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A UNIFORM ESTIMATE FOR FOURIER RESTRICTION TO SIMPLE CURVES

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ABSTRACT. We prove a uniform Fourier extension-restriction estimate for a certain class of curves in $\mathbb{R}^d$.

1. INTRODUCTION

Let γ be a curve in $\mathbb{R}^d$ given by

$$\gamma(t) = \Big(t, \frac{t^2}{2}, \dots, \frac{t^{d-1}}{(d-1)!}, \phi(t)\Big) \tag{1.1}$$

where $\phi \in C^{(d+2)}(a,b)$ and $\phi^{(j)}(t) > 0$ for $t \in (a,b)$, $j = 0,1,2,\dots,d+2$. Such curves are termed *simple* in [5]. Write $\omega(t)$ for $\phi^{(d)}(t)$. The purpose of this note is to prove a uniform Fourier extension-restriction theorem for affine arclength measure on curves (1.1):

Theorem 1.1. *Suppose λ is the measure on γ given by $d\lambda = \omega(t)^{2/(d^2+d)}dt$. If $1 \le p < d+2$ and $\frac{1}{p} + \frac{d(d+1)}{2}\frac{1}{q} = 1$, then there is $C(p,d)$ such that the following estimate holds:*

$$\|\widehat{f\,d\lambda}\|_q \le C(p,d)\,\|f\|_{L^p(\lambda)}.$$

If $d = 2$ this is just the theorem in [7], a result originally established in slightly more generality, but with a more complicated proof, by Sjölin in [9]. This note is the result of an attempt to apply the method from [7] in higher dimensions. Theorem 1.1, which is an immediate consequence of Theorems 1.2 and 1.3 below, is somewhat analogous to the result of [4] on two fronts: it is a direct consequence of a geometric inequality (Theorem 1.2 here) combined with a fairly simple argument (Theorem 1.3 here), and its range of exponents p is the (probably suboptimal) range obtained by Christ [3]. For a better range of p when ω satisfies a certain auxiliary condition, see Theorem 1.1 in [2]. For some of the history of the problem of restricting Fourier transforms to curves, see [1].

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Theorem 1.2. *There exists $C = C(d)$ such that the estimate*

(1.2)

$$\int_a^b \Big(\int_{\{t_1 \le t_i < b\}} \chi_F\big(\gamma(t_1) + \cdots + \gamma(t_d)\big) \Big[\prod_{i=2}^d \omega(t_i) \Big]^{2/(d^2+d)} dt_2 \cdots dt_d \Big)^{(d+2)/2} \times \omega(t_1)^{2/(d^2+d)}\, dt_1 \le C(d)\, m_d(F)$$

holds for Borel $F \subset \mathbb{R}^d$.

The next result is Theorem 7 in [8].

Theorem 1.3. *Suppose that λ is a nonnegative Borel measure on $\mathbb{R}^d$ satisfying the inequality*

$$\int \Big(\int_{\{\tau(y_1) \le \tau(y_i)\}} \chi_F(y_1 + y_2 + \cdots + y_m)\, d\lambda(y_2) \cdots d\lambda(y_m) \Big)^{\frac{m+2}{2}} d\lambda(y_1) \le c\ m_d(F)$$

for some nonnegative integer $m \ge 3$, some real-valued Borel function τ on supp(λ), and all Borel $F \subseteq \mathbb{R}^d$. Then the adjoint restriction estimate

$$\|\widehat{f d\lambda}\|_q \le C(c,p)\ \|f\|_{L^p(\lambda)}$$

holds whenever $\frac{1}{p} + \frac{m(m+1)}{2}\frac{1}{q} = 1$ and $1 \le p < m+2$.

The next section contains the proof of Theorem 1.2, while §3 contains the proofs of certain lemmas used in §2.

2. Proof of Theorem 1.2

The idea for proving Theorem 1.2, from [8], is to regard (1.2) as an $L^{(d+2)/2,1} \to L^{(d+2)/2}$ estimate for a certain operator T and to establish (1.2) by establishing the dual $L^{(d+2)/d} \to L^{(d+2)/d,\infty}$ estimate for T^*. Let $J(t_1, \dots, t_d)$ be the absolute value of the Jacobian determinant for the map

$$(t_1, \dots t_d) \mapsto \gamma(t_1) + \cdots + \gamma(t_d).$$

Then

$$\begin{aligned} \langle f, T^* g \rangle &= \langle Tf, g \rangle \\ &= \int_a^b \Big(\int_{\{t_1 \le t_i < b\}} f\big(\gamma(t_1) + \cdots + \gamma(t_d)\big) \Big[\prod_{i=2}^d \omega(t_i) \Big]^{2/(d^2+d)} dt_2 \cdots dt_d \Big) \\ &\qquad \times g(t_1)\, \omega(t_1)^{2/(d^2+d)}\, dt_1 \\ &= \int_{\{a < t_1 \le t_i < b\}} \Big(\frac{g(t_1)}{J(t_1, \dots, t_d)} \Big[\prod_{i=1}^d \omega(t_i) \Big]^{2/(d^2+d)} \Big) f\big(\gamma(t_1) + \cdots + \gamma(t_d)\big) \\ &\qquad \times J(t_1, \dots, t_d)\, dt_d \cdots dt_1 \end{aligned}$$

so that the desired $L^{(d+2)/d} \to L^{(d+2)/d,\infty}$ estimate for nonnegative g,

$$\lambda^{(d+2)/d} m_d\{T^* g \ge \lambda\} \le C(d) \int_a^b g(t_1)^{(d+2)/d} \omega(t_1)^{2/(d^2+d)}\, dt_1 \quad (\lambda > 0),$$

will follow from the estimate

(2.1)

$$\lambda^{(d+2)/d}\int \chi_{\widetilde{E}}(t_1,\dots,t_d)\,J(t_1,\dots,t_d)\,dt_d\cdots dt_1 \leq C(d)\int_a^b g(t_1)^{(d+2)/d}\,\omega(t_1)^{2/(d^2+d)}\,dt_1,$$

where

$$\widetilde{E}=\Big\{(t_1,\dots,t_d): a<t_1\leq\cdots\leq t_d<b,\ \frac{g(t_1)}{J(t_1,\dots,t_d)}\Big[\prod_{i=1}^d \omega(t_i)\Big]^{2/(d^2+d)}\geq\lambda\Big\}.$$

(The change of variables implicit in this argument can be justified as in [6], p. 549.) By absorbing λ into g we can assume $\lambda=1$. Thus (2.1) will follow from integrating the inequality

$$\int \chi_{\widetilde{E}}(t_1,\dots,t_d)\,J(t_1,\dots,t_d)\,dt_d\cdots dt_2 \leq C(d)\,g(t_1)^{(d+2)/d}\,\omega(t_1)^{2/(d^2+d)} \tag{2.2}$$

with respect to t_1. Lemma 2.3 in [2] shows that there is a nonnegative function $\psi=\psi(u;t_1,\dots,t_d)$ supported in $[t_1,t_d]$ such that

$$J(t_1,\dots,t_d)=\int_{t_1}^{t_d}\omega(u)\,\psi(u;t_1,\dots,t_d)\,du \tag{2.3}$$

and so (2.2) will follow from the inequality

$$\int \chi_{\widetilde{E}}(t_1,\dots,t_d)\,J(t_1,\dots,t_d)\,dt_d\cdots dt_2 \leq C(d)\,c^{(d+2)/d}\,\omega(t_1)^{2/(d^2+d)}, \tag{2.4}$$

to hold for $c>0$ and $t_1\in(a,b)$, where now

$$\widetilde{E}=\Big\{(t_1,\dots,t_d): a<t_1\leq\cdots\leq t_d<b,\ \int_{t_1}^{t_d}\omega(u)\,\psi(u;t_1,\dots,t_d)\,du\leq c\Big[\prod_{i=1}^d\omega(t_i)\Big]^{2/(d^2+d)}\Big\}.$$

