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CONSTRUCTING SEPARATED SEQUENCES IN BANACH SPACES

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Abstract. A construction of separated sequences in the unit sphere of a Banach space is given. If a space X admits an equivalent nearly uniformly convex norm or c_0 is not finitely representable in X, then lower bounds for separation constants of sequences are strictly greater than 1. This gives a partial answer to a problem posed by J. Diestel.

1. Introduction

Separated sequences appear in various problems of Banach space theory. One of them is the packing problem (see [20]). Separated sequences are also used in the construction of measures of noncompactness, which in turn are important tools in metric fixed point theory (see [2]). In this paper we consider the separation measure of noncompactness of the unit ball of a Banach space which is also called Kottman's constant. A recent application of this constant is given in [19].

In [10], J. Elton and E. Odell proved that the unit sphere S_X of an infinite dimensional Banach space X contains a sequence (x_n) such that

$$\operatorname{sep}(x_n) = \inf_{i \neq j} \|x_i - x_j\| > 1.$$

Presenting this theorem in [7], J. Diestel posed the problem for which Banach spaces X there is an $s > 1$ such that given any infinite dimensional subspace Y of X, one can find a sequence (y_n) in S_Y with $\operatorname{sep}(y_n) > s$. The greatest s with this property will be denoted by $s(X)$. Partial answers to Diestel's problem were obtained in [29], [6] and [25]. The results given in those papers guarantee the existence of sequences with the separation constants bounded from below by some coefficients or values of some moduli, in particular the modulus of convexity.

In [22], it was proved that there exists a constant $c > 1$ such that for each nonreflexive Banach space one can find a sequence (x_n) in S_X for which $\operatorname{sep}(x_n) > c$. In this paper we introduce new coefficients, and using an idea from [22], we establish lower bounds for $s(X)$. In contrast to the constants considered in [29], [6] and [25], one of our lower bounds for $s(X)$ is isomorphically invariant. Our results give new answers to Diestel's problem. They show in particular that $s(X) > 1$ if X admits an equivalent nearly uniformly convex norm or if c_0 is not finitely representable in X. This covers the case when X is superreflexive. We find the values of $s(L_p)$ and

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show that our estimates for $s(X)$ are sharper than some of the estimates obtained in [6] and [25].

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2. Preliminaries

In this paper we consider only infinite dimensional Banach spaces. Let X be such a space. By S_X and B_X we denote its unit sphere and closed unit ball, respectively. By a subspace of a Banach space we mean a closed linear subspace. In the sequel we will use the following theorem, whose proof can be found in [4].

Theorem 1. *Let (x_n) be a bounded sequence in a Banach space X. Then there exists a subsequence (y_n) of (x_n) such that for all scalars $\alpha_1, \dots, \alpha_m$ the following limit exists:*

$$L(\alpha_1, \dots, \alpha_m) = \lim_{\substack{n_1 < \dots < n_m \\ n_1 \to \infty}} \left\| \sum_{i=1}^{m} \alpha_i y_{n_i} \right\|. \tag{1}$$

Let X be a Banach space. By $\mathcal{N}(X)$ we denote the set of all sequences (y_n) in X such that all limits (1) exist, (y_n) converges weakly to 0 and (y_n) does not have a norm-convergent subsequence. Let us recall that a Banach space X is said to have the Schur property if weakly convergent sequences in X are norm convergent. A space X does not have the Schur property if and only if $\mathcal{N}(X) \neq \emptyset$. Additionally by $\mathcal{N}_1(X)$ we denote the set of all sequences $(y_n) \in \mathcal{N}(X)$ of norm-one vectors.

We put $e_n = (0, \dots, 0, 1, 0, \dots)$, where 1 occupies the nth place and the remaining coordinates are 0. Let $(y_n) \in \mathcal{N}(X)$. Given scalars $\alpha_1, \dots, \alpha_m$, we put

$$\left| \sum_{i=1}^{m} \alpha_i e_i \right| = \lim_{\substack{n_1 < \dots < n_m \\ n_1 \to \infty}} \left\| \sum_{i=1}^{m} \alpha_i y_{n_i} \right\|.$$

