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ON QUASI-METRIC AND METRIC SPACES

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ABSTRACT. Given a space X with a quasi-metric ρ it is known that the so-called p-chain approach can be used to produce a metric in X equivalent to ρ^p for some $0 < p \le 1$, hence also a quasi-metric $\tilde{\rho}$ equivalent to ρ with better properties. We refine this result and obtain an exponent p which is, in general, optimal.

1. Introduction

A quasi-metric on a nonempty set X is a mapping $\rho : X \times X \to [0, \infty)$ which satisfies the following conditions:

(i) for every $x, y \in X$, $\rho(x, y) = 0$ if and only if $x = y$;
(ii) for every $x, y \in X$, $\rho(x, y) = \rho(y, x)$;
(iii) there is a constant $K \ge 1$ such that for every $x, y, z \in X$,

$$\rho(x, y) \le K\big(\rho(x, z) + \rho(z, y)\big).$$

The pair (X, ρ) is then called a quasi-metric space; if $K = 1$, then ρ is a metric and (X, ρ) is a metric space.

Condition (iii) can be replaced by

(iii)$'$ there is a constant $K_o \ge 1$ such that for every $x, y, z \in X$,

$$\rho(x, y) \le K_o \max\{\rho(x, z), \rho(z, y)\},$$

which is equivalent to (iii) if we do not care about constants entering into both conditions, but is slightly more restrictive than (iii) if we do: (iii)$'$ implies (iii) with $K = K_o$, while (iii) implies (iii)$'$ with $K_o = 2K$. It should be pointed out that in the area of general topology a quasi-metric is often understood as a mapping ρ which violates the symmetry condition (ii) rather than the triangle inequality (i.e. (i) and (iii) with $K = 1$ are assumed to hold). In the present paper we adhere to the definition given above.

Two quasi-metrics ρ_1 and ρ_2 on X are said to be equivalent if $c^{-1}\rho_2(x, y) \le \rho_1(x, y) \le c\rho_2(x, y)$ with some $c > 0$ independent of $x, y \in X$.

Macías and Segovia proved [8, Theorem 2] that given a quasi-norm ρ it is possible to construct a quasi-metric ρ' equivalent to ρ and such that the quasi-metric balls related to ρ' are open in the topology $\mathcal{F}_{\rho'} = \mathcal{F}_\rho$; see Section 2 for the definition of

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$\mathcal{F}_\rho$. More precisely, they proved that for a given ρ there exist a quasi-metric ρ', a number $0<\alpha<1$ and $C>0$ such that ρ' is equivalent to ρ and

$$|\rho'(x,z)-\rho'(y,z)|\le C\rho'(x,y)^\alpha\big(\max\{\rho'(x,z),\rho'(y,z)\}\big)^{1-\alpha},\quad x,y,z\in X.$$

A direct computation then shows that the above inequality indeed implies the fact that the quasi-metric balls related to ρ' are open in $\mathcal{F}_{\rho'}=\mathcal{F}_\rho$.

Then Aimar, Iaffei and Nitti [1] furnished a direct proof of the aforementioned result. In [1] a construction of Frink [5] was adapted to produce from a given ρ, with appropriately chosen p, $0<p<1$, the metric d_p (see (2.1)) equivalent to ρ^p. The question of finding some p so that ρ^p is equivalent to a metric is also discussed in [6, p. 110]. The analogous result for quasi-normed spaces is called the Aoki-Rolewicz theorem; see [2], [10], and also [7] and [9].

The aim of this paper is to refine the result of [1]; see the comments at the end of Section 2 that explain the refinement. Also, we take the opportunity to furnish an example and make some remarks related to the question, when is it that quasi-metric balls related to a given quasi-metric ρ are open sets in the topology induced in X by ρ.

Quasi-metric spaces are naturally involved in a part of harmonic analysis related to the theory of *spaces of homogeneous type* (see [3, Chapter 6] as an introduction to this theory) and enjoy continued interest. To be more specific let us mention that if we extend a fundamental theory of Calderón-Zygmund singular integral operators to a more abstract setting, it turns out that the essential arguments are measure theoretic rather than Fourier analytic. The fundamental notion here is that of a space of homogeneous type which is, first of all, a quasi-metric space equipped in addition with a regular Borel measure that respects the quasi-metric in an appropriate way.

