ON QUASI-METRIC AND METRIC SPACES

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Abstract. Given a space $X$ with a quasi-metric $\rho$ it is known that the so-called $p$-chain approach can be used to produce a metric in $X$ equivalent to $\rho^p$ for some $0 < p \leq 1$, hence also a quasi-metric $\bar{\rho}$ equivalent to $\rho$ with better properties. We refine this result and obtain an exponent $p$ which is, in general, optimal.

1. Introduction

A quasi-metric on a nonempty set $X$ is a mapping $\rho : X \times X \to [0, \infty)$ which satisfies the following conditions:

(i) for every $x, y \in X$, $\rho(x, y) = 0$ if and only if $x = y$;
(ii) for every $x, y \in X$, $\rho(x, y) = \rho(y, x)$;
(iii) there is a constant $K \geq 1$ such that for every $x, y, z \in X$,
$$\rho(x, y) \leq K(\rho(x, z) + \rho(z, y)).$$

The pair $(X, \rho)$ is then called a quasi-metric space; if $K = 1$, then $\rho$ is a metric and $(X, \rho)$ is a metric space.

Condition (iii) can be replaced by

(iii)$'$ there is a constant $K_\alpha \geq 1$ such that for every $x, y, z \in X$,
$$\rho(x, y) \leq K_\alpha \max\{\rho(x, z), \rho(z, y)\},$$

which is equivalent to (iii) if we do not care about constants entering into both conditions, but is slightly more restrictive than (iii) if we do: (iii)$'$ implies (iii) with $K = K_\alpha$, while (iii) implies (iii)$'$ with $K_\alpha = 2K$. It should be pointed out that in the area of general topology a quasi-metric is often understood as a mapping $\rho$ which violates the symmetry condition (ii) rather than the triangle inequality (i.e. (i) and (iii) with $K = 1$ are assumed to hold). In the present paper we adhere to the definition given above.

Two quasi-metrics $\rho_1$ and $\rho_2$ on $X$ are said to be equivalent if $c^{-1}\rho_2(x, y) \leq \rho_1(x, y) \leq c\rho_2(x, y)$ with some $c > 0$ independent of $x, y \in X$.

Macías and Segovia proved [8, Theorem 2] that given a quasi-norm $\rho$ it is possible to construct a quasi-metric $\rho'$ equivalent to $\rho$ and such that the quasi-metric balls related to $\rho'$ are open in the topology $F_{\rho'} = F_{\rho}$; see Section 2 for the definition of...
F\rho. More precisely, they proved that for a given \( \rho \) there exist a quasi-metric \( \rho' \), a number \( 0 < \alpha < 1 \) and \( C > 0 \) such that \( \rho' \) is equivalent to \( \rho \) and

\[
|\rho'(x, z) - \rho'(y, z)| \leq C\rho'(x, y)^\alpha \left( \max\{\rho'(x, z), \rho'(y, z)\} \right)^{1-\alpha}, \quad x, y, z \in X.
\]

A direct computation then shows that the above inequality indeed implies the fact that the quasi-metric balls related to \( \rho' \) are open in \( F\rho' = F\rho \).

Then Aimar, Iaffei and Nitti [1] furnished a direct proof of the aforementioned result. In [1] a construction of Frink [5] was adapted to produce from a given \( \rho \), with appropriately chosen \( p \), \( 0 < p < 1 \), the metric \( d_p \) (see (2.1)) equivalent to \( \rho^p \). The question of finding some \( p \) so that \( \rho^p \) is equivalent to a metric is also discussed in [6, p. 110]. The analogous result for quasi-normed spaces is called the Aoki-Rolewicz theorem; see [2], [10], and also [7] and [9].

The aim of this paper is to refine the result of [1]; see the comments at the end of Section 2 that explain the refinement. Also, we take the opportunity to furnish an example and make some remarks related to the question, when is it that quasi-metric balls related to a given quasi-metric \( \rho \) are open sets in the topology induced in \( X \).