This formula gives us a norm $|\cdot|$ on the space of all sequences of scalars with finitely many nonzero terms. Its completion is called a spreading model for the sequence (y_n). The sequence (e_n) is an unconditional basis of E. Actually, we have

$$\left| \sum_{i \in I} \alpha_i e_i \right| \leq \left| \sum_{i \in J} \alpha_i e_i \right|$$

for all finite sets $I \subset J \subset \mathbb{N}$ and all scalars α_i (see [3]). It follows that

$$\left| \sum_{i=1}^{m} \theta_i \alpha_i e_i \right| \leq 2 \left| \sum_{i=1}^{m} \alpha_i e_i \right| \tag{2}$$

for every $\theta_i \in \{-1, 1\}$. Obviously (e_n) is invariant under spreading, i.e.,

$$\left| \sum_{i=1}^{m} \alpha_i e_i \right| = \left| \sum_{i=1}^{m} \alpha_i e_{n_i} \right|$$

for all indices $n_1 < \dots < n_m$.

Let (x_n) be a sequence in a Banach space X. Let us recall that the separation constant of (x_n) is defined by the formula

$$\operatorname{sep}(x_n) = \inf_{i \neq j} \|x_i - x_j\|.$$

Next, given a nonempty bounded set $A \subset X$, we put

$$\beta(A) = \sup\{\operatorname{sep}(x_n) \colon (x_n) \subset A\}.$$

This number is called the separation measure of noncompactness of A. Clearly,

$$\beta(A) = \sup\left\{\lim_{\substack{n<m\\ n\to\infty}} \|x_n - x_m\|\right\},$$

where the supremum is taken over all sequences (x_n) in A for which the above limit exists. Let us mention that $\beta(S_X) = \beta(B_X)$ (see [1]). The coefficient $K(X) = \beta(B_X)$ is called Kottman's constant. We have $K(l_p) = 2^{1/p}$ and $K(L_p(0,1)) = \max\left\{2^{1/p}, 2^{1-1/p}\right\}$ for all $1 \le p < \infty$ (see [2, p. 31]).

Diestel's problem can now be formulated in the following way: for which spaces X does

$$s(X) = \inf\{K(Y) \colon Y \text{ is an infinite dimensional subspace of } X\} > 1?$$

Simple examples of such spaces are c_0 and l_1. From [23, Proposition 2.a.2] and James' distortion theorem (see [15]) it follows that $s(c_0) = 2 = s(l_1)$.

We introduce a coefficient which enables us to estimate $s(X)$ from below. Assume that X does not have the Schur property. Given a sequence $(x_n) \in \mathcal{N}_1(X)$, we put

$$l(x_n) = \limsup_{m\to\infty} \lim_{\substack{n_1<\cdots<n_{2^m}\\ n_1\to\infty}} \left\|\sum_{i=1}^{2^m} x_{n_i}\right\|^{\frac{1}{m}}.$$

Next, we set

$$\lambda(X) = \inf\{l(x_n) \colon (x_n) \in \mathcal{N}_1(X)\}.$$

Clearly, $1 \le \lambda(X) \le 2$, and we will show that $\lambda(X) \le s(X)$.

Consider the case when X has a basis (z_n). Let us recall that a sequence (y_n) is a block basic sequence of (z_n) if there exist a sequence (α_n) of scalars and a sequence $0 = n_1 < n_2 < \cdots$ of integers such that

$$y_k = \sum_{i=n_k+1}^{n_{k+1}} \alpha_i z_i$$

for every k. For every sequence $(x_n) \in \mathcal{N}(X)$ there exist a block basic sequence (y_n) of (z_n) and a subsequence (x_{m_k}) such that $\lim_{k\to\infty} \|x_{m_k} - y_k\| = 0$ (see [23, p. 7]). It follows that when evaluating $\lambda(X)$ it is enough to consider block basic sequences. Applying this remark to the space c_0, we see that $\lambda(c_0) = 1 < s(c_0)$.

Let us recall that the modulus of convexity of a Banach space X is defined as

$$\delta(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| \colon x, y \in B_X,\ \|x-y\| \ge \varepsilon\right\},$$

where $\varepsilon \in [0,2]$. The norm $\|\cdot\|$ is said to be uniformly convex if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A Banach space X is superreflexive if and only if X admits an equivalent uniformly convex norm (see [11]). Then X also admits an equivalent norm such that the corresponding modulus of convexity is of power type q for some $q \ge 2$; i.e., there exists $C > 0$ such that $\delta(\varepsilon) \ge C\varepsilon^q$ for every $\varepsilon \in [0,2]$ (see [30]).

