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CASTELNUOVO-MUMFORD REGULARITY AND THE REDUCTION NUMBER OF SOME MONOMIAL CURVES

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Abstract. We compare the Castelnuovo-Mumford regularity and the reduction number of some classes of monomial projective curves with at most one singular point. Furthermore, for smooth monomial curves we prove an upper bound on the regularity which is stronger than the one given by L'vovsky.

1. Introduction

Let K be a field and S be the submonoid of $\mathbf{N}^2$ generated by a set

$$\mathcal{A} := \{(\alpha, 0), (\alpha - a_1, a_1), \ldots, (\alpha - a_c, a_c), (0, \alpha)\},$$

where $0 < a_1 < \cdots < a_c < \alpha$ is a sequence of relatively prime integers. The ring $K[S] = K[t_1^\alpha, t_1^{\alpha-a_1}t_2^{a_1}, ..., t_1^{\alpha-a_c}t_2^{a_c}, t_2^\alpha] \subset K[t_1, t_2]$ is isomorphic to the coordinate ring of a monomial curve of degree α in $\mathbb{P}^{c+1}$. By a famous result of Gruson-Lazarsfeld-Peskine [6], the Castelnuovo-Mumford regularity $\operatorname{reg} K[S]$ is bounded by $\alpha - c$. In view of the Eisenbud-Goto conjecture [4] we call the latter number the Eisenbud-Goto bound. Moreover, developing the method of [6], L'vovsky gave in [9, Proposition 5.5] a new bound which is much better than $\alpha - c$. In terms of gaps (see Definition 2.1), $\alpha - c = \sum(\sharp L) + 1$, where L runs over all gaps of $\mathcal{A}$, while L'vovsky's bound is $\sharp L + \sharp L' + 1$, where L, L' are the longest and the second longest gaps of $\mathcal{A}$. Since the local cohomology modules of $K[S]$ can be completely described in terms of S (see, e.g., [3, 11] or Lemma 2.2), it is of interest to ask for a combinatorial proof of the Eisenbud-Goto bound and/or L'vovsky's bound. In the case of smooth monomial curves, Herzog and Hibi recently gave a combinatorial proof of the Eisenbud-Goto bound in [7], but that proof does not work for L'vovsky's bound. Our first result is to provide a new bound on the Castelnuovo-Mumford regularity (Theorem 2.7), which is even better than L'vovsky's bound. The proof is purely combinatorial, but unfortunately it works only for smooth monomial curves.

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Our original motivation came from the fact that $r(S) \leq \alpha - c$, where $r(S)$ is the reduction number of $K[S]$ with respect to the natural minimal reduction of $K[S]$, and there is a rather close relation between $r(S)$ and $\operatorname{reg} K[S]$, namely $r(S) \leq \operatorname{reg} K[S] \leq \max\{2r(S) - 2, 1\}$ (see [12] and [8, Theorem 1.1 and Theorem 3.1]). Therefore, it is natural to ask

Question. When does

$$(Q) \qquad r(S) = \operatorname{reg} K[S]$$

hold?

Whenever this question has a positive answer, it gives a combinatorial proof of the Eisenbud-Goto bound for the regularity of monomial curves. Unfortunately this equality does not hold in general. We can even construct a counterexample from the class of smooth monomial curves (see Example 3.2). Nevertheless, we can give some classes of monomial curves, for which the above question has a positive answer.

This paper has three sections. In Section 2 we first show that in order to establish L'vovsky's bound (by a combinatorial method) it suffices to study the first local cohomology module. Then we provide a new bound on the Castelnuovo-Mumford regularity for smooth monomial curves. In Section 3 we first give a combinatorial interpretation of Question (Q). Based on it we provide Example 3.2, where $r(S) < \operatorname{reg} K[S]$. Then we give four classes of monomial curves for which these two invariants agree. Monomial curves in this paper have at most one singular point.

2. A new bound for the Castelnuovo-Mumford regularity

We will speak about the monomial curve defined by $0 < a_1 < \ldots < a_c < a_{c+1} := \alpha$, and this shall mean that the second coordinates of the elements of $\mathcal{A}$ are $0, a_1, \ldots, a_c, a_{c+1}$. Note that this curve is *smooth* if and only if $a_1 = 1$ and $a_c = \alpha - 1$, and it *has at most one singular point* if and only if either $a_1 = 1$ or $a_c = \alpha - 1$.

Notation. The degree of an element of the group $G \subseteq \mathbb{Z}^2$ generated by S is defined as the sum of its entries, divided by α. For subsets $\mathcal{B}, C$ of $\mathbf{N}^d$ we set $\mathcal{B} + \mathcal{C} := \{b + c |\ b \in \mathcal{B}, c \in \mathcal{C}\}$, $m\mathcal{B} = \mathcal{B} + \cdots + \mathcal{B}$ (m times). We denote by $\mathcal{B}_i$ the set of i-th coordinates of elements of $\mathcal{B}$. Finally, we set $\mathbf{g}_i = (\alpha - i, i)$, where $0 \leq i \leq \alpha$, and we also write $\mathbf{e}_1 := \mathbf{g}_0$ and $\mathbf{e}_2 := \mathbf{g}_\alpha$.