Homogeneity allows absorbing c into ω, so we can assume $c=1$. With $t_1>a$ fixed, then, and with

(2.5)

$$E=\Big\{(t_2,\dots,t_d): t_1\leq t_2\leq\cdots\leq t_d<b,\ J(t_1,\dots,t_d)\leq\Big[\prod_{i=1}^d\omega(t_i)\Big]^{2/(d^2+d)}\Big\},$$

inequality (2.4), and so (1.2), will follow from

$$\int \chi_E(t_2,\dots,t_d)\Big[\prod_{i=2}^d\omega(t_i)\Big]^{2/(d^2+d)}dt_d\cdots dt_2\leq C(d). \tag{2.6}$$

To begin the proof of (2.6), let $J\subset\mathbb{Z}$ be an interval of integers such that $\{2^j: j\in J\}$ is the set of dyadic values assumed by ω on (a,b). For each $j\in J$, choose $a_j\in(a,b)$ such that $\omega(a_j)=2^j$. If J has a least element, say $j_{\min}$, we let $a_{j_{\min}-1}=a$ and append $j_{\min}-1$ to J. If J has a greatest element, we make a similar accommodation. Then, writing $I_j=[a_j,a_{j+1})\cap(a,b)$, we obtain a partition $\{I_j\}_{j\in J}$ of (a,b).

The proof of (2.6) simplifies significantly in the case $d = 2$. For the sake of the reader who would see the main idea of the proof but be spared its unfortunately complicated notation, we will first sketch the proof in that special case. So assume, for the moment, that $d = 2$. Since $\psi(u; t_1, t_2) = \chi_{(t_1,t_2)}(u)$, (2.3) shows that (2.5) becomes

$$E = \Big\{t_2 : t_2 \geq t_1,\ \int_{t_1}^{t_2} \omega(u)\, du \leq [\omega(t_1)\omega(t_2)]^{1/3}\Big\}, \tag{2.7}$$

while (2.6) becomes

$$\int \chi_E(t_2)\, \omega(t_2)^{1/3}\, dt_2 \leq C. \tag{2.8}$$

Let $E_j = E \cap I_j$ and $s_j = \sup E_j$. Then, with j_1 chosen so that $t_1 \in I_{j_1}$, there are the inequalities

$$\int_{E_j} \omega(u)\, du \leq \int_{t_1}^{s_j} \omega(u)\, du \leq [\omega(t_1)\omega(s_j)]^{1/3} \lesssim (2^{j_1}\, 2^j)^{1/3}$$

(by (2.7) and the definitions of s_j and E_j). Thus

$$\int_{E_j} \omega(u)^{1/3}\, du \sim 2^{-2j/3} \int_{E_j} \omega(u)\, du \lesssim (2^{j_1}\, 2^{-j})^{1/3}.$$

Since $j \geq j_1$, (2.8) now follows by summing a geometric series.

We return now to the proof of the general case of (2.6). With $t_1 \in (a, b)$ fixed, say $t_1 \in I_{j_1}$, with E as in (2.5), and for integers $j_2 \leq j_3 \leq \cdots \leq j_d$ in J with $j_2 \geq j_1$, we set

$$E_{j_2\cdots j_d} = \{(t_2, \ldots, t_d) \in E : (t_2, \ldots, t_d) \in I_{j_2} \times \cdots \times I_{j_d}\}.$$

The desired estimate (2.6) will follow from

$$\sum_{j_2 \geq j_1} \cdots \sum_{j_d \geq j_{d-1}} \big(2^{j_2+\cdots+j_d}\big)^{2/(d^2+d)} m_{d-1}\big(E_{j_2\cdots j_d}\big) \leq C(d). \tag{2.9}$$

To establish (2.9) it is enough to show that, for each $(j_2, \ldots, j_d)$ figuring in the sum in (2.9), we have

$$\big(2^{j_2+\cdots+j_d}\big)^{1/(d-1)} m_{d-1}\big(E_{j_2\cdots j_d}\big)^{d/2} \leq C(d)\, \big(2^{j_1+j_2+\cdots+j_d}\big)^{2/(d^2+d)}. \tag{2.10}$$

In fact, some algebra shows that (2.10) is equivalent to

$$\big(2^{j_2+\cdots+j_d}\big)^{2/(d^2+d)} m_{d-1}\big(E_{j_2\cdots j_d}\big) \leq C(d)\, 2^{4j_1/(d^3+d^2)} 2^{-4(j_2+\cdots+j_d)/[(d^3+d^2)(d-1)]}$$

and so, given (2.10), (2.9) follows by summing a geometric series.

Moving towards the proof of (2.10), fix $(j_2, \ldots, j_d)$. In what follows we will often write $j(l)$ instead of j_l. Let $p_1 < p_2 < \cdots p_{k-1}$ be the indices i in $\{1, 2, \ldots, d-1\}$ for which $j(i+1) - j(i) \geq 2$ and set $p_0 = 0$ and $p_k = d$. Define $\ell_1, \ldots, \ell_k$ by $\ell_n = p_n - p_{n-1} - 1$ $(n = 1, \ldots, k)$, and observe that $\ell_1 + \cdots + \ell_k = d - k$. Then

$$\{j(1), j(2), \ldots, j(d)\}$$
$$= \{j(p_0+1), j(p_0+2), \ldots, j(p_1); j(p_1+1), \ldots, j(p_2); \ldots; j(p_{k-1}+1), \ldots, j(p_k)\}$$

where if $j(i)$ and $j(i+1)$ are separated by a semicolon, then $j(i+1) - j(i) \geq 2$, and otherwise $0 \leq j(i+1) - j(i) \leq 1$. Next we construct k subintervals J_n of (a, b) by setting, for $n = 1, \ldots, k$,

$$J_n = I_{j(p_{n-1}+1)} \cup I_{j(p_{n-1}+2)} \cup \cdots \cup I_{j(p_n)} \tag{2.11}$$

so that, recalling the definition of I_j, the endpoints c_n and d_n of J_n are given by $c_n = a_{j(p_{n-1}+1)}$ and $d_n = a_{j(p_n)+1}$. Note that $c_1 < d_1 < c_2 < d_2 < \cdots < c_k < d_k$ (see (2.13) below) and that if $(t_2, \ldots, t_d) \in E_{j_2 \ldots j_d}$, then

$$c_n \leq t_{p_{n-1}+1} \leq t_{p_{n-1}+2} \leq \cdots \leq t_{p_n} \leq d_n. \tag{2.12}$$