2. Main result

Let (X,ρ) be a quasi-metric space. Given p, $0<p\le 1$, define $d_p: X\times X\to[0,\infty)$ by letting

$$d_p(x,y)=\inf\left\{\sum_{j=1}^n \rho(x_{j-1},x_j)^p : x=x_0,x_1,\dots,x_n=y,\quad n\ge 1\right\}. \tag{2.1}$$

Clearly, d_p is symmetric and satisfies the triangle inequality

$$d_p(x,y)\le d_p(x,z)+d_p(z,y),\quad x,y,z\in X;$$

in addition, $d_p\le\rho^p$. It is reasonable to refer to this process of producing d_p from ρ as the *p-chain approach*. It was shown in [1] that with p chosen properly, $d_p(x,y)\ne 0$ for $x\ne y$; thus d_p becomes a metric, in fact equivalent to ρ^p.

A similar approach in the context of quasi-normed spaces is known; see [2], [10], [7] and [9]. Although the argument which is presented in [9] to prove a "quasi-normed" result analogous to that from the proposition below doesn't seem to be directly applicable in the quasi-metric case, it gives a hint about how p should be chosen.

Proposition. *Let (X,ρ) be a quasi-metric space and let $0<p\le 1$ be given by $(2K)^p=2$. Then d_p obtained from ρ by the p-chain approach is a metric on X*

equivalent to ρ^p. *In other words,* $\tilde{\rho} = \tilde{\rho}(p) = d_p^{1/p}$ *is a quasi-metric on* X *equivalent to* ρ *and satisfying, in addition, the so-called* p*-triangle inequality*

$$\tilde{\rho}(x, y) \le \left(\tilde{\rho}(x, z)^p + \tilde{\rho}(z, y)^p\right)^{1/p}, \quad x, y, z \in X.$$

The same conclusions hold if ρ *satisfies (i), (ii) and (iii)*$'$ *with* $K_o \ge 2$ *and if* $0 < p \le 1$ *is then determined by* $K_o^p = 2$.

Proof. Clearly, it is sufficient to consider the case where ρ satisfies (iii)$'$ with $K_o \ge 2$. It has already been mentioned that $d = d_p$ given by (2.1) is symmetric, satisfies the triangle inequality and verifies the left hand side of the inequalities

$$d(x, y) \le \rho(x, y)^p \le 4\, d(x, y), \quad x, y \in X. \tag{2.2}$$

Showing the right hand side of (2.2) will complete the proof that d is a metric (equivalent to ρ^p). Obviously, the statements concerning $\tilde{\rho}$ then follow.

We prove by induction on n that for any given sequence of $n+1$ points $x = x_0, x_1, \ldots, x_n = y$, $n \ge 2$,

$$\rho(x, y)^p \le 2\Big(\rho(x_0, x_1)^p + 2\sum_{j=1}^{n-2} \rho(x_j, x_{j+1})^p + \rho(x_{n-1}, x_n)^p\Big) \tag{2.3}$$

(if $n = 2$, then the middle term on the right hand side of (2.3) is absent). Consequently, $\rho(x, y)^p \le 4\, d(x, y)$ follows.

If $n = 2$ and three points x, x_1, y are given, then using $K_o^p = 2$ gives

$$\begin{aligned} \rho(x, y)^p &\le K_o^p \max\{\rho(x, x_1)^p, \rho(x_1, y)^p\} \\ &= 2 \max\{\rho(x, x_1)^p, \rho(x_1, y)^p\}. \end{aligned}$$

Observe that, as a consequence, we obtain the starting point for the induction. Assume now that the induction hypothesis holds, (i.e. (2.3) is satisfied), and consider a sequence of $n+2$ points $x = x_0, x_1, \ldots, x_{n+1} = y$. Let m be the largest number among $\{0, 1, \ldots, n\}$ with the property

$$\rho(x, y)^p \le 2\rho(x_m, y)^p. \tag{2.4}$$

Since $\rho(x, y)^p \le 2 \max\{\rho(x, x_{m+1})^p, \rho(x_{m+1}, y)^p\}$, therefore

$$\rho(x, y)^p \le 2\rho(x, x_{m+1})^p \tag{2.5}$$

(this is clear if $m \le n - 1$ and obvious for $m = n$). Combining (2.4) and (2.5) gives

$$\begin{aligned} \rho(x, y)^p &\le 2 \min\{\rho(x, x_{m+1})^p, \rho(x_m, y)^p\} \\ &\le \rho(x, x_{m+1})^p + \rho(x_m, y)^p. \end{aligned}$$

If it happens that $m = 0$ or $m = n$, then the first inequality above readily gives the required conclusion, i.e. (2.3) with n replaced by $n+1$, and there is actually no need to invoke the induction hypothesis. Assume therefore that $1 \le m \le n-1$. Then, applying the induction hypothesis to the sequences $x = x_0, x_1, \ldots, x_{m+1}$ and