Each element of $\mathcal{A}$ can be interpreted as an integer point of the segment $[\mathbf{e}_1, \mathbf{e}_2]$.

Definition 2.1. A maximal set L of consecutive integer points of the segment $[\mathbf{e}_1, \mathbf{e}_2]$ not belonging to $\mathcal{A}$ is called a *gap* of $\mathcal{A}$; that is, L has the form $L = \{\mathbf{g}_i, \mathbf{g}_{i+1}, ..., \mathbf{g}_j\} \subset [\mathbf{e}_1, \mathbf{e}_2] \setminus \mathcal{A}$ such that $\mathbf{g}_{i-1}, \mathbf{g}_{j+1} \in \mathcal{A}$. If L is a gap of $\mathcal{A}$, its *length* is $\sharp L$.

If L is a gap of $\mathcal{A}$, then L_i is also said to be a gap of $\mathcal{A}_i$ $(i = 1, 2)$.

Let

$$\operatorname{end}(H^i_{\mathfrak{m}}(K[S])) := \max\{n |\ H^i_{\mathfrak{m}}(K[S])_n \neq 0\}$$

(with the convention that $\operatorname{end}(H^i_{\mathfrak{m}}(K[S])) = -\infty$ if $H^i_{\mathfrak{m}}(K[S])) = 0$). Then the Castelnuovo-Mumford regularity is the number

$$\operatorname{reg}(K[S]) := \max\{\operatorname{end}(H^1_{\mathfrak{m}}(K[S])) + 1,\ \operatorname{end}(H^2_{\mathfrak{m}}(K[S])) + 2\}.$$

This invariant is always positive. Applying [6] we get that $\operatorname{reg} K[S] \leq \sum(\sharp L)+1$, where L runs over all gaps of $\mathcal{A}$. L'vovsky's bound [9, Proposition 5.5] says that $\operatorname{reg} K[S] \leq \sharp L + \sharp L' + 1$, where L, L' are the longest and the second longest gaps of $\mathcal{A}$ (if there is only one gap, we set $L' = \emptyset$). We first show that in order to establish L'vovsky's bound (by a combinatorial method) it suffices to study the first local cohomology module $H^1_{\mathfrak{m}}(K[S])$.

Lemma 2.2. *Let S_i be the numerical semigroup generated by the i-th coordinates of the elements of $\mathcal{A}$ ($i = 1, 2$) and let $S' = G \cap (S_1 \times S_2)$. Then as $\mathbb{Z}$-graded modules we have*

(i) $H^2_{\mathfrak{m}}(K[S]) \cong K[G \cap ((\mathbb{Z} \setminus S_1) \times (\mathbb{Z} \setminus S_2))]$.

(ii) $H^1_{\mathfrak{m}}(K[S]) \cong K[S' \setminus S]$.

Proof. (i) Clearly $S - \mathbb{N}\mathbf{e}_1 = \{\mathbf{u} \in G |\ u_2 \in S_2\}$ and $S - \mathbb{N}\mathbf{e}_2 = \{\mathbf{u} \in G |\ u_1 \in S_1\}$. By [11, Corollary 3.8] we get

$$H^2_{\mathfrak{m}}(K[S]) \cong K[G \setminus ((S - \mathbb{N}\mathbf{e}_1) \cup (S - \mathbb{N}\mathbf{e}_2))] = K[G \cap ((\mathbb{Z} \setminus S_1) \times (\mathbb{Z} \setminus S_2))].$$

(ii) See [3, Lemma 2.6]. □

Recall that the *conductor* of a numerical semigroup Γ is the smallest integer c such that Γ contains all integer numbers from c. We cite a result from [1] here:

Lemma 2.3 ([1, Th. 3.1.1])**.** *Let Γ be the numerical semigroup generated by a sequence of relatively prime integers $0 < a_1 < a_2 < \cdots < a_n$ ($n \geq 2$). Then the conductor of Γ is at most $(a_1 - 1)(a_n - 1)$.*

Definition 2.4. We say the subset $m\mathcal{A}$, where m is a positive integer, is *full* if $m\mathcal{A} = \{(m\alpha, 0), (m\alpha - 1, 1), \ldots, (0, m\alpha)\}$.

The following result says that for any monomial curve, $\operatorname{end}(H^2_{\mathfrak{m}}(K[S])) + 2$ is at most the sum of lengths of the first and the last gaps of $\mathcal{A}$ minus 1.