We will need the facts that if $n = 2, \ldots k$, then

$$d_{n-1} \leq a_{j(p_{n-1}+1)-1} < c_n \tag{2.13}$$

and

$$c_n - a_{j(p_{n-1}+1)-1} \gtrsim d_n - c_n. \tag{2.14}$$

(Throughout this note, the constants implied by symbols such as $\lesssim$ can easily be checked to depend only on d.) To see (2.14), note that because $\omega(a_j) = 2^j$ and ω' is nondecreasing we have

$$(a_{j+1} - a_j)\,\omega'(a_j) \leq \int_{a_j}^{a_{j+1}} \omega'(u)\,du = 2^j = 2\int_{a_{j-1}}^{a_j} \omega'(u)\,du \leq 2\,(a_j - a_{j-1})\,\omega'(a_j)$$

so that $(a_{j+1} - a_j) \leq 2\,(a_j - a_{j-1})$ and therefore

$$(a_{j+p} - a_{j+p-1}) \leq 2^p\,(a_j - a_{j-1}). \tag{2.15}$$

Now, by definition of p_{n-1}, $j(p_{n-1}) + 1 \leq j(p_{n-1} + 1) - 1$, and so $a_{j(p_{n-1}+1)-1}$ lies between $d_{n-1} = a_{j(p_{n-1})+1}$ and $c_n = a_{j(p_{n-1}+1)}$ in the sense of (2.13). Also, according to (2.11), $J_n = (c_n, d_n)$ is (up to endpoints) the union of no more than d intervals $I_j = [a_j, a_{j+1})$. By choice of the p_n, each interval but the first in the union in (2.11) is either identical to or contiguous to the one on its left. Since the first of these intervals is $(a_{j(p_{n-1}+1)}, a_{j(p_{n-1}+1)+1})$, (2.15) implies (2.14).

We now outline the proof of (2.10), beginning with a lemma (the proofs of the lemmas will be given in §3):

Lemma 2.1. *Suppose $t_1 < \cdots < t_d$ and $(\alpha_i, \beta_i) \subset (t_i, t_{i+1})$. Write $\Delta_i = \beta_i - \alpha_i$ and suppose*

$$f = \sum_{i=1}^{d-1} \nu_i\,\chi_{(\alpha_i,\beta_i)}$$

where $\nu_i \geq 0$. Fix $p \in \{1, \ldots, d-1\}$. Suppose

$$\{e_i : i = 1, 2, \ldots d-1, i \neq p\} = \{1, 2, \ldots, d-2\}.$$

Then

$$\int_{t_1}^{t_d} f(u)\,\psi(u; t_1, \ldots, t_d)\,du \gtrsim \nu_p\,\Delta_p^{d-1} \prod_{\substack{1 \leq i \leq d-1 \\ i \neq p}} \Delta_i^{e_i}.$$

With t_1 and $j_2, \ldots, j_d$ fixed and with $(t_2, \ldots, t_d) \in E_{j_2 \ldots j_d}$, we will apply Lemma 2.1 to a collection $\mathcal{I}$ of intervals (α_i, β_i) specified as follows: for $n = 1, \ldots, k$ and $i = p_{n-1} + 1, \ldots, p_n - 1$, the ℓ_n intervals (t_i, t_{i+1}) will be in $\mathcal{I}$; additionally, for $n = 2, \ldots, k$, the intervals $\tilde{J}_n$ defined by

$$\tilde{J}_n = (a_{j(p_{n-1}+1)-1}, c_n) \subset (t_{p_{n-1}}, t_{p_{n-1}+1}) \tag{2.16}$$

will be in $\mathcal{I}$ (we set $\widetilde{J}_1 = \emptyset$). Observe that there are integers $m_1 \leq m_2 \leq \cdots \leq m_k$ such that if

$$\mathcal{J}_n = \tilde{J}_n \cup \Big(\bigcup_{i=p_{n-1}+1}^{p_n-1} (t_i, t_{i+1}) \Big),$$

then

$$\omega \sim 2^{m_n} \text{ on } \mathcal{J}_n \ (n = 1, \dots, k). \tag{2.17}$$

(This is true because, according to (2.12), $\mathcal{J}_n$ is contained in the union of at most $\ell_n + 1 \le d$ contiguous intervals I_j, and $\omega \sim 2^j$ on I_j.) Then (2.10) can be written as

$$\begin{aligned} &2^{[\ell_1 m_1 + (\ell_2+1)m_2 + \cdots + (\ell_k+1)m_k]/(d-1)} m_{d-1}\big(E_{j_2\cdots j_d}\big)^{d/2} \\ &\qquad \lesssim 2^{2[(\ell_1+1)m_1 + (\ell_2+1)m_2 + \cdots + (\ell_k+1)m_k]/(d^2+d)}. \end{aligned} \tag{2.18}$$

Similarly, the inequality

$$J(t_1, \dots, t_d) \le \Big[\prod_{i=1}^{d} \omega(t_i)\Big]^{2/(d^2+d)}$$

in (2.5) implies that

$$J(t_1, \dots, t_d) \lesssim 2^{2[(\ell_1+1)m_1 + \cdots + (\ell_k+1)m_k]/(d^2+d)}. \tag{2.19}$$

Now, as we will see below, Lemma 2.1, (2.17), (2.19), and

$$J(t_1, \dots, t_d) = \int_{t_1}^{t_d} \omega(u)\, \psi(u; t_1, \dots, t_d)\, du$$

will yield certain estimates of the form

$$2^{m_n} \Delta_p^{d-1} \prod_{\substack{1 \le i \le d-1 \\ i \ne p}} \Delta_i^{e_i} \lesssim 2^{2[(\ell_1+1)m_1 + (\ell_2+1)m_2 + \cdots + (\ell_k+1)m_k]/(d^2+d)} \tag{2.20}$$

when $(\alpha_p, \beta_p) \subset \mathcal{J}_n$. A weighted geometric mean of these estimates (2.20) will give

$$\begin{aligned} &2^{[\ell_1 m_1 + (\ell_2+1)m_2 + \cdots + (\ell_k+1)m_k]/(d-1)}\, W_1^{d/(\ell_1+1)} \prod_{n=2}^{k} \Big(\rho_n^{(d+\ell_n)/2} W_n^{(d-1)/(\ell_n+1)}\Big) \\ &\qquad\qquad \lesssim 2^{2[(\ell_1+1)m_1 + (\ell_2+1)m_2 + \cdots + (\ell_k+1)m_k]/(d^2+d)}, \end{aligned} \tag{2.21}$$

where ρ_n is the length of $\tilde{J}_n$ and with the W_n's given by

$$W_n = W(t_{p_{n-1}+1}, \dots, t_{p_n})$$

where, for $s_1 \le \cdots \le s_m$,

$$W(s_1, \dots, s_m) = \sup\Big\{\prod_{i=1}^{m-1} (s_{i+1} - s_i)^{e_i} : \{e_1, \dots, e_{m-1}\} = \{1, \dots, m-1\}\Big\}.$$

Lemma 2.3 below will allow the choice of $(t_2, \dots, t_d) \in E_{j_2 \dots j_d}$ such that

$$m_{d-1}\big(E_{j_2\cdots j_d}\big)^{d/2} \lesssim W_1^{d/(\ell_1+1)} \prod_{n=2}^{k} \Big(\rho_n^{(d+\ell_n)/2} W_n^{(d-1)/(\ell_n+1)}\Big). \tag{2.22}$$

With (2.21) this will yield (2.18).