$x_m, x_{m+1}, \dots, x_{n+1} = y$ (both of length $\le n+1$) gives

$$\begin{aligned}\rho(x,y)^p &\le \rho(x,x_{m+1})^p + \rho(x_m,y)^p \\ &\le 2\Big(\rho(x_0,x_1)^p + 2\sum_{j=1}^{m-1}\rho(x_j,x_{j+1})^p + \rho(x_m,x_{m+1})^p\Big) \\ &\quad + 2\Big(\rho(x_m,x_{m+1})^p + 2\sum_{j=m+1}^{n-1}\rho(x_j,x_{j+1})^p + \rho(x_n,x_{n+1})^p\Big) \\ &= 2\Big(\rho(x_0,x_1)^p + 2\sum_{j=1}^{n-1}\rho(x_j,x_{j+1})^p + \rho(x_n,x_{n+1})^p\Big).\end{aligned}$$

This completes the induction step and thus the proof of the Proposition. □

If (X,ρ) is a quasi-metric space, then $\mathcal{F}_\rho$, the topology in X induced by ρ, is canonically defined by means of the theory of *uniform structures*; in case ρ is a metric this procedure leads to the usual metric topology in X. We refer the reader to the monograph [4], where in Chapter 8 this way of introducing a topology is discussed.

The uniform structure U_ρ generated by ρ is defined to consist of all subsets $V \subset X \times X$, symmetric in the sense that $(x,y) \in V$ if and only if $(y,x) \in V$ and containing a set of the form $R_\epsilon = \{(x,y) : \rho(x,y) < \epsilon\}$ for some $\epsilon > 0$ (in particular V contains the diagonal $\{(x,x) : x \in X\}$). Since the countable family $\{R_{1/n}\}_{n\ge 1}$ is a *basis for the uniform structure* U_ρ, it follows from a general result (see [4, Chapter 8, Theorem 9]) that the topology $\mathcal{F}_\rho$ generated by U_ρ in X is metrizable.

Given $r > 0$ and $x \in X$, let

$$B(x,r) = \{y \in X : \rho(x,y) < r\}$$

be the *quasi-metric ball* related to ρ of radius r and with center x. According to the procedure of defining a topology by means of a uniform structure (see [4, Chapter 8]), in this case $G \subset X$ is defined to be open, i.e. $G \in \mathcal{F}_\rho$, if and only if for every $x \in G$ there exists $r > 0$ such that $B(x,r) \subset G$ (at this point one easily checks directly that the topology axioms are satisfied for such a definition). It is clear that if ρ_1 is a quasi-metric equivalent to ρ, then $\mathcal{F}_{\rho_1} = \mathcal{F}_\rho$; also, for any $a > 0$, ρ^a is a quasi-metric as well and $\mathcal{F}_{\rho^a} = \mathcal{F}_\rho$. Thus the Proposition furnishes a direct argument showing that $\mathcal{F}_\rho$ is metrizable [1], [8].

The quasi-metric balls themselves need not be open (unless ρ is a genuine metric) as the following simple example shows.

Example. Let $X = \{0,1,2,\dots\}$. Given $\epsilon > 0$, we define $\rho = \rho_\epsilon$ on $X \times X$ in the following way. For $0 \le n < m$, we set $\rho(n,m)$ as

$$\rho(0,1) = 1, \qquad \rho(0,m) = 1+\epsilon \quad \text{if } m \ge 2,$$
$$\rho(1,m) = \frac{1}{m}, \qquad \rho(n,m) = \frac{1}{n} + \frac{1}{m} \quad \text{if } n \ge 2.$$

We then extend ρ onto $X \times X$ by putting $\rho(n,n) = 0$ for any $n \ge 0$ and $\rho(n,m) = \rho(m,n)$ if $0 \le m < n$. We will show that

$$\rho(k,n) \le (1+\epsilon)\big(\rho(k,m) + \rho(m,n)\big), \quad k,m,n \in X, \tag{2.6}$$

and thus ρ is a quasi-metric with $K = 1+\epsilon$. It is clear that it suffices to check (2.6) for pairwise distinct k, m, n only. Let L and R denote the left and the right hand sides of the inequality (2.6), respectively. If one of k, n, m is 0, then $L \le 1+\epsilon \le R$. If none of k, m, n are 0, then we consider subcases. First, assume 1 appears among k, m, n. If $k = 1$, then $L = \frac{1}{n}$ while $R = (1+\epsilon)(\frac{2}{m} + \frac{1}{n})$. Similarly if $n = 1$. If $m = 1$, then $L = \frac{1}{k} + \frac{1}{n}$ while $R = (1+\epsilon)(\frac{1}{k} + \frac{1}{n})$. Next, assume 1 does not appear among k, n, m. Then $L = \frac{1}{k} + \frac{1}{n}$ and $R = (1+\epsilon)(\frac{1}{k} + \frac{2}{m} + \frac{1}{n})$. This finishes our checking that ρ is indeed a quasi-metric with constant $K = 1+\epsilon$.