Quasi-metric spaces are naturally involved in a part of harmonic analysis related to the theory of spaces of homogeneous type (see [3, Chapter 6] as an introduction to this theory) and enjoy continued interest. To be more specific let us mention that if we extend a fundamental theory of Calderón-Zygmund singular integral operators to a more abstract setting, it turns out that the essential arguments are measure theoretic rather than Fourier analytic. The fundamental notion here is that of a space of homogeneous type which is, first of all, a quasi-metric space equipped in addition with a regular Borel measure that respects the quasi-metric in an appropriate way.

2. Main result

Let \((X, \rho)\) be a quasi-metric space. Given \( p, 0 < p \leq 1 \), define \( d_p : X \times X \to [0, \infty) \) by letting

\[
(2.1) \quad d_p(x, y) = \inf \left\{ \sum_{j=1}^{n} \rho(x_{j-1}, x_j)^p : x = x_0, x_1, \ldots, x_n = y, \quad n \geq 1 \right\}.
\]

Clearly, \( d_p \) is symmetric and satisfies the triangle inequality

\[
d_p(x, y) \leq d_p(x, z) + d_p(z, y), \quad x, y, z \in X;
\]

in addition, \( d_p \leq \rho^p \). It is reasonable to refer to this process of producing \( d_p \) from \( \rho \) as the \( p \)-chain approach. It was shown in [1] that with \( p \) chosen properly, \( d_p(x, y) \neq 0 \) for \( x \neq y \); thus \( d_p \) becomes a metric, in fact equivalent to \( \rho^p \).

A similar approach in the context of quasi-normed spaces is known; see [2], [10], [7] and [9]. Although the argument which is presented in [9] to prove a “quasi-normed” result analogous to that from the proposition below doesn’t seem to be directly applicable in the quasi-metric case, it gives a hint about how \( p \) should be chosen.

**Proposition.** Let \((X, \rho)\) be a quasi-metric space and let \( 0 < p \leq 1 \) be given by \(((2K)^p)^p = 2 \). Then \( d_p \), obtained from \( \rho \) by the \( p \)-chain approach is a metric on \( X \).
equivalent to \( \rho^p \). In other words, \( \hat{\rho}(p) = d_{\rho}^{1/p} \) is a quasi-metric on \( X \) equivalent to \( \rho \) and satisfying, in addition, the so-called \( p \)-triangle inequality

\[
\hat{\rho}(x, y) \leq (\hat{\rho}(x, z) + \hat{\rho}(z, y))^{1/p}, \quad x, y, z \in X.
\]

The same conclusions hold if \( \rho \) satisfies (i), (ii) and (iii)' with \( K_o \geq 2 \) and if \( 0 < p \leq 1 \) is then determined by \( K_p^o = 2 \).

**Proof.** Clearly, it is sufficient to consider the case where \( \rho \) satisfies (iii)' with \( K_o \geq 2 \).

It has already been mentioned that \( d = d_\rho \) given by (2.1) is symmetric, satisfies the triangle inequality and verifies the left hand side of the inequalities

\[
(2.2) \quad d(x, y) \leq \rho(x, y)^p \leq 4d(x, y), \quad x, y \in X.
\]

Showing the right hand side of (2.2) will complete the proof that \( d \) is a metric (equivalent to \( \rho^p \)). Obviously, the statements concerning \( \hat{\rho} \) then follow.

We prove by induction on \( n \) that for any given sequence of \( n + 1 \) points \( x = x_0, x_1, \ldots, x_n = y, \ n \geq 2 \),

\[
(2.3) \quad \rho(x, y)^p \leq 2\left(\rho(x_0, x_1)^p + 2 \sum_{j=1}^{n-2} \rho(x_j, x_{j+1})^p + \rho(x_{n-1}, x_n)^p\right)
\]

(if \( n = 2 \), then the middle term on the right hand side of (2.3) is absent). Consequently, \( \rho(x, y)^p \leq 4d(x, y) \) follows.