To give the details missing from the argument in the preceding paragraph we will need a lemma whose statement requires the introduction of some more notation: $\mathcal{A}_{d-1}$ will stand for the convex hull in $\mathbb{R}^{d-1}$ of the set of all permutations of the

$(d-1)$-tuple $(1,2,\ldots,d-1)$. Recall that $\ell_1+\cdots+\ell_k+k-1=d-1$. If $\ell_1>0$, $\mathcal{A}'_{d-1}$ is defined to be the collection of all permutations of $(d-1)$-tuples

$$(d-1)\Big(\frac{1}{\ell_1}(1,\ldots,\ell_1);\frac{1}{\ell_2+1}(1,\ldots,\ell_2);\ldots;\frac{1}{\ell_k+1}(1,\ldots,\ell_k);\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{k-1\text{ times}}\Big).$$

Note that if $k=1$, then $\mathcal{A}'_{d-1}=\mathcal{A}_{d-1}$. For $k\geq 2$, define $\mathcal{A}''_{d-1}$ to be the collection of all permutations of $(d-1)$-tuples

$$(d-1)\Big(\frac{1}{\ell_1+1}(1,\ldots,\ell_1);\ldots;\frac{1}{\ell_k+1}(1,\ldots,\ell_k);\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{k-2\text{ times}};1\Big).$$

(To simplify the notation, and since no confusion will result from doing so, we suppress the dependence of $\mathcal{A}'_{d-1}$ and $\mathcal{A}''_{d-1}$ on k and the ℓ_n's.)

Lemma 2.2. *The inclusions $\mathcal{A}'_{d-1}$, $\mathcal{A}''_{d-1}\subset\mathcal{A}_{d-1}$ hold.*

Moving towards (2.21), fix $n'\in\{2,\ldots,k\}$. We will show that

$$2^{m_{n'}}\prod_{n=1}^{k}W_n^{(d-1)/(\ell_n+1)}\Big(\prod_{\substack{1\leq n\leq k\\ n\neq 1,n'}}\rho_n^{(d-1)/2}\Big)\rho_{n'}^{d-1} \lesssim 2^{2[(\ell_1+1)m_1+\cdots+(\ell_k+1)m_k]/(d^2+d)}. \tag{2.23}$$

Recall the definitions of the intervals (α_i,β_i), whose lengths Δ_i are the numbers ρ_n $(n=2,\ldots,k)$ along with the numbers $t_{i+1}-t_i$ for $i=p_{n-1}+1,\ldots,p_n-1$ and $n=1,\ldots,k$. Since

$$W_n=\prod_{i=p_{n-1}+1}^{p_n-1}(t_{i+1}-t_i)^{e_i^n},$$

for some choice of $\{e_i^n\}$ with

$$\{e_i^n\}_{i=p_{n-1}+1}^{p_n-1}=\{1,\ldots,\ell_n\},$$

it follows that

$$\prod_{n=1}^{k}W_n^{(d-1)/(\ell_n+1)}\Big(\prod_{\substack{1\leq n\leq k\\ n\neq 1,n'}}\rho_n^{(d-1)/2}\Big)\rho_{n'}^{d-1}=\prod_{i=1}^{d-1}\Delta_i^{\sigma(i)}, \tag{2.24}$$

where the vector $\sigma=\big(\sigma(i)\big)$ is in $\mathcal{A}''_{d-1}$ and where, if i_0 is the index for which $(\alpha_{i_0},\beta_{i_0})=\widetilde{J}_{n'}$, then $\sigma(i_0)=d-1$ and so $\Delta_{i_0}^{\sigma(i_0)}=\rho_{n'}^{d-1}$. By Lemma 2.2, the vector σ is a convex combination

$$\sigma=\sum_q\lambda_q\tau_q \tag{2.25}$$

of vectors $\tau_q=\big(\tau_q(i)\big)$, each of which is a permutation of $(1,\ldots,d-1)$. Further, since $\sigma(i_0)=d-1$, we have $\tau_q(i_0)=d-1$ for each q. Now it follows from Lemma 2.1 that if the c_i's are nonnegative, then

$$c_{i_0}\,\Delta_{i_0}^{d-1}\prod_{\substack{1\leq i\leq d-1\\ i\neq i_0}}\Delta_i^{\tau_q(i)}\lesssim\int_{t_1}^{t_d}\sum_{i=1}^{d-1}c_i\,\chi_{(\alpha_i,\beta_i)}(u)\,\psi(u;t_1,\ldots,t_d)\,du \tag{2.26}$$

for each q. From (2.24), (2.25), and (2.26) it then follows that

$$c_{i_0}\prod_{n=1}^{k} W_n^{(d-1)/(\ell_n+1)}\Big(\prod_{\substack{1\le n\le k\\ n\neq 1,n'}}\rho_n^{(d-1)/2}\Big)\rho_{n'}^{d-1} \lesssim \int_{t_1}^{t_d}\sum_{i=1}^{d-1} c_i\,\chi_{(\alpha_i,\beta_i)}(u)\,\psi(u;t_1,\dots,t_d)\,du.$$

Since (2.17) implies that $\omega\sim 2^{m_{n'}}$ on $\widetilde{J}_{n'}=(\alpha_{i_0},\beta_{i_0})$, we have

(2.27)

$$2^{m_{n'}}\prod_{n=1}^{k} W_n^{(d-1)/(\ell_n+1)}\Big(\prod_{\substack{1\le n\le k\\ n\neq 1,n'}}\rho_n^{(d-1)/2}\Big)\rho_{n'}^{d-1} \lesssim \int_{t_1}^{t_d} 2^{m_{n'}}\chi_{\widetilde{J}_{n'}}(u)\,\psi(u;t_1,\dots,t_d)\,du$$
$$\lesssim \int_{t_1}^{t_d}\omega(u)\,\psi(u;t_1,\dots,t_d)\,du = J(t_1,\dots,t_d)\lesssim 2^{2[(\ell_1+1)m_1+\cdots+(\ell_k+1)m_k]/(d^2+d)}$$

by (2.3) and (2.19). This is (2.23). Analogous to (2.23) we will also need, in the case $\ell_1>0$, the estimate

$$2^{m_1}W_1^{(d-1)/\ell_1}\prod_{n=2}^{k} W_n^{(d-1)/(\ell_n+1)}\prod_{n=2}^{k}\rho_n^{(d-1)/2} \lesssim 2^{2[(\ell_1+1)m_1+\cdots+(\ell_k+1)m_k]/(d^2+d)}. \tag{2.28}$$

As before,

$$W_1^{(d-1)/\ell_1}\prod_{n=2}^{k} W_n^{(d-1)/(\ell_n+1)}\prod_{n=2}^{k}\rho_n^{(d-1)/2} = \prod_{i=1}^{d-1}\Delta_i^{\sigma(i)}$$

where now σ is in $\mathcal{A}'_{d-1}$. With $\sigma=\sum_q\lambda_q\tau_q$ as in (2.25), Lemma 2.1 gives

$$2^{m_1}\prod_{i=1}^{d-1}\Delta_i^{\tau_q(i)}\lesssim\int_{t_1}^{t_d}\sum_{1}^{d-1}2^{m_1}\,\chi_{(\alpha_i,\beta_i)}(u)\,\psi(u;t_1,\dots,t_d)\,du$$

for each q. This leads, as before, to

$$2^{m_1}W_1^{(d-1)/\ell_1}\prod_{n=2}^{k} W_n^{(d-1)/(\ell_n+1)}\prod_{n=2}^{k}\rho_n^{(d-1)/2} \lesssim \int_{t_1}^{t_d}\sum_{1}^{d-1}2^{m_1}\,\chi_{(\alpha_i,\beta_i)}(u)\,\psi(u;t_1,\dots,t_d)\,du.$$

Since $\omega\gtrsim 2^{m_1}$ on $[t_1,t_d]$, (2.28) follows as in (2.27).