3. Results

Lemma 2. *Let X be a Banach space, $(x_n) \in \mathcal{N}(X)$ and $\varepsilon_i \in \{-1,1\}$ for every $i \in \mathbb{N}$. Then*

$$\limsup_{m\to\infty} \lim_{\substack{n_1<\cdots<n_{2^m}\\ n_1\to\infty}} \left\|\sum_{i=1}^{2^m} \varepsilon_i x_{n_i}\right\|^{\frac{1}{m}} = \limsup_{m\to\infty} \lim_{\substack{n_1<\cdots<n_{2^m}\\ n_1\to\infty}} \left\|\sum_{i=1}^{2^m} x_{n_i}\right\|^{\frac{1}{m}}.$$

Proof. Let E be the spreading model for the sequence (x_n). From (2) we see that

$$\frac{1}{2}\left|\sum_{i=1}^{2^m} e_i\right| \le \left|\sum_{i=1}^{2^m} \varepsilon_i e_i\right| \le 2\left|\sum_{i=1}^{2^m} e_i\right|.$$

Hence

$$\limsup_{m\to\infty}\left|\sum_{i=1}^{2^m} \varepsilon_i e_i\right|^{\frac{1}{m}} = \limsup_{m\to\infty}\left|\sum_{i=1}^{2^m} e_i\right|^{\frac{1}{m}},$$

which is the conclusion of the lemma. □

Let X be a Banach space and (x_n) be a sequence in S_X with $\operatorname{sep}(x_n) > 0$. By Rosenthal's theorem ([32]; see also [7, p. 201]) either (x_n) has a subsequence equivalent to the standard basis of l_1 or there is a subsequence (x_{n_k}) such that $(x_{n_{2k-1}} - x_{n_{2k}})$ converges weakly to 0. In the first case, from James' distortion theorem it follows that $K(X) = 2$. Since the second case is impossible if X has the Schur property, for such space we have $s(X) = 2$.

Theorem 3. *Let X be a Banach space without the Schur property. Then $s(X) \ge \lambda(X)$.*

Proof. Our result can be derived from Krivine's theorem ([21]; see also [27, Theorem 3]), but we give a short proof which shows a direct way of constructing separated sequences. Let Y be an infinite dimensional subspace of X. By the preceding remark we can assume that Y does not have the Schur property and we take a sequence $(y_n) \in \mathcal{N}_1(Y)$. Consequently, the limits given by formula (1) exist, and we put

$$a_m = L(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2^m}), \quad b_m = L(\varepsilon_1, \ldots, \varepsilon_{2^{m-1}}, -\varepsilon_{2^{m-1}+1}, \ldots, -\varepsilon_{2^m}),$$

where $\varepsilon_i = (-1)^i$ for every $i \in \mathbb{N}$. Observe that for all indices $n_1 < \cdots < n_{2^m+1}$ we have

$$\left|\left\|\sum_{i=1}^{2^m} \varepsilon_i y_{n_i}\right\| - \left\|\sum_{i=1}^{2^{m-1}} \varepsilon_i y_{n_i} - \sum_{i=2^{m-1}+1}^{2^m} \varepsilon_i y_{n_{i+1}}\right\|\right| \le \|y_{n_{2^m-1}}\| + \|y_{n_{2^m+1}}\| = 2.$$

It follows that

$$|a_m - b_m| \le 2 \tag{3}$$

for every $m \in \mathbb{N}$.

It suffices to consider the case when $\lambda = \lambda(X) > 1$. From Lemma 2 we see that

$$\limsup_{m\to\infty} (a_m)^{\frac{1}{m}} \ge \lambda.$$

Since the sequence (a_m) is nondecreasing, $\lim_{m\to\infty} a_m = +\infty$. Given $\gamma \in (0, r)$, we can therefore find $m_0 \in \mathbb{N}$ so that

$$a_n \geq \frac{2}{\gamma} \tag{4}$$

for every $n > m_0$.

We will show that there exists $N > m_0$ such that

$$a_N \geq (\lambda - \gamma) a_{N-1}. \tag{5}$$

Suppose that this is not the case. Then $a_m < (\lambda - \gamma) a_{m-1}$ for every $m > m_0$ which easily yields $a_m < (\lambda - \gamma)^{m-m_0} a_{m_0}$. Consequently,

$$(a_m)^{\frac{1}{m}} < (\lambda - \gamma)^{1-\frac{m_0}{m}} (a_{m_0})^{\frac{1}{m}},$$

and passing to the limit superior with $m \to \infty$, we obtain a contradiction.

From (3), (5), and (4) we now see that

$$\frac{b_N}{a_{N-1}} \geq \frac{a_N}{a_{N-1}} - \frac{2}{a_{N-1}} \geq \lambda - 2\gamma.$$

We put

$$x_k = \sum_{i=1}^{N-1} \varepsilon_i y_{k(N-1)+i}$$

for $k \in \mathbb{N}$. Then

$$a_{N-1} = \lim_{n\to\infty} \|x_n\|, \quad b_N = \lim_{\substack{n<m \\ n\to\infty}} \|x_n - x_m\|.$$

We can assume that $\|x_k\| > 0$ for every $k \in \mathbb{N}$ and set $z_k = \frac{1}{\|x_k\|} x_k$. It is easy to see that

$$\frac{b_N}{a_{N-1}} = \lim_{\substack{n<m \\ n\to\infty}} \|z_n - z_m\| \leq K(Y).$$

This shows that $K(Y) \geq \lambda - 2\gamma$, and passing to the limit with $\gamma \to 0$, we obtain the inequality $K(Y) \geq \lambda$. □

It is easy to see that if X is a Banach space without the Schur property and a space Y is isomorphic to X, then $\lambda(Y) = \lambda(X)$. So we actually have the estimate

$$\inf\{s(Y) \colon Y \text{ is isomorphic to } X\} \geq \lambda(X).$$

Assume that X has the following property: there exist constants $p > 1$ and $C > 0$ such that every sequence $(x_n) \in \mathcal{N}(X)$ has a subsequence (x_{n_k}) such that

$$C \left(\sum_{k=1}^{m} |\alpha_k|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{k=1}^{m} \alpha_k x_{n_k} \right\|$$

for all scalars $\alpha_1, \ldots, \alpha_m$. Then $\lambda(X) \geq 2^{1/p}$. The class of such spaces contains all infinite dimensional spaces which admit equivalent nearly uniformly convex norms (see [31]). Let us recall that a norm $\|\cdot\|$ in X is nearly uniformly convex provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if (x_n) is a sequence in B_X with $\operatorname{sep}(x_n) \geq \varepsilon$, then $\|x\| \leq 1 - \delta$ for some $x \in \operatorname{conv}\{x_n \colon n \in \mathbb{N}\}$ (see [14]).

Corollary 4. *If a Banach space X admits an equivalent nearly uniformly convex norm, then $s(X) > 1$.*

A uniformly convex norm is nearly uniformly convex, so we see in particular that $s(X) > 1$ if X is superreflexive. We will extend this result to the class of all C-convex spaces, i.e., spaces X such that c_0 is not finitely representable in X.

Let us recall that a Banach space X has M-cotype p if there exists a constant C such that

$$\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{\frac{1}{p}} \leq C \max_{\varepsilon_k \in \{-1,1\}} \left\|\sum_{k=1}^{n} \varepsilon_k x_k\right\|$$

for all vectors $x_1, \ldots, x_n \in X$. If $(x_n) \in \mathcal{N}_1(X)$, then using (2), we obtain the following inequality in the spreading model for (x_n):

$$\begin{aligned} 2^{\frac{m}{p}} &= \left(\sum_{k=1}^{2^m} |e_k|^p\right)^{\frac{1}{p}} \leq C \lim_{\substack{n_1 < \cdots < n_{2^m} \\ n_1 \to \infty}} \max_{\varepsilon_k \in \{-1,1\}} \left\|\sum_{i=1}^{2^m} \varepsilon_i x_{n_i}\right\| \\ &= C \max_{\varepsilon_k \in \{-1,1\}} \left|\sum_{i=1}^{2^m} \varepsilon_i e_i\right| \leq 2C \left|\sum_{k=1}^{2^m} e_k\right| \end{aligned}$$

for every m. This shows that $\lambda(X) \geq 2^{1/p}$.

Corollary 5. *If a Banach space X has M-cotype p for some $p \geq 1$, then $s(X) \geq 2^{1/p}$.*

Assume that the modulus of convexity of X is of power type p for some $p \geq 2$. Then X has cotype p (see [12] or [24, Theorem 1.e.16]), so it has M-cotype p and consequently, $s(X) \geq 2^{1/p}$. If X is C-convex, then it has M-cotype p for some $p \geq 1$ (see [28] or [18, Theorem 5.2.3]), so Corollary 5 gives us also the following result.

Corollary 6. *If a Banach space X is C-convex, then $s(X) > 1$.*

A space X is C-convex if and only if $C(m, X) > 1$ for some $m \in \mathbb{N}$, where

$$C(m, X) = \inf\left\{\max\left\{\left\|\sum_{i=1}^{m} \varepsilon_i x_i\right\| : \varepsilon_i \in \{-1, 1\}\right\} : \|x_i\| \geq 1\right\}$$

(see [28] or [18, p. 65]). From Dvoretzki's theorem (see [9]) it follows that $C(m, X) \leq \sqrt{m}$ for every m. To avoid this limitation we modify the definition of $C(m, X)$ in the following way:

$$C_1(m, X) = \inf\left\{\max\left\{\lim_{\substack{n_1 < \cdots < n_m \\ n_1 \to \infty}} \left\|\sum_{i=1}^{m} \varepsilon_i x_{n_i}\right\| : \varepsilon_i \in \{-1, 1\}\right\} : (x_n) \in \mathcal{N}_1(X)\right\}.$$

Some properties of these coefficients are analogous to the properties of $C(m, X)$. In particular, we have the following result (compare to [18, Theorem 5.2.2]).

Lemma 7. *Let X be a Banach space without the Schur property. Then*

$$C_1(mk, X) \geq C_1(m, X) C_1(k, X)$$

for every k, m.

Proof. Let $(x_n) \in \mathcal{N}_1(X)$ and $m, k \in \mathbb{N}$. There are $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, such that

$$\lim_{j \to \infty} \|y_j\| = \lim_{\substack{n_1 < \cdots < n_m \\ n_1 \to \infty}} \left\|\sum_{i=1}^{m} \varepsilon_i x_{n_i}\right\| \geq C_1(m, X),$$

where

$$y_j = \sum_{i=1}^{m} \varepsilon_i x_{mj+i}.$$

We can assume that $\|y_j\| > 0$ for every j and put $z_j = \frac{1}{\|y_j\|} y_j$. It is easy to see that $(z_n) \in \mathcal{N}_1(X)$. Consequently, there exist $\theta_i \in \{-1, 1\}$, $i = 1, \ldots, k$, such that

$$\lim_{\substack{n_1 < \cdots < n_k \\ n_1 \to \infty}} \left\| \sum_{i=1}^{k} \theta_i z_{n_i} \right\| \geq C_1(k, X).$$

Then, in the spreading model E for (x_n) we have

$$\left| \sum_{j=0}^{k-1} \sum_{i=1}^{m} \theta_j \varepsilon_i e_{mj+i} \right| = \lim_{\substack{n_1 < \cdots < n_k \\ n_1 \to \infty}} \left\| \sum_{i=1}^{k} \theta_i y_{n_i} \right\| = \lim_{n \to \infty} \|y_n\| \lim_{\substack{n_1 < \cdots < n_k \\ n_1 \to \infty}} \left\| \sum_{i=1}^{k} \theta_i z_{n_i} \right\|$$
$$\geq C_1(m, X) C_1(k, X).$$

This clearly gives us the desired inequality. □

If $(x_n) \in \mathcal{N}_1(X)$, then using (2), we obtain the following inequality in the spreading model for (x_n):

$$C_1(2^m, X) \leq 2 \left| \sum_{i=1}^{2^m} e_i \right|$$

for every m. From Lemma 7 we see that the sequence $(C_1(2^n, X)^{1/n})$ is convergent. Therefore, the above inequality gives us the following theorem.

Theorem 8. *Let X be a Banach space without the Schur property. Then*

$$C_1(2^n, X)^{\frac{1}{n}} \leq \lim_{m \to \infty} C_1(2^m, X)^{\frac{1}{m}} \leq \lambda(X)$$

for every n.

Clearly, $C(n, X) \leq C_1(n, X)$, and in particular we obtain the estimate $C(2, X) \leq \lambda(X)$. Observe that

$$C(2, X) = \inf \{\max \{\|x + y\|, \|x - y\|\} : x, y \in S_X\}.$$

This coefficient is also denoted by $g(X)$ (see [13]). In [25], it was shown that $K(X) \geq g(X)$. From Theorem 3 we see that $s(X) \geq \lambda(X) = \lambda(Y)$ for any space Y isomorphic to X. This and Theorem 8 give us the following estimate.

Corollary 9. *Let X be a Banach space. Then*

$$s(X) \geq \sup \{g(Y) \colon Y \text{ is isomorphic to } X\}.$$

Let $1 < p < \infty$. Considering block basic sequences of the standard basis of l_p, we see that $C_1(m, l_p) = m^{1/p}$ for every m. In particular $C_1(2, l_p) = 2^{1/p}$, which is also the value of $K(l_p) = s(l_p) = \lambda(l_p)$. Consequently, if a Banach space X contains isometric copies of l_p with p arbitrarily large, then $s(X) = 1$. In particular $s(C([0, 1])) = 1$, but it is also easy to construct a reflexive space X with $s(X) = 1$. Namely, it suffices to put

$$X = \left(\sum_{k=1}^{\infty} l_{p_k} \right)_{l_2},$$

where $p_k > 1$ for all k and $\lim_{k \to \infty} p_k = +\infty$.

For spaces X without the Schur property the weakly convergent sequence coefficient $WCS(X)$ is defined. We have the formula

$$WCS(X) = \inf \left\{ \lim_{\substack{n<m \\ n\to\infty}} \|x_n - x_m\| \right\},$$

where the infimum is taken over all weakly null sequences (x_n) in S_X for which the above limit exists (see [2, p. 120]). Clearly, $WCS(X) \le C_1(2, X)$.

Using our estimates, we can compute $s(L_p(0,1))$. If $1 \le p < \infty$, then $WCS(L_p(0,1)) = \min\{2^{1/p}, 2^{1-1/p}\}$ (see [2, Theorem 6.3]). Consider the case when $p \ge 2$. Then we have $WCS(L_p(0,1)) = 2^{1/p}$. Since $L_p(0,1)$ contains an isometric copy of l_p, we get $WCS(L_p(0,1)) = C_1(2, L_p(0,1)) = \lambda(L_p(0,1)) = s(L_p(0,1)) = 2^{1/p}$.

Assume now that $1 \le p < 2$. Then $WCS(L_p(0,1)) = 2^{1-1/p}$, and this value is attained for the sequence of Rademacher functions. We therefore see that $WCS(L_p(0,1)) = C_1(2, L_p(0,1))$. But $L_p(0,1)$ has cotype 2 (see [24, p. 73]), so Corollary 5 shows that $\lambda(L_p(0,1)) \ge \sqrt{2}$. Moreover, $L_p(0,1)$ contains an isometric copy of l_2 (see [17, p. 16]). From this and Theorem 3 we obtain $s(L_p(0,1)) = \lambda(L_p(0,1)) = \sqrt{2}$. In particular we see that $s(L_p(0,1)) < K(L_p(0,1))$ if $p \ne 2$.

In [6], it was shown that

$$K(X) \ge d = \inf_{x\in S_X} \sup \left\{ \inf_{z\in S_Z} \|x+z\| \right\},$$

where the supremum is taken over all subspaces Z of X with finite codimension. From [26, Theorem 4] it follows that if X does not have the Schur property, then

$$d \le \inf_{x\in S_X} \inf \left\{ \limsup_{n\to\infty} \|x+y_n\| \colon (y_n) \in \mathcal{N}_1(X) \right\} \le WCS(X).$$

We have already observed that there are spaces X with $WCS(X) < s(X)$. There are also spaces such that $WCS(X) < g(X) < C_1(2, X)$. As an example we take $X = l_{2,\infty}$. This is the space l_2 with the norm

$$\|x\| = \max\left\{\|x^+\|_2, \|x^-\|_2\right\},$$

where $x \in l_2$ and $\|\cdot\|_2$ is the standard norm of l_2. Here we treat l_2 as a lattice with the standard order, so $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$. Let (e_n) be the standard basis of l_2. Then $\|e_n\| = 1 = \|e_n - e_m\|$ for every n and $m \ne n$. Consequently, $WCS(X) = 1$. Results in [16] and [5] show that $g(X) = 2\sqrt{2}/(1+\sqrt{2}) > WCS(X)$.

We will show that $s(X) = K(X) = C_1(2, X) = \sqrt{2} > g(X)$. When computing $C_1(2, X)$ it is enough to consider block basic sequences (y_k) of (e_n), and the same is true for $K(X)$ (see [8]). The vectors y_k are pairwise disjoint, and it is easy to see that

$$\|y_i - y_j\| \le \max\{\|y_i + y_j\|, \|y_i - y_j\|\} = \left(\|y_i\|^2 + \|y_j\|^2\right)^{\frac{1}{2}}$$

if $i \ne j$. It follows that $K(X) \le \sqrt{2} = C_1(2, X)$. In view of Theorems 3 and 8, this shows that $K(X) = s(X) = C_1(2, X) = \sqrt{2}$.

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