $N \in \mathcal{N}_A$ such that $f_{n_k}(x) \to u_A(x)$ for every $x \in A \setminus N$.*

If Y is a trace space of X, then $f_A \in Y$ is unique in Y. Therefore we can define the *trace operator* $\mathrm{Tr}_{X,Y} : X \to Y$ by $\mathrm{Tr}_{X,Y} f := f_A$.

Assume that Y is a trace space of X. Then the following are immediate consequences.

- $\mathrm{Tr}_{X,Y} : X \to Y$ is linear.
- $\mathrm{Tr}_{X,Y}\, u = u|_A$ if $u \in C(B) \cap X$.

Example 4.2. Let $\Omega \subset \mathbb{R}^N$ be an open set, $p \in (1, \infty)$, A be a subset of Ω, $X := W^{1,p}(\Omega)$ and let $Y := F(A, \mathcal{N}_p)$ where $\mathcal{N}_p$ is the nullspace which consists of all Cap_p-polar sets contained in A. Then Y is a trace space of X and the (unique) trace operator $\mathrm{Tr} : X \to Y$ is given by

$$\mathrm{Tr}\, f := \tilde{f}|_A,$$

where $\tilde{f}$ denotes the Cap_p-quasi continuous representative of f. In fact, if $u_n \in C(\Omega) \cap W^{1,p}(\Omega)$ converges to f in $W^{1,p}(\Omega)$, then there exists a subsequence $(u_{n_k})_k$ which converges Cap_p-quasi everywhere to $\tilde{f}$.

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $X \subset W^{1,p}(\Omega)$ be a subspace. In order to get a trace for functions in X on the boundary $\partial\Omega$ of Ω we will consider X as a subspace of $F(\overline{\Omega}, \mathcal{N})$, where $\mathcal{N}$ is given by

$$\mathcal{N} := \left\{ A \subset \overline{\Omega} : \lambda(A \cap \Omega) = 0 \right\}$$

and λ is the Lebesgue measure in $\mathbb{R}^N$. This leads to two main difficulties:

(1) Since in general $C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is not dense in $W^{1,p}(\Omega)$, we cannot take $X = W^{1,p}(\Omega)$. Therefore we will consider $X := \tilde{W}^{1,p}(\Omega)$, where $\tilde{W}^{1,p}(\Omega)$ is defined to be the closure of $C_c(\overline{\Omega}) \cap W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$. Note that $W^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega)$ whenever the boundary of Ω is smooth.

(2) For the range space Y it is in general not possible to take the space $Y := F(\partial\Omega, \mathcal{N}_p)$, where $\mathcal{N}_p$ denotes the set of all Cap_p-polar subsets of $\partial\Omega$. In fact (see [4, Example 2.5.5]) there exist a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$, a compact set $A \subset \partial\Omega$ with $\mathrm{Cap}_p(A) > 0$ and a sequence $f_n \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ such that $f_n \equiv 1$ on A and $f_n \to 0$ in $W^{1,p}(A)$. Hence the trace of the zero-function cannot be defined up to a Cap_p-polar set. This problem can be solved via the relative capacity introduced in the following section.

5. The relative capacity

In this section we will introduce the relative p-capacity and we will cite the theorems which we will need in the following. The relative 2-capacity $\mathrm{Cap}_{2,\Omega}$ was first introduced by W. Arendt and M. Warma [3] to study the Robin Laplacian on 'arbitrary' domains. For the extension to $p \in (1, \infty)$ we refer to [6].

Definition 5.1. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in (1, \infty)$. Then the relative p-capacity $\mathrm{Cap}_{p,\Omega} : \mathcal{P}(\overline{\Omega}) \to [0, \infty]$ is defined by

$$\mathrm{Cap}_{p,\Omega}(A) := \inf \left\{ \|u\|^p_{W^{1,p}(\Omega)} : u \in Y_{p,\Omega}(A) \right\},$$

where $Y_{p,\Omega}(A)$ consists of all functions $u \in \tilde{W}^{1,p}(\Omega)$ such that $u \geq 1$ almost everywhere on $O \cap \Omega$ for a neighborhood O of A.

Theorem 5.2 ([5, Theorem 3.22] or [6, Theorem 3.27]). *For every $u \in \tilde{W}^{1,p}(\Omega)$ there is a $\mathrm{Cap}_{p,\Omega}$-quasi continuous representative $\tilde{u} : \overline{\Omega} \to \mathbb{R}$ which is unique up to a $\mathrm{Cap}_{p,\Omega}$-polar set in $\overline{\Omega}$.*

This theorem suggests that there exists a trace operator from $\tilde{W}^{1,p}(\Omega)$ into $Y := F(\partial\Omega, \mathcal{N}_p(\Omega))$ (given by $\mathrm{Tr}\, u := \tilde{u}|_{\partial\Omega}$), where $\mathcal{N}_p(\Omega)$ consists of all $\mathrm{Cap}_{p,\Omega}$-polar sets in $\partial\Omega$. In fact, this follows from

Theorem 5.3 ([5, Consequence of Theorem 3.24] or [6, Theorem 3.29]). *Let $\Omega \subset \mathbb{R}^N$ be an open set, $p \in (1,\infty)$ and let $u_n \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ be a convergent sequence in $W^{1,p}(\Omega)$ with limit u. Then there is a subsequence $(u_{n_k})_k$ such that $u_{n_k} \to \tilde{u}$ $\mathrm{Cap}_{p,\Omega}$-quasi everywhere on $\overline{\Omega}$.*

In many concrete partial differential equations and variational problems it is useful to know that the trace operator maps $\tilde{W}^{1,p}(\Omega)$ into $L^q(\partial\Omega, \mu)$ for some $q \geq 1$ and some Borel measure μ on $\partial\Omega$. Obviously, a necessary condition for that is the validity of the following implication:

$$\mathrm{Cap}_{p,\Omega}(A) = 0 \implies \mu(A) = 0 \qquad \text{for all Borel sets } A \subset \partial\Omega. \tag{5.1}$$

We will call a Borel measure μ on $\partial\Omega$ $\mathrm{Cap}_{p,\Omega}$-*admissible* if (5.1) holds. To get a large class of admissible measures the following relation between the p-capacity and the relative p-capacity will be important.

Theorem 5.4 ([5, Consequence of Theorem 3.14] or [6, Theorem 3.20]). *Let $\Omega \subset \mathbb{R}^N$ be a $W^{1,p}$-extension domain. Then there exists a constant $C > 0$ such that*

$$\mathrm{Cap}_p(A) \leq C \cdot \mathrm{Cap}_{p,\Omega}(A) \leq C \cdot \mathrm{Cap}_p(A)$$

for every set $A \subset \overline{\Omega}$.

6. Traces for upper/lower Ahlfors measures

In this section we show that if $\Omega \subset \mathbb{R}^N$ is a bounded $W^{1,p}$-extension domain $(1 < p < N)$ and μ is an upper d-Ahlfors measure on $\partial\Omega$ with $0 < N - d < p$, then there exists a continuous trace operator from $W^{1,p}(\Omega)$ into some Besov spaces on the boundary $\partial\Omega$. Note that we do not assume that Ω is a so-called (ε, δ)-domain as required in [7, Theorem 10.6].

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^N$ be a $W^{1,p}$-extension domain and let $E_1, E_2 : W^{1,p}(\Omega) \to \mathcal{W}^{1,p}(\mathbb{R}^N)$ be two (possibly nonlinear) extension operators. Then $E_1 f = E_2 f$ Cap_p-quasi everywhere on $\overline{\Omega}$ for every $f \in W^{1,p}(\Omega)$.*

Proof. Let $f \in W^{1,p}(\Omega)$ and let $f_j := E_j f$, $j = 1, 2$. Then f_1 and f_2 are $\mathrm{Cap}_{p,\Omega}$-quasi continuous on $\overline{\Omega}$ and $f_1 = f_2$ a.e. on Ω. By Theorem 5.2 we get that $f_1 = f_2$ $\mathrm{Cap}_{p,\Omega}$-q.e. on $\overline{\Omega}$, and hence by Theorem 5.4 it follows that $f_1 = f_2$ Cap_p-q.e. on $\overline{\Omega}$. □

Remark 6.2. We can reformulate Theorem 6.1 as follows. Let $\Omega \subset \mathbb{R}^N$ be a bounded $W^{1,p}$-extension domain and let $u_1, u_2 \in \mathcal{W}^{1,p}(\mathbb{R}^N)$. If $u_1 = u_2$ almost everywhere on Ω, then $u_1 = u_2$ Cap_p-quasi everywhere on $\overline{\Omega}$.

The existence of a trace operator is given in the following.

Corollary 6.3. *If Ω is a $W^{1,p}$-extension domain and $S \subset \overline{\Omega}$, then there exists a unique trace operator* $\mathrm{Tr} : W^{1,p}(\Omega) \to F(S, \mathcal{N}_p)$*, where* $\mathcal{N}_p$ *consists of all* Cap_p*-polar sets in S, given by*

$$\mathrm{Tr}\, f := Ef|_S$$

for every extension operator $E : W^{1,p}(\Omega) \to \mathcal{W}^{1,p}(\mathbb{R}^N)$.

Definition 6.4. Let $S \subset \mathbb{R}^N$ be a compact set and let $d \in (0, N)$. Then we call a Borel measure μ with $\mathrm{supp}(\mu) \subset S$ an *upper d-Ahlfors measure* on S if there exist constants $M, R_0 > 0$ such that

$$\mu(B(x,r)) \leq Mr^d \quad \text{for all } 0 < r < R_0 \text{ and } x \in S. \tag{6.1}$$

If (6.1) holds with $\leq$ replaced by $\geq$, then we will call μ a *lower d-Ahlfors measure*. Note that for $s := N - d$ an upper/lower d-Ahlfors measure on S is called an upper/lower s-Ahlfors measure in [7].

Remark 6.5. Let $p \in (1, N)$ and $S \subset \mathbb{R}^N$ be compact. If μ is an upper d-Ahlfors measure on S with $N - p < d < N$, then

$$\mathrm{Cap}_p(A) = 0 \quad \Longrightarrow \quad \mathcal{H}^d(A) = 0 \quad \Longrightarrow \quad \mu(A) = 0$$

for all Borel sets $A \subset S$. Here $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure on $\mathbb{R}^N$.

The following result follows from D. Danielli, N. Garofalo and D. Nhieu; see the beautiful Theorem 8.6 in [7] and A. Jonsson [10, Theorem 3].

Theorem 6.6 (Interior sharp trace inequality)**.** *Let $S \subset \mathbb{R}^N$ be a compact set, $p \in (1, N)$, $N - p < d < N$ and μ be an upper d-Ahlfors measure on S. Then for every $0 < \beta \leq 1 - (N-d)/p$ there exists a constant $C > 0$ such that*

$$\|f\|_{B^p_\beta(F, d\mu)} \leq C \cdot \|f\|_{W^{1,p}(\mathbb{R}^N)}$$

for all $f \in W^{1,p}(\mathbb{R}^N)$.

Here the *Besov space* $B^p_\beta(S, d\mu)$ is defined to be the vector space of all $f \in L^p(S, d\mu)$ for which

$$N^p_\beta(f, S, d\mu)^p := \int_S \int_S \frac{|f(x) - f(y)|^p}{|x-y|^{\beta p + d}}\, d\mu(y)\, d\mu(x) < \infty.$$

The norm on $B^p_\beta(S, d\mu)$ is given by $\|\cdot\|_{B^p_\beta(S,d\mu)} := \|\cdot\|_{L^p(S,d\mu)} + N^p_\beta(\cdot, S, d\mu)$. Now we can prove the following trace theorem.

Theorem 6.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $W^{1,p}$-extension domain, $S \subset \overline{\Omega}$ closed, $p \in (1, N)$, $N - p < d < N$ and μ be an upper d-Ahlfors measure on S. Then for every $0 < \beta \leq 1 - (N-d)/p$ there exists a constant $C > 0$ such that*

$$\|\mathrm{Tr}\, f\|_{B^p_\beta(S,d\mu)} \leq C \cdot \|f\|_{W^{1,p}(\Omega)} \qquad \textit{for every } f \in W^{1,p}(\Omega).$$

Proof. Let $E : W^{1,p}(\Omega) \to \mathcal{W}^{1,p}(\mathbb{R}^N)$ be a linear and bounded extension operator. It follows from Theorem 6.6, using that $Ef = \mathrm{Tr}\, f$ μ-a.e. on S, that

$$\|\mathrm{Tr}\, f\|_{B^p_\beta(S,d\mu)} \leq C_2 \cdot \|Ef\|_{W^{1,p}(\Omega)} \leq C \cdot \|f\|_{W^{1,p}(\Omega)}$$

for all $f \in W^{1,p}(\Omega)$. $\square$

The following result follows from D. Danielli, N. Garofalo and D. Nhieu; see Theorem 11.1 in [7]

Theorem 6.8 (Embedding a Besov space into a Lebesgue space). *Let $S \subset \mathbb{R}^N$ be a compact set, $\beta \in (0,1)$, $p \geq 1$ and let μ be a lower d-Ahlfors measure on S with $d \in (\beta p, N)$. Then for $q := pd/(d-\beta p) > p$ there exists a constant $C > 0$ such that*

$$\|f\|_{L^q(S,d\mu)} \leq C \cdot \|f\|_{B^p_\beta(S,d\mu)} \qquad \text{for every } f \in B^p_\beta(S, d\mu).$$

Now we can prove the following trace result.

Proposition 6.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $W^{1,p}$-extension domain, $S \subset \overline{\Omega}$ closed, $p \in (1, N)$, $N - p < d < N$ and μ be a d-Ahlfors measure on S. Then there exists a constant $C > 0$ such that*

$$\|\operatorname{Tr} f\|_{L^q(S,d\mu)} \leq C \cdot \|f\|_{W^{1,p}(\Omega)}$$

for every $f \in W^{1,p}(\Omega)$ and $p \leq q \leq pd/(N-p)$.

Proof. It is sufficient to prove the limit case $q := pd/(N-p)$. We get from Theorem 6.7 with $\beta := 1 - (N-d)/p \in (0,1)$ that there exists a constant C_1 such that

$$\|\operatorname{Tr} f\|_{B^p_\beta(S,d\mu)} \leq C_1 \cdot \|f\|_{W^{1,p}(\Omega)} \tag{6.2}$$

for all $f \in W^{1,p}(\Omega)$. From Theorem 6.8, using that $q = pd/(d - \beta p)$ and $\beta p < d$, we get that there exists a constant $C_2 > 0$ such that

$$\|\operatorname{Tr} f\|_{L^q(S,d\mu)} \leq C_2 \cdot \|\operatorname{Tr} f\|_{B^p_\beta(S,d\mu)}\,. \tag{6.3}$$

Combining (6.2) and (6.3) proves the claim. □

Remark 6.10. In the case when Ω is a bounded Lipschitz domain, $S = \partial\Omega$ and μ is the surface measure $\sigma = \mathcal{H}^{N-1}|_{\partial\Omega}$, then we get that there exists a constant $C > 0$ such that

$$\|\operatorname{Tr} f\|_{L^q(\partial\Omega,d\sigma)} \leq C \cdot \|f\|_{W^{1,p}(\Omega)}$$

for every $f \in W^{1,p}(\Omega)$ and $1 \leq q \leq p(N-1)/(N-p)$. This is a well-known result; see Adams [2, Theorem 5.22].

7. Traces for upper Ahlfors measures

If one is only interested in traces of Sobolev functions in L^q-spaces, there is no need to pass through a Besov space as we will show here. Note that the heart of all these trace results is contained in Theorems 5.4 and 6.1. We will use the following theorem, which goes back to D. R. Adams (see [1, Theorem 7.2.2]).

Theorem 7.1. *Let $1 < p < q < \infty$ and μ be a (positive) Radon measure on $\mathbb{R}^N$. Then the following are equivalent.*

1. *$I : \mathcal{W}^{1,p}(\mathbb{R}^N) \to L^q(\mathbb{R}^N, d\mu)$ is well-defined and continuous.*
2. *There exists a constant $C > 0$ such that for all $x \in \mathbb{R}^N$ and $r \in (0,1]$,*

$$\mu(B(x,r)) \leq C \cdot \operatorname{Cap}_p(B(x,r))^{q/p}.$$

From this beautiful theorem we get immediately the following result.

Theorem 7.2. *Let $1 < p < q < \infty$, $\Omega \subset \mathbb{R}^N$ be a bounded $W^{1,p}$-extension domain, $S \subset \overline{\Omega}$ closed and μ be a (positive) Radon measure on S. Then the following are equivalent.*

(1) $\mathrm{Tr} : \mathcal{W}^{1,p}(\Omega) \to L^q(S, d\mu)$ *is well-defined and continuous.*
(2) *There exists a constant $C > 0$ such that for all $x \in S$ and $r \in (0, 1]$,*

$$\mu(B(x, r)) \le C \cdot \mathrm{Cap}_p(B(x, r))^{q/p}.$$

For concrete applications one is interested in having a sufficient and easy to verify condition on the measure μ to get the continuity of the trace operator. This is obtained in the following.