Now (2.21) will follow by considering a particular weighted geometric mean of the estimates (2.28) and (2.23). In fact, given the computations

$$\frac{\ell_1}{d-1}+\frac{\ell_2+1}{d-1}+\frac{\ell_3+1}{d-1}+\cdots+\frac{\ell_k+1}{d-1}=1,$$
$$\frac{d-1}{\ell_1}\frac{\ell_1}{d-1}+\frac{d-1}{\ell_1+1}\Big(\frac{\ell_2+1}{d-1}+\frac{\ell_3+1}{d-1}+\cdots+\frac{\ell_k+1}{d-1}\Big)=\frac{d}{\ell_1+1},$$

and

$$(d-1)\frac{\ell_{n'}+1}{d-1}+\frac{d-1}{2}\Big(1-\frac{\ell_{n'}+1}{d-1}\Big)=\frac{d+\ell_{n'}}{2},\quad n'=2,\dots,k,$$

(2.21) is an immediate consequence of (2.23) and (2.28).

Now the proof of (2.18) will be complete when we have explained how to choose $(t_2, \dots, t_d) \in E_{j_2 \dots j_d}$ so that (2.22) holds. We will need another lemma and some more notation: recall that

$$c_1 < d_1 < c_2 < d_2 < \cdots < c_k < d_k.$$

Let $\delta_n = d_n - c_n$. Recall that $1 \leq p_1 < p_2 < \cdots < p_k = d$, $p_0 = 0$, $\ell_n = p_n - p_{n-1} - 1$, and that

$$c_n \leq t_{p_{n-1}+1} < \cdots < t_{p_n} \leq d_n \tag{2.29}$$

for $n = 1, \dots k$. With $t_{p_{n-1}+1} \in [c_n, d_n)$, write $\mathbf{t_n}$ for an ℓ_n-tuple $(t_{p_{n-1}+2}, \dots, t_{p_n})$ satisfying (2.29) and $\mathbf{t}$ for the $(\ell_1 + \cdots + \ell_k = d - k)$-tuple $(\mathbf{t_1}, \dots, \mathbf{t_k})$.

Lemma 2.3. *The inequality*

$$m_{d-k}\big(\{\mathbf{t} : W_1^{d/(\ell_1+1)} \prod_{n=2}^{k} W_n^{(d-1)/(\ell_n+1)} \leq \mu\}\big) \lesssim \mu^{2/d} \prod_{n=2}^{k} \delta_n^{\ell_n/d}$$

holds for $\mu > 0$.

Writing

$$(t_2, \dots, t_d) = (t_{p_1+1}, \dots, t_{p_{k-1}+1}; \mathbf{t})$$

with $\mathbf{t}$ as above, we use (2.29) to choose

$$(t'_{p_1+1}, \dots, t'_{p_{k-1}+1}) \in \prod_{n=2}^{k} [c_n, d_n]$$

such that

$$m_{d-k}\big(\{\mathbf{t} : (t'_{p_1+1}, \dots, t'_{p_{k-1}+1}; \mathbf{t}) \in E_{j_2 \cdots j_d}\}\big) \geq \frac{m_{d-1}(E_{j_2 \cdots j_d})}{\prod_{n=2}^{k} \delta_n}.$$

Let $c_1(d) > 0$ be sufficiently small. Then if $\mu > 0$ is such that

$$\mu^{2/d} \prod_{n=2}^{k} \delta_n^{\ell_n/d} = c_1(d) \frac{m_{d-1}(E_{j_2 \cdots j_d})}{\prod_{n=2}^{k} \delta_n},$$

it follows from Lemma 2.3 that there is

$$(t_2, \dots, t_d) \in E_{j_2 \cdots j_d}$$

such that

$$W_1^{d/(\ell_1+1)} \prod_{n=2}^{k} W_n^{(d-1)/(\ell_n+1)} > \mu = c_2(d) \frac{m_{d-1}(E_{j_2 \cdots j_d})^{d/2}}{\prod_{n=2}^{k} \delta_n^{(d+\ell_n)/2}}.$$

Recalling that ρ_n is the length of $\tilde{J}_n$, so that $\delta_n = d_n - c_n \lesssim \rho_n$ by (2.16) and (2.14), (2.22) follows.

3. Proofs of lemmas

Proof of Lemma 2.1. The proof is by induction on d. Since $\psi(u;t_1,t_2)=\chi_{[t_1,t_2]}(u)$, the case $d=2$ is clear. Fix $p\in\{1,\dots,d-1\}$ and then a choice of $\{e_i\}_{i\neq p}$ so that $\{e_i\}_{i\neq p}=\{1,\dots,d-2\}$. Let q satisfy $e_q=1$. We will give the argument in the case $q<p$, the case $q>p$ being similar. Let m_i be the midpoint of $[\alpha_i,\beta_i]$. Define intervals $I_1,\dots I_{d-1}$ and $J_1,\dots J_{d-1}$ as follows:

$$I_j=(\alpha_i,m_i),\ J_i=(m_i,\beta_i)\text{ if } i<q,$$
$$I_q=(\alpha_q,m_q),\ J_q=\emptyset,$$
$$I_i=(m_i,\beta_i),\ J_i=(\alpha_i,m_i)\text{ if } i>q.$$

We use the identity

$$\psi(u;t_1,\dots,t_d)=\int_{t_1}^{t_2}\cdots\int_{t_{d-1}}^{t_d}\psi(u;s_1,\dots,s_{d-1})\,ds_1\cdots ds_{d-1},$$

a consequence of the proof of Lemma 2.3 in [2], and the fact that $\psi(u,s_1,\dots,s_{d-1})$ is supported on $[s_1,s_{d-1}]$ to write

(3.1)
$$\int_{t_1}^{t_d} f(u)\,\psi(u;t_1,\dots,t_d)\,du = \int_{t_1}^{t_2}\cdots\int_{t_{d-1}}^{t_d}\int_{s_1}^{s_{d-1}} f(u)\,\psi(u;s_1,\dots,s_{d-1})\,du\,ds_1\cdots ds_{d-1}.$$

Then

$$(3.1)\geq\int_{I_1}\cdots\int_{I_{d-1}}\int_{s_1}^{s_{d-1}}\sum_{\substack{1\leq i\leq d-1\\ i\neq q}}\nu_i\,\chi_{J_i}(u)\,\psi(u;s_1,\dots,s_{d-1})\,du\,ds_1\cdots ds_{d-1}.$$

Now if $s_i\in I_i$ for $i=1,\dots,d-1$, then $J_i\subset(s_i,s_{i+1})$ if $i<q$ and $J_i\subset(s_{i-1},s_i)$ if $i>q$. Thus, assuming the lemma for $d-1$ and noting that $\{e_i-1\}_{i\neq p,q}=\{1,\dots,d-3\}$, it follows that

$$\int_{s_1}^{s_{d-1}}\sum_{\substack{1\leq i\leq d-1\\ i\neq q}}\nu_i\,\chi_{J_i}(u)\,\psi(u;s_1,\dots,s_{d-1})\,du\gtrsim\nu_p\,\Delta_p^{d-2}\prod_{\substack{1\leq i\leq d-1\\ i\neq p,q}}\Delta_i^{e_i-1}$$

and so

$$(3.1)\gtrsim\nu_p\,\Big(\prod_{i=1}^{d-1}\Delta_i\Big)\Delta_p^{d-2}\prod_{\substack{1\leq i\leq d-1\\ i\neq p,q}}\Delta_i^{e_i-1}=\nu_p\,\Delta_p^{d-1}\prod_{\substack{1\leq i\leq d-1\\ i\neq p}}\Delta_i^{e_i},$$

completing the proof of Lemma 2.1. □

Proof of Lemma 2.2. Lemma 2.2 is the statement that if

$$\ell_1+\cdots+\ell_k+k-1=r,$$

then **(a)** if $\ell_1>0$, $k\geq 2$, and $\mathcal{A}'_r$ is the collection of all permutations of r-tuples

$$r\Big(\frac{1}{\ell_1}(1,\dots,\ell_1);\frac{1}{\ell_2+1}(1,\dots,\ell_2);\dots;\frac{1}{\ell_k+1}(1,\dots,\ell_k);\underbrace{\frac12,\cdots,\frac12}_{k-1\text{ times}}\Big), \tag{3.2}$$

we have $\mathcal{A}_r' \subset \mathcal{A}_r$, and **(b)** if $k \geq 2$ and $\mathcal{A}_r''$ is the collection of all permutations of r-tuples

$$r\Big(\frac{1}{\ell_1+1}(1,\ldots,\ell_1);\ldots;\frac{1}{\ell_k+1}(1,\ldots,\ell_k);\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{k-2 \text{ times}};1\Big),$$

we have $\mathcal{A}_r'' \subset \mathcal{A}_r$. We will show these inclusions by induction on k. We require the following two facts, which we establish at the end of the proof of this lemma:

$$\Big(\frac{1}{\ell_{k-1}+1}(1,\ldots,\ell_{k-1});\frac{1}{\ell_k+1}(1,\ldots,\ell_k)\Big) \in \Big(\frac{1}{\ell_{k-1}+\ell_k+1}\Big)\mathcal{A}_{\ell_{k-1}+\ell_k} \tag{3.3}$$

and

$$\Big(\frac{1}{\ell_{k-1}+1}(1,\ldots,\ell_{k-1});\frac{1}{\ell_k+1}(1,\ldots,\ell_k);\frac{1}{2}\Big) \in \Big(\frac{1}{\ell_{k-1}+\ell_k+2}\Big)\mathcal{A}_{\ell_{k-1}+\ell_k+1}. \tag{3.4}$$

If $k=2$ and $\ell_1 > 1$, a vector (3.2) can be written as

$$r\Big(\frac{1}{\ell_1}(1,\ldots,\ell_1-1);1;\frac{1}{\ell_2+1}(1,\ldots,\ell_2);\frac{1}{2}\Big)$$

and therefore, by (3.4), as a linear combination of permutations of vectors

$$r\Big(\frac{1}{\ell_1+\ell_2+1}(1,\ldots,\ell_1+\ell_2);1\Big) \in \mathcal{A}_r.$$

And if $k=2$ and $\ell_1 = 1$, then (3.2) can be written as

$$r\Big(1;\frac{1}{\ell_2+1}(1,\ldots,\ell_2);\frac{1}{1+1}(1)\Big)$$

and therefore, by (3.3), as a linear combination of permutations of the vector

$$r\Big(1;\frac{1}{\ell_2+2}(1,\ldots,\ell_2+1)\Big) \in \mathcal{A}_r.$$

Thus **(a)** holds for $k=2$. The fact that **(b)** holds for $k=2$ follows similarly from (3.3). So assume that $k \geq 3$ and that **(a)** and **(b)** hold with $k-1$ in place of k.

To show that **(b)** holds for k, fix a vector

$$r\Big(\frac{1}{\ell_1+1}(1,\ldots,\ell_1);\ldots;\frac{1}{\ell_{k-1}+1}(1,\ldots,\ell_{k-1});\frac{1}{\ell_k+1}(1,\ldots,\ell_k);\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{k-2 \text{ times}};1\Big) \tag{3.5}$$

in $\mathcal{A}_r''$. It follows from (3.4) that (3.5) can be written as a convex combination of permutations of vectors

$$\begin{aligned} r\Big(\frac{1}{\ell_1+1}(1,\ldots,\ell_1);\ldots;&\frac{1}{\ell_{k-2}+1}(1,\ldots,\ell_{k-2}); \\ &\frac{1}{\ell_{k-1}+\ell_k+2}(1,\ldots,\ell_{k-1}+\ell_k+1);\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{k-3 \text{ times}};1\Big), \end{aligned} \tag{3.6}$$

and our induction assumption implies that each permutation of (3.6) is in $\mathcal{A}_r$. This establishes **(b)** for k.

To see that **(a)** holds for k, note that the argument above shows that a vector

$$r\Big(\frac{1}{\ell_1}(1,\dots,\ell_1);\dots;\frac{1}{\ell_{k-1}+1}(1,\dots,\ell_{k-1});\frac{1}{\ell_k+1}(1,\dots,\ell_k);\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{k-1\text{ times}}\Big)$$

of the form (3.2) can be written as a convex combination of permutations of vectors

$$r\Big(\frac{1}{\ell_1}(1,\dots,\ell_1);\dots;\frac{1}{\ell_{k-2}+1}(1,\dots,\ell_{k-2}); \frac{1}{\ell_{k-1}+\ell_k+2}(1,\dots,\ell_{k-1}+\ell_k+1);\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{k-2\text{ times}}\Big)$$

and so, by the induction assumption, is in $\mathcal{A}_r$.

It remains to establish (3.3) and (3.4). We require an alternate description of $\mathcal{A}_r$. Let $\mathcal{A}_r^*$ be the set

$$\Big\{(a_1,\dots,a_r):\sum_{j=1}^r a_j=\frac{r(r+1)}{2}\text{ and }\sum_{j\in\mathcal{E}}a_j\ge\frac{|\mathcal{E}|(|\mathcal{E}|+1)}{2}\text{ if }\mathcal{E}\subset\{1,\dots,r\}\Big\}.$$

We want to show that $\mathcal{A}_r=\mathcal{A}_r^*$, and it is enough to show that each extreme point of the convex set $\mathcal{A}_r^*$ is a permutation of $(1,\dots,r)$. So assume that $(a_1,\dots,a_r)$ is an extreme point of $\mathcal{A}_r^*$. Without loss of generality we may also assume $a_1\le a_2\le\cdots\le a_r$.