If \( n = 2 \) and three points \( x, x_1, y \) are given, then using \( K_p^o = 2 \) gives

\[
\rho(x, y)^p \leq K_p^o \max\{\rho(x, x_1)^p, \rho(x_1, y)^p\} = 2\max\{\rho(x, x_1)^p, \rho(x_1, y)^p\}.
\]

Observe that, as a consequence, we obtain the starting point for the induction. Assume now that the induction hypothesis holds, (i.e. (2.3) is satisfied), and consider a sequence of \( n + 2 \) points \( x = x_0, x_1, \ldots, x_{n+1} = y \). Let \( m \) be the largest number among \( \{0, 1, \ldots, n\} \) with the property

\[
(2.4) \quad \rho(x, y)^p \leq 2\rho(x_m, y)^p.
\]

Since \( \rho(x, y)^p \leq 2\max\{\rho(x, x_{m+1})^p, \rho(x_{m+1}, y)^p\} \), therefore

\[
(2.5) \quad \rho(x, y)^p \leq 2\rho(x, x_{m+1})^p
\]

(this is clear if \( m \leq n - 1 \) and obvious for \( m = n \)). Combining (2.4) and (2.5) gives

\[
\rho(x, y)^p \leq 2\min\{\rho(x, x_{m+1})^p, \rho(x_m, y)^p\} \leq \rho(x, x_{m+1})^p + \rho(x_m, y)^p.
\]

If it happens that \( m = 0 \) or \( m = n \), then the first inequality above readily gives the required conclusion, i.e. (2.3) with \( n \) replaced by \( n + 1 \), and there is actually no need to invoke the induction hypothesis. Assume therefore that \( 1 \leq m \leq n - 1 \). Then, applying the induction hypothesis to the sequences \( x = x_0, x_1, \ldots, x_{m+1} \) and
\[ x_m, x_{m+1}, \ldots, x_{n+1} = y \] (both of length \( \leq n + 1 \)) gives
\[
\rho(x, y)^p \leq \rho(x, x_{m+1})^p + \rho(x_m, y)^p
\]
\[
\leq 2 \left( \rho(x_0, x_1)^p + 2 \sum_{j=1}^{m-1} \rho(x_j, x_{j+1})^p + \rho(x_m, x_{m+1})^p \right)
\]
\[
+ 2 \left( \rho(x_m, x_{m+1})^p + 2 \sum_{j=m+1}^{n-1} \rho(x_j, x_{j+1})^p + \rho(x_n, x_{n+1})^p \right)
\]
\[
= 2 \left( \rho(x_0, x_1)^p + 2 \sum_{j=1}^{n-1} \rho(x_j, x_{j+1})^p + \rho(x_n, x_{n+1})^p \right).
\]

This completes the induction step and thus the proof of the Proposition. \(\square\)

If \((X, \rho)\) is a quasi-metric space, then \(F_\rho\), the topology in \(X\) induced by \(\rho\), is canonically defined by means of the theory of \textit{uniform structures}; in case \(\rho\) is a metric this procedure leads to the usual metric topology in \(X\). We refer the reader to the monograph [4], where in Chapter 8 this way of introducing a topology is discussed.

The uniform structure \(U_\rho\) generated by \(\rho\) is defined to consist of all subsets \(V \subseteq X \times X\), symmetric in the sense that \((x, y) \in V\) if and only if \((y, x) \in V\) and containing a set of the form \(R_\epsilon = \{(x, y) : \rho(x, y) < \epsilon\}\) for some \(\epsilon > 0\) (in particular \(V\) contains the diagonal \(\{(x, x) : x \in X\}\)). Since the countable family \(\{R_{1/n}\}_{n \geq 1}\) is a \textit{basis for the uniform structure} \(U_\rho\), it follows from a general result (see [4, Chapter 8, Theorem 9]) that the topology \(F_\rho\) generated by \(U_\rho\) in \(X\) is metrizable.