Corollary 2.5. (i) $\operatorname{end}(H^2_{\mathfrak{m}}(K[S])) \leq a_1 + \alpha - a_c - 3$.

(ii) *If the curve is smooth, then $S' = G \cap \mathbb{N}^2$ (note that $G \cap \mathbb{N}^2$ is the set of $(u_1, u_2) \in \mathbb{N}^2$ such that $u_1 + u_2$ is a multiple of α) and*

$$\begin{aligned} \operatorname{reg} K[S] &= \max\{\deg \mathbf{u} |\ \mathbf{u} \in (S' \setminus S) \cup \{0\}\} + 1 \\ &= \min\{m > 0 |\ m\mathcal{A} \text{ is full}\}. \end{aligned}$$

Proof. (i) Let $\mathbf{u} \in (\mathbb{Z} \setminus S_1) \times (\mathbb{Z} \setminus S_2)$. Then by Lemma 2.3

$$\deg \mathbf{u} \leq [(a_1 - 1)(\alpha - 1) + (\alpha - a_c - 1)(\alpha - 1) - 2]/\alpha < a_1 + \alpha - a_c - 2.$$

Hence the first statement follows from Lemma 2.2(i).

(ii) If the curve is smooth, then by (i) $\operatorname{end}(H^2_{\mathfrak{m}}(K[S])) + 2 \leq 1$. In this case $S_1 = S_2 = \mathbb{N}$. Hence $S' = G \cap \mathbb{N}^2$ and the equalities follow from Lemma 2.2(ii). □

Thus, in order to give a new proof for L'vovsky's bound it suffices to study when an element of S' belongs to S. Let Γ be a numerical semigroup generated by the numbers $a_1 < a_2 < \cdots < a_n$. If $x \in \Gamma$ we define the *degree of* x in Γ (w.r.t. a given subset of generators) as follows:

$$\delta_\Gamma := \min\{p_1 + \cdots + p_n |\ x = p_1 a_1 + \cdots + p_n a_n \text{ and } p_i \in \mathbb{N}\}.$$

For short we shall write $\delta_i(x)$ for $\delta_{S_i}(x)$, where $S_1,\ S_2$ are numerical semigroups defined in Lemma 2.2. The following result is elementary, but it is in some cases useful to decide whether or not $x \in S$ for some $x \in S'$.

Lemma 2.6 ([3, Lemma 3.1])**.** *Keep the notation of Lemma* 2.2*. Let* $\mathbf{u} := (u_1, u_2) \in S'$*. Then* $\mathbf{u} \in S \iff \deg(\mathbf{u}) \geq \delta_1(u_1) \iff \deg(\mathbf{u}) \geq \delta_2(u_2)$*.*

In the next theorem we give a new bound on the Castelnuovo-Mumford regularity of smooth curves which is clearly much better than L'vovsky's bound.

Theorem 2.7. *Assume that the smooth monomial curve is defined by* $0 < a_1 < \cdots < a_c < \alpha$*. Let* $\varepsilon = \max\{i|\ 1, ..., i, \alpha - 1, ..., \alpha - i \in \{a_1, \ldots, a_c\}\} \geq 1$*. Denote by* $\lambda(\mathcal{A})$ *the longest length of a gap of* $\mathcal{A}$*. Then*

$$\operatorname{reg} K[S] \leq \left[\frac{\lambda(\mathcal{A}) - 1}{\varepsilon}\right] + 2.$$

Proof. Note that in our situation we have

$$K[S] \text{ is Cohen-Macaulay } \iff S = S' \iff \mathcal{A} \text{ is full.}$$

If these equivalent conditions hold, then $\operatorname{reg} K[S] = 1$ and the claim is clear. Assume $S' \neq S$. Recall that $\mathbf{g}_i = (\alpha - i, i)$. Let $\mathbf{u} \in S' \setminus S$. Then we can write $\mathbf{u} = \mathbf{y} + p\mathbf{e}_1 + q\mathbf{e}_2$, where $p, q \in \mathbb{N}$, $\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2$ and $y_1 + y_2 = \alpha$. Since $\mathbf{u} \notin S$, $\mathbf{y} + i\mathbf{e}_1 \notin S$ for all $i = 0, ..., p$. Note that

$$\mathbf{y} + i\mathbf{e}_1 = (i - j)\mathbf{g}_t + j\mathbf{g}_{t+1} + (y_1 + ti + j, y_2 - ti - j),$$

where $0 \leq j \leq i \leq p$ and $0 \leq t \leq \varepsilon - 1$. By the definition of ε we have $\mathbf{g}_0, ..., \mathbf{g}_\varepsilon \in S$. Hence for all i, j, t as above we must have $(y_1 + ti + j, y_2 - ti - j) \notin S$. This means that

$$(y_1, \alpha - y_1), (y_1 + 1, \alpha - (y_1 + 1)), \ldots, (y_1 + \varepsilon p, \alpha - (y_1 + \varepsilon p)) \notin \mathcal{A}. \tag{1}$$