Corollary 7.3. *Let $p \in (1, N)$ and $\Omega \subset \mathbb{R}^N$ be a bounded $W^{1,p}$-extension domain and $S \subset \overline{\Omega}$ be closed. If μ is an upper d-Ahlfors measure on S with $d \in (N - p, N)$, then*

$$\mathrm{Tr} : W^{1,p}(\Omega) \to L^q(S, d\mu)$$

is continuous for $q := dp/(N - p) > p$ and

$$\mathrm{Tr} : W^{1,p}(\Omega) \to L^s(S, d\mu)$$

is compact for all $s \in [1, q)$.

Proof. Using that there exists a constant $C_1 > 0$ such that $\mathrm{Cap}_p(B(x, r)) \ge C_1 \cdot r^{N-p}$ for all $x \in \mathbb{R}^N$ and $r > 0$, we get that

$$\mu(B(x, r)) \le Mr^d \le C_2 \cdot \mathrm{Cap}_p(B(x, r))^{d/(N-p)} = C_2 \cdot \mathrm{Cap}_p(B(x, r))^{q/p}$$

for all $x \in \partial\Omega$ and $r \in (0, 1]$. Hence by Theorem 7.2 we get the continuity of $\mathrm{Tr} : W^{1,p}(\Omega) \to L^q(\partial\Omega, d\mu)$. The compactness follows from [1, Theorem 7.3.2]; note that $\mu(\partial\Omega) < \infty$. □

Corollary 7.4. *Let $p \ge N \ge 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded $W^{1,p}$-extension domain, $S \subset \overline{\Omega}$ be a closed set, μ be an upper d-Ahlfors measure on S with $d \in (0, N)$ and let $q \in [1, \infty)$. Then there exists a linear and compact trace operator*

$$\mathrm{Tr} : W^{1,p}(\Omega) \to L^q(S, d\mu)$$

such that $\mathrm{Tr}\, u = u|_S$ for all $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$.

Proof. Let $s \in (1, N)$ be such that $d \in (N - s, N)$ and $r := ds/(N - s) > q$ and let B be an open ball such that $\overline{\Omega} \subset B$. Denote by E an extension operator from $W^{1,p}(\Omega) \to W^{1,p}(B)$ and let $\mathrm{Tr}_2 : W^{1,s}(B) \to L^r(S, d\mu)$ be given by Corollary 7.3. Then Tr given by

$$W^{1,p}(\Omega) \xrightarrow{E} W^{1,p}(B) \hookrightarrow W^{1,s}(B) \xrightarrow{\mathrm{Tr}_2} L^{(r+q)/2}(S, d\mu) \hookrightarrow L^q(S, d\mu)$$

is compact. □

8. Final remarks

The following interesting result, which gives a relation between $W^{1,p}$-extension domains and trace operators, is obtained by P. Harjulehto [9, Theorem 4.3]:

Theorem 8.1. *Let $p \in (N - 1, \infty)$, $\Omega \subset \mathbb{R}^N$ be a bounded domain such that its boundary is d-regular, $d \in [N - 1, N)$. Then the following are equivalent:*

(1) Ω *is a $W^{1,p}$-extension domain.*

(2) *There exists a bounded linear trace operator*

$$T : W^{1,p}(\Omega) \to B^p_{1-(N-d)/p}(\partial\Omega, d\mathcal{H}^d)$$

such that $Tu = \lim_{r\to 0} \lambda(\Omega(x,r))^{-1} \int_{\Omega(x,r)} u(x)\, dx$ $\mathcal{H}^d$*-a.e. on* $\partial\Omega$ *and Lipschitz continuous functions are dense in* $W^{1,p}(\Omega)$.

Here $\Omega(x,r) := \Omega \cap B(x,r)$.

For further and related results we refer the reader to Adams and Hedberg [1], Hajłasz and Martio [8], Jonsson [10], Maz'ya [11], Wallin [12] and the references therein.

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