Our first step will be to show that

$$a_1<a_2<\cdots<a_d. \tag{3.7}$$

To this end, assume that $a_s=a_{s+1}$ for some $s\in\{1,\dots,d-1\}$. We will show that if either $s\in\mathcal{E}$ and $s+1\notin\mathcal{E}$ or $s+1\in\mathcal{E}$ and $s\notin\mathcal{E}$, then

$$\sum_{j\in\mathcal{E}}a_j>\frac{|\mathcal{E}|(|\mathcal{E}|+1)}{2}. \tag{3.8}$$

If (3.8) holds, it will follow that there is $\delta>0$ such that the vector obtained from $(a_1,\dots,a_r)$ by replacing a_s and a_{s+1} by $a_s+\eta$ and $a_{s+1}-\eta$ is in $\mathcal{A}_r$ whenever $|\eta|<\delta$. This implies that $(a_1,\dots,a_r)$ is not extreme. To show (3.8) we begin with an observation:

$$\text{if } a_k=a_{k+1},\ \text{then } \sum_{j=1}^k a_j>\frac{k(k+1)}{2}. \tag{3.9}$$

(To see (3.9), observe that the assumption $\sum_{j=1}^k a_j=\frac{k(k+1)}{2}$ and the inequality $\sum_{j=1}^{k+1}a_j\ge\frac{(k+1)(k+2)}{2}$ together imply that $a_{k+1}\ge k+1$ and so $a_k\ge k+1$ if $a_k=a_{k+1}$. Then

$$\sum_{j=1}^k a_j=a_k+\sum_{j=1}^{k-1}a_j\ge(k+1)+\frac{(k-1)k}{2},$$

contradicting $\sum_{j=1}^k a_j=\frac{k(k+1)}{2}$.) Returning to (3.8), we will write $k=|\mathcal{E}|$ and consider three cases:

Case I ($k < s$): Here we will show that if $s \in \mathcal{E}$ or $s+1 \in \mathcal{E}$, then (3.8) holds. Assuming $s \in \mathcal{E}$, it follows that

$$\frac{k(k+1)}{2} \le \sum_{j=1}^{k} a_j = \sum_{j=1}^{k-1} a_j + a_k \le \sum_{j=1}^{k-1} a_j + a_s \le \sum_{j\in\mathcal{E}} a_j.$$

Thus if (3.8) fails, then $a_k = a_{k+1}$ (since $a_k = a_s$) and $\sum_{j=1}^{k} a_j = k(k+1)/2$, contradicting (3.9). The case $s+1 \in \mathcal{E}$ is similar.

Case II ($k = s$): We will just observe that if $a_s = a_{s+1}$, then (3.8) holds. In fact, since

$$\sum_{j\in\mathcal{E}} a_j \ge \sum_{j=1}^{k} a_j,$$

this follows immediately from (3.9).

Case III ($k > s$): We will show that if either $s \notin \mathcal{E}$ or $s+1 \notin \mathcal{E}$, then (3.8) holds. Write $\mathcal{E} = \{j_1, j_2, \dots, j_k\}$ with $j_1 < \cdots < j_k$. Since $k > s$ and either $s \notin \mathcal{E}$ or $s+1 \notin \mathcal{E}$, it follows that $j_k > k$. Then, since $1 \le j_1, \dots, k \le j_k$, the inequality

$$\frac{k(k+1)}{2} \le \sum_{l=1}^{k} a_l \le \sum_{l=1}^{k} a_{j_l} = \sum_{j\in\mathcal{E}} a_j$$

shows that if (3.8) fails, then $a_k = a_{j_k}$ and $\sum_{j=1}^{k} a_j = k(k+1)/2$, again resulting in a contradiction of (3.9). Thus (3.8), and so (3.7), are established.

Now suppose that $(a_1, \dots, a_r)$ is extreme and (3.7) holds. If $a_j \ge j$ for $j = 1, \dots, r$, then the condition

$$\sum_{j=1}^{r} a_j = r(r+1)/2$$

forces $(a_1, \dots, a_r) = (1, \dots, r)$. But if we have $a_j < j$ for any j, we can choose t such that $a_t < t$ and $a_j \ge j$ for $j = 1, \dots, t-1$ (the condition $a_1 \ge 1$ implies that $t > 1$). Since

$$\sum_{j=1}^{t} a_j \ge t(t+1)/2 \tag{3.10}$$

we can choose s with $s \le t-1$, $a_s > s$, and $a_j = j$ if $s < j < t$. Thus

$$(a_1, \dots, a_r) = (a_1, \dots, a_{s-1}, a_s, s+1, \dots, t-1, a_t, a_{t+1}, \dots, a_r).$$

It follows from (3.10) and $a_t < t$ that

$$\sum_{j=1}^{p} a_j > p(p+1)/2$$

for $p = s, \cdots, t-1$. Thus there is $\delta > 0$ such that if $|\eta| < \delta$, then

$$(a_1, \dots, a_{s-1}, a_s + \eta, s+1, \dots, t-1, a_t - \eta, a_{t+1}, \dots, a_r) \in \mathcal{A}_r^*, \tag{3.11}$$

where we have used the fact that (3.7) implies that the entries of the vector in (3.11) are nondecreasing if δ is small enough. Then $(a_1, \dots, a_r)$ cannot be an extreme point of $\mathcal{A}_r^*$. Thus $a_j \ge j$ for $j = 1, \dots, r$ and so $(a_1, \dots, a_r) = (1, \dots, r)$ as desired.

We return to the proofs for (3.3) and (3.4). Since $\mathcal{A}_{\ell_{k-1}+\ell_k} = \mathcal{A}^*_{\ell_{k-1}+\ell_k}$, (3.3) will follow from checking that if $m \le \ell_{k-1}$ and $n \le \ell_k$, then

$$\frac{\ell_{k-1}+\ell_k+1}{\ell_{k-1}+1}\frac{m(m+1)}{2} + \frac{\ell_{k-1}+\ell_k+1}{\ell_k+1}\frac{n(n+1)}{2} \ge \frac{(n+m)(n+m+1)}{2}. \tag{3.12}$$

This inequality is equivalent to the inequality

$$m(m+1)\ell_k(\ell_k+1) + n(n+1)\ell_{k-1}(\ell_{k-1}+1) \ge 2mn(\ell_{k-1}+1)(\ell_k+1). \tag{3.13}$$

And the easily checked inequality

$$\sqrt{m(m+1)n(n+1)\ell_{k-1}(\ell_{k-1}+1)\ell_k(\ell_k+1)} \ge mn(\ell_{k-1}+1)(\ell_k+1)$$

shows that (3.13) follows from the inequality between arithmetic and geometric means.

Similarly, to show that (3.4) holds it is enough to check that if $m \le \ell_{k-1}$ and $n \le \ell_k$, then the inequalities

$$\frac{\ell_{k-1}+\ell_k+2}{\ell_{k-1}+1}\ \frac{m(m+1)}{2} + \frac{\ell_{k-1}+\ell_k+2}{\ell_k+1}\ \frac{n(n+1)}{2} \ge \frac{(n+m)(n+m+1)}{2} \tag{3.14}$$

and

$$\begin{aligned} \frac{\ell_{k-1}+\ell_k+2}{\ell_{k-1}+1}\ \frac{m(m+1)}{2} + \frac{\ell_{k-1}+\ell_k+2}{\ell_k+1}\ \frac{n(n+1)}{2} + \frac{\ell_{k-1}+\ell_k+2}{2} \\ \ge \frac{(n+m+1)(n+m+2)}{2} \end{aligned} \tag{3.15}$$

hold. Since (3.14) follows from (3.12), it is enough to establish (3.15). Now inequality (3.12) is equivalent to

$$\frac{m(m+1)}{2(\ell_{k-1}+1)} + \frac{n(n+1)}{2(\ell_k+1)} \ge \frac{(n+m)(n+m+1)}{2(\ell_{k-1}+\ell_k+1)},$$

while (3.15) is equivalent to

$$\begin{aligned} \frac{m(m+1)}{2(\ell_{k-1}+1)} + \frac{n(n+1)}{2(\ell_k+1)} + \frac{1}{2} &\ge \frac{(n+m+1)(n+m+2)}{2(\ell_{k-1}+\ell_k+2)} \\ &= \Big(\frac{(n+m)(n+m+1)}{2} + (n+m+1)\Big) \\ &\times \Big(\frac{1}{\ell_{k-1}+\ell_k+1} - \frac{1}{(\ell_{k-1}+\ell_k+1)(\ell_{k-1}+\ell_k+2)}\Big). \end{aligned}$$

Thus it is enough to show that

$$\begin{aligned} \frac{1}{2} + \frac{(n+m)(n+m+1)}{2(\ell_{k-1}+\ell_k+1)(\ell_{k-1}+\ell_k+2)} + \frac{n+m+1}{(\ell_{k-1}+\ell_k+1)(\ell_{k-1}+\ell_k+2)} \\ \ge \frac{n+m+1}{\ell_{k-1}+\ell_k+1}. \end{aligned}$$

This is equivalent to the inequality

$$(\ell_{k-1}+\ell_k+1) + \frac{(n+m+1)(n+m+2)}{\ell_{k-1}+\ell_k+2} \ge 2(n+m+1),$$

which follows from the arithmetic-geometric mean inequality and the easily checked

$$(n+m+1)(\ell_{k-1}+\ell_k+2) \le (\ell_{k-1}+\ell_k+1)(n+m+2),$$

itself a consequence of $m \le \ell_{k-1}$, $n \le \ell_k$. □

Proof of Lemma 2.3. Recall that $c_n \le t_{p_{n-1}+1} < \cdots < t_{p_n} \le d_n$ and that $W_n = W(t_{p_{n-1}+1}, \dots, t_{p_n})$. It follows that $W_n \lesssim \delta_n^{\ell_n(\ell_n+1)/2}$ and so

(3.16)

$$m_{d-k}\big(\{\mathbf{t} : W_1^{d/(\ell_1+1)} \prod_{n=2}^{k} W_n^{(d-1)/(\ell_n+1)} \le \mu\}\big)$$
$$\lesssim \sum_{2^{p_2} \le \delta_2^{\ell_2(\ell_2+1)/2}} \cdots \sum_{2^{p_k} \le \delta_k^{\ell_k(\ell_k+1)/2}} \prod_{n=2}^{k} \Big|\{W_n \le 2^{p_n}\}\Big|$$
$$\cdot \Big|\{W_1 \le \big(\mu / \prod_{n=2}^{k} 2^{p_n(d-1)/(\ell_n+1)}\big)^{(\ell_1+1)/d}\}\Big|,$$

where

$$|\{W_n \le \lambda\}| = m_{\ell_n}\big(\{(t_{p_{n-1}+2}, \dots, t_{p_n}) : W(t_{p_{n-1}+1}, t_{p_{n-1}+2}, \dots, t_{p_n}) \le \lambda\}\big).$$

We will need the estimate

$$|\{W_n \le \lambda\}| \lesssim \lambda^{2/(\ell_n+1)}. \tag{3.17}$$

To show (3.17) and, more generally, to show that

(3.18)

$$m_p\big(\{(s_1, \dots, s_p) : 0 \le s_1 \le \cdots \le s_p, W(0, s_1, \dots, s_p) \le \lambda\}\big) \le C(p)\, \lambda^{2/(p+1)},$$

we will argue by induction on p. (Inequality (3.18) is an analog of a result, Proposition 2.4 (i) in [1], from [5] and [6].) The case $p = 1$ is clear, so assume that (3.18) holds. If we make the change of variables

$$u_1 = s_1, u_2 = s_2 - s_1, \dots, u_p = s_p - s_{p-1},$$

then

$$m_p\big(\{(s_1, \dots, s_p) : 0 \le s_1 \le \cdots \le s_p,\, W(0, s_1, \dots, s_p) \le \lambda\}\big)$$
$$= p!\, m_p\big(\{(u_1, \dots, u_p) : 0 \le u_1 \le \cdots \le u_p,\, \prod_{j=1}^{p} (u_j)^j \le \lambda\}\big).$$

Now

$$m_{p+1}\big(\{(u_1, \dots, u_{p+1}) : 0 \le u_1 \le \cdots \le u_{p+1},\, \prod_{j=1}^{p+1} (u_j)^j \le \lambda\}\big)$$
$$= \int_0^\infty m_p\big(\{(u_1, \dots, u_p) : 0 \le u_1 \le \cdots \le u_p \le u_{p+1},$$
$$\prod_{j=1}^{p} u_j^j \le \lambda/(u_{p+1})^{p+1}\}\big)\, du_{p+1}.$$

Thus the estimate

$$m_p\big(\{(u_1, \dots, u_p) : 0 \le u_1 \le \cdots \le u_{p+1},\, \prod_{j=1}^{p} (u_j)^j \le \lambda/(u_{p+1})^{p+1}\}\big)$$
$$\le C(p)\, \min\{(u_{p+1})^p, (\lambda/(u_{p+1})^{p+1})^{2/(p+1)}\},$$

a consequence of (3.18), shows that (3.18) holds with p replaced by $p + 1$.

Now, using (3.17) and some algebra, we have

$$(3.16) \lesssim \mu^{2/d} \sum_{2^{p_2} \le \delta_2^{\ell_2(\ell_2+1)/2}} \cdots \sum_{2^{p_k} \le \delta_k^{\ell_k(\ell_k+1)/2}} \prod_{n=2}^{k} 2^{2p_n/[d(\ell_n+1])} \lesssim \mu^{2/d} \prod_{n=2}^{k} \delta_n^{\ell_n/d}.$$

This gives the desired conclusion and therefore completes the proof of Lemma 2.3. □

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References

[1] J.-G. Bak, D. Oberlin and A. Seeger, Restriction of Fourier transforms to curves and related oscillatory integrals, *Amer. J. Math.* **131** (2009), 277–311. MR2503984

[2] ———, Restriction of Fourier transforms to curves. II: some classes with vanishing torsion, *J. Australian Math. Soc.* **85** (2008), 1–28. MR2460861

[3] M. Christ, On the restriction of the Fourier transform to curves: endpoint results and the degenerate case, *Trans. Amer. Math. Soc.* **287** (1985), 223–238. MR766216 (87b:42018)

[4] S. Dendrinos and J. Wright, Fourier restriction to polynomial curves I: a geometric inequality, preprint.

[5] S.W. Drury and B. Marshall, Fourier restriction theorems for curves with affine and Euclidean arclengths, *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 111–125. MR764500 (87b:42019)

[6] ———, Fourier restriction theorems for degenerate curves, *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 541–553. MR878901 (88e:42026)

[7] D. Oberlin, Fourier restriction for affine arclength measures in the plane, *Proc. Amer. Math. Soc.* **129** (2001), 3303–3305. MR1845006 (2002g:42013)

[8] ———, Some convolution inequalities and their applications, *Trans. Amer. Math. Soc.* **354** (2002), 2541–2556. MR1885663 (2002j:42016)

[9] P. Sjölin, Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in R^2, *Studia Math.* **51** (1974), 169–182. MR0385437 (52:6299)

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