Given \(r > 0\) and \(x \in X\), let
\[
B(x, r) = \{y \in X : \rho(x, y) < r\}
\]
be the \textit{quasi-metric ball} related to \(\rho\) of radius \(r\) and with center \(x\). According to the procedure of defining a topology by means of a uniform structure (see [4, Chapter 8]), in this case \(G \subseteq X\) is defined to be open, i.e. \(G \subseteq F_\rho\), if and only if for every \(x \in G\) there exists \(r > 0\) such that \(B(x, r) \subseteq G\) (at this point one easily checks directly that the topology axioms are satisfied for such a definition). It is clear that if \(\rho_1\) is a quasi-metric equivalent to \(\rho\), then \(F_{\rho_1} = F_\rho\); also, for any \(a > 0\), \(\rho^a\) is a quasi-metric as well and \(F_{\rho^a} = F_\rho\). Thus the Proposition furnishes a direct argument showing that \(F_\rho\) is metrizable [1, 8].

The quasi-metric balls themselves need not be open (unless \(\rho\) is a genuine metric) as the following simple example shows.

**Example.** Let \(X = \{0, 1, 2, \ldots\}\). Given \(\epsilon > 0\), we define \(\rho = \rho_\epsilon\) on \(X \times X\) in the following way. For \(0 \leq n < m\), we set \(\rho(n, m)\) as
\[
\rho(0, 1) = 1, \quad \rho(0, m) = 1 + \epsilon \quad \text{if } m \geq 2,
\]
\[
\rho(1, m) = \frac{1}{m}, \quad \rho(n, m) = \frac{1}{n} + \frac{1}{m} \quad \text{if } n \geq 2.
\]
We then extend \(\rho\) onto \(X \times X\) by putting \(\rho(n, n) = 0\) for any \(n \geq 0\) and \(\rho(n, m) = \rho(m, n)\) if \(0 \leq m < n\). We will show that
\[
(2.6) \quad \rho(k, n) \leq (1 + \epsilon) \left( \rho(k, m) + \rho(m, n) \right), \quad k, m, n \in X,
\]
and thus $\rho$ is a quasi-metric with $K = 1 + \epsilon$. It is clear that it suffices to check (2.6) for pairwise distinct $k, m, n$ only. Let $L$ and $R$ denote the left and the right hand sides of the inequality (2.6), respectively. If one of $k, n, m$ is 0, then $L \leq 1 + \epsilon \leq R$. If none of $k, m, n$ are 0, then we consider subcases. First, assume 1 appears among $k, m, n$. If $k = 1$, then $L = \frac{1}{n}$ while $R = (1 + \epsilon)(\frac{2}{m} + \frac{1}{n})$. Similarly if $n = 1$. If $m = 1$, then $L = \frac{1}{k} + \frac{1}{n}$ while $R = (1 + \epsilon)(\frac{1}{k} + \frac{2}{m} + \frac{1}{n})$. Next, assume 1 does not appear among $k, n, m$. Then $L = \frac{1}{k} + \frac{1}{n}$ and $R = (1 + \epsilon)(\frac{1}{k} + \frac{2}{m} + \frac{1}{n})$. This finishes our checking that $\rho$ is indeed a quasi-metric with constant $K = 1 + \epsilon$.

Now, note that $B(0, 1 + \epsilon/2) = \{0, 1\}$ while $B(1, \eta)$ contains infinitely many elements, for any $\eta > 0$. Hence, none of $B(1, \eta)$ are contained in $B(0, 1 + \epsilon/2)$, which shows that $B(0, 1 + \epsilon/2)$ is not open.

The metric $d_\rho$, produced from $\rho = \rho_\epsilon$ by the $p$-chain approach, can be computed directly. Let $p$ be defined as in the Proposition, with $1 + \epsilon$ in place of $K$. It is not hard to see that with the notation $x_0 = ((1 + \epsilon)^p - 1)^{-1/p}$, $d_\rho$ is given by

$$
(2.7) \quad d_\rho(k, n) = \begin{cases} 
1 + \max\{k, n\}^{-p}, & \text{if } \min\{k, n\} = 0 \text{ and } \max\{k, n\} > x_0, \\
\rho(k, n)^p, & \text{otherwise}.
\end{cases}
$$

It is easily seen that if for a quasi-metric space $(X, \rho)$ there exists a function $K(\epsilon), \epsilon > 0$, such that $K(\epsilon) \to 1$ as $\epsilon \to 0^+$, and

$$
\forall \ x, y, z \in X \quad \rho(y, z) < \epsilon \rho(x, y) \Rightarrow \rho(x, z) \leq K(\epsilon)(\rho(x, y) + \rho(y, z)),
$$

then the quasi-metric balls $B(x, r), x \in X, r > 0$, are open sets in the topology $\mathcal{F}_p$.

In particular, if a quasi-metric $\rho$ which, for some $p, 0 < p \leq 1$, satisfies the $p$-triangle inequality

$$
(2.8) \quad \rho(x, z) \leq (\rho(x, y)^p + \rho(y, z)^p)^{1/p}, \quad x, y, z \in X,
$$

then $K(\epsilon) = (1 + \epsilon)^{1/p}$ is appropriate. Consequently, a quasi-metric space $(X, \rho)$ with $\rho$ satisfying (2.8) has all its quasi-metric balls open; hence this also happens for the quasi-metric $\bar{\rho} = \bar{\rho}(p)$ from the Proposition.

It is clear that the quasi-metric space $(X, \rho)$ from the Example must fail to satisfy (2.8) for any $p, 0 < p \leq 1$. To see this by a direct argument note that for $x = 0, y = 1$ and $z = n \geq 2$ we have $\rho(0, n) = 1 + \epsilon$, while

$$
(\rho(0, 1)^p + \rho(1, n)^p)^{1/p} = (1 + n^{-p})^{1/p} \to 1, \quad \text{as } n \to \infty.
$$

Given a quasi-metric space $(X, \rho)$, let $K(\rho)$ denote the smallest constant $K$ for which (iii) holds, and let $\rho(\rho)$ denote the largest $p \in (0, 1]$ for which (2.8) holds; if such $p$ does not exist, then we set $p(\rho) = 0$. For instance, for the quasi-metric $\rho_\epsilon$ from the example we have $K(\rho_\epsilon) = 1 + \epsilon$ and $p(\rho_\epsilon) = 0$. Also, let $\bar{\rho}(\rho)$ denote the supremum of the set of $p \in (0, 1]$ with the property that there exists a quasi-metric $\bar{\rho}$ equivalent to $\rho$ and such that (2.8) is satisfied with $\bar{\rho}$ replacing $\rho$. Equivalently,

$$
\bar{\rho}(\rho) = \sup\{p \in (0, 1] : d_\rho \text{ is a metric equivalent to } \rho^p\}.
$$

It follows from the Proposition that for any given $(X, \rho)$ one has

$$
(2.9) \quad \bar{\rho}(\rho) \geq \frac{1}{\log_2(2K(\rho))}.
$$
Note that in [1] the weaker estimate
\[ p(\rho) \geq \frac{1}{(\log_2(3K(\rho)))^2} \]
was proved. A simple example of the usual \( \ell^p \) spaces, \( 0 < p < 1 \), shows that the estimate \( \tilde{p}(\rho) \) cannot, in general, be improved.

It can happen, however, that the inequality (2.9) is strict. In fact, for the quasimetric \( \rho_e \) from the above example one can apply the \( p \)-chain approach for any \( 0 < p \leq 1 \) and obtain a metric. Thus for this rather pathological example, actually \( \tilde{p}(\rho) = 1 \).

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References


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