Now, note that $B(0, 1+\epsilon/2) = \{0, 1\}$ while $B(1, \eta)$ contains infinitely many elements, for any $\eta > 0$. Hence, none of $B(1, \eta)$ are contained in $B(0, 1+\epsilon/2)$, which shows that $B(0, 1+\epsilon/2)$ is not open.

The metric d_p, produced from $\rho = \rho_\epsilon$ by the p-chain approach, can be computed directly. Let p be defined as in the Proposition, with $1+\epsilon$ in place of K. It is not hard to see that with the notation $x_0 = \big((1+\epsilon)^p - 1\big)^{-1/p}$, d_p is given by

$$d_p(k,n) = \begin{cases} 1 + \max\{k,n\}^{-p}, & \text{if } \min\{k,n\} = 0 \text{ and } \max\{k,n\} > x_0, \\ \rho(k,n)^p, & \text{otherwise.} \end{cases} \tag{2.7}$$

It is easily seen that if for a quasi-metric space (X, ρ) there exists a function $K(\epsilon)$, $\epsilon > 0$, such that $K(\epsilon) \to 1$ as $\epsilon \to 0^+$, and

$$\forall\, x, y, z \in X \quad \rho(y,z) < \epsilon\rho(x,y) \Rightarrow \rho(x,z) \le K(\epsilon)\big(\rho(x,y) + \rho(y,z)\big),$$

then the quasi-metric balls $B(x, r)$, $x \in X$, $r > 0$, are open sets in the topology $\mathcal{F}_\rho$.

In particular, if a quasi-metric ρ which, for some p, $0 < p \le 1$, satisfies the p-triangle inequality

$$\rho(x,z) \le \big(\rho(x,y)^p + \rho(y,z)^p\big)^{1/p}, \quad x, y, z \in X, \tag{2.8}$$

then $K(\epsilon) = (1+\epsilon^p)^{1/p}$ is appropriate. Consequently, a quasi-metric space (X, ρ) with ρ satisfying (2.8) has all its quasi-metric balls open; hence this also happens for the quasi-metric $\tilde\rho = \tilde\rho(p)$ from the Proposition.

It is clear that the quasi-metric space (X, ρ) from the Example must fail to satisfy (2.8) for any p, $0 < p \le 1$. To see this by a direct argument note that for $x = 0, y = 1$ and $z = n \ge 2$ we have $\rho(0, n) = 1+\epsilon$, while

$$\big(\rho(0,1)^p + \rho(1,n)^p\big)^{1/p} = \big(1 + n^{-p}\big)^{1/p} \to 1, \qquad \text{as } n \to \infty.$$

Given a quasi-metric space (X, ρ), let $K(\rho)$ denote the smallest constant K for which (iii) holds, and let $p(\rho)$ denote the largest $p \in (0,1]$ for which (2.8) holds; if such p does not exist, then we set $p(\rho) = 0$. For instance, for the quasi-metric ρ_ϵ from the example we have $K(\rho_\epsilon) = 1+\epsilon$ and $p(\rho_\epsilon) = 0$. Also, let $\tilde p(\rho)$ denote the supremum of the set of $p \in (0,1]$ with the property that there exists a quasi-metric $\tilde\rho$ equivalent to ρ and such that (2.8) is satisfied with $\tilde\rho$ replacing ρ. Equivalently,

$$\tilde p(\rho) = \sup\{p \in (0,1] : d_p \quad \text{is a metric equivalent to } \rho^p\}.$$

It follows from the Proposition that for any given (X, ρ) one has

$$\tilde p(\rho) \ge \frac{1}{\log_2(2K(\rho))}. \tag{2.9}$$

Note that in [1] the weaker estimate

$$\tilde{p}(\rho) \geq \frac{1}{(\log_2(3K(\rho)))^2}$$

was proved. A simple example of the usual ℓ^p spaces, $0 < p < 1$, shows that the estimate (2.9) cannot, in general, be improved.

It can happen, however, that the inequality (2.9) is strict. In fact, for the quasi-metric ρ_ϵ from the above example one can apply the p-chain approach for any $0 < p \leq 1$ and obtain a metric. Thus for this rather pathological example, actually $\tilde{p}(\rho) = 1$.

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