Similarly, the condition $y + q\mathbf{e}_2 \notin S$ forces

$$(y_1, \alpha - y_1), (y_1 - 1, \alpha - (y_1 - 1)), \ldots, (y_1 - \varepsilon q, \alpha - (y_1 - \varepsilon q)) \notin \mathcal{A}.$$

Therefore, if L denotes the gap containing $\mathbf{y}$, then $\sharp L \geq 1 + (p + q)\varepsilon$. Since $\deg \mathbf{u} = p + q + 1$, we get

$$\deg \mathbf{u} \leq \left[\frac{\sharp L - 1}{\varepsilon}\right] + 1 \leq \left[\frac{\lambda(\mathcal{A}) - 1}{\varepsilon}\right] + 1. \tag{2}$$

By Corollary 2.5(ii), the claim follows. □

In the next section, using the reduction number, we can give a lower bound for the Castelnuovo-Mumford regularity in terms of the length of a certain gap of $\mathcal{A}$ (see Proposition 3.4). There we also give a sufficient condition to attain the above upper bound. In general, even for smooth monomial curves, the relation between the Castelnuovo-Mumford regularity and lengths of gaps of $\mathcal{A}$ is quite complicated; see Proposition 3.5.

3. Relation between the Castelnuovo-Mumford regularity and the reduction number

Note that $\mathfrak{q} := (t_1^\alpha, t_2^\alpha)$ is a minimal reduction of the maximal homogeneous ideal $\mathfrak{m}$ of $K[S]$. Let $r(S)$ denote the reduction number of $\mathfrak{m}$ with respect to $\mathfrak{q}$, that is, $r(S) = \min\{r \in \mathbb{N}|\ \mathfrak{m}^{r+1} = \mathfrak{q}\mathfrak{m}^r\}$. Note that one can easily compute $r(S)$ as follows:

$$r(S) = \min\{r \in \mathbb{N} | (r + 1)\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2\} + r\mathcal{A}\} \geq 1.$$

One has a rather close relation between $r(S)$ and $\operatorname{reg} K[S]$, namely

$$r(S) \leq \operatorname{reg} K[S] \leq \max\{2r(S) - 2,\ 1\} \tag{3}$$

(by [12] and [8, Theorem 3.1]). (The lower inequality holds for any graded algebra.) From many computations we thought that for monomial curves one always has $r(S) = \operatorname{reg} K[S]$. However, later we found a counterexample. In order to explain the way to find it we need an auxiliary combinatorial result.

Assume that the given monomial curve is smooth, i.e. $a_1 = 1$ and $a_c = \alpha - 1$. We say that $\mathcal{A}$ has (P_1) if for every $m \in \mathbb{N}$ one has

$$m\mathcal{A} \text{ not full} \Rightarrow m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\} \text{ not full.}$$

The following lemma contains a combinatorial translation of (Q):

Lemma 3.1. *Assume that the given monomial curve is smooth. Then $\mathcal{A}$ has $(P_1) \iff \operatorname{reg} K[S] = r(S)$.*

Proof. $\Longrightarrow$: Let $m = \operatorname{reg} K[S] - 1$. By Corollary 2.5, $m\mathcal{A}$ is not full, but $(m+1)\mathcal{A}$ is full. If $m \geq r(S)$, then by the definition of $r(S)$ we have $(m+1)\mathcal{A} = m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\}$ and $m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\}$ is full. This contradicts the property (P_1) of $\mathcal{A}$. Hence $m \leq r(S) - 1$ or equivalently $\operatorname{reg} K[S] \leq r(S)$, which forces $\operatorname{reg} K[S] = r(S)$.

$\Longleftarrow$: Assume that $m\mathcal{A}$ is not full. If $m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\}$ is full, then $(m+1)\mathcal{A} = m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\}$. This means $r(S) \leq m$. Since $\operatorname{reg}(S) = r(S) \leq m$, by Corollary 2.5, $m\mathcal{A}$ is full, a contradiction. □

Using this combinatorial description and Macaulay 2 ([10]) we find the following counterexample:

Example 3.2. In general $\operatorname{reg} K[S] = r(S)$ does not hold, not even for smooth monomial curves; e.g., if the curve is defined by $\{0, 1, 2, 5, 13, 14, 16, 17\}$, then $r(S) = 3 < 4 = \operatorname{reg} K[S]$.

In this example, one can check that $3\mathcal{A}$ is not full, but $3\mathcal{A} + \{\mathbf{e_1}, \mathbf{e_2}\}$ is full.

In the rest of the paper we will give some sufficient conditions under which we still have $\operatorname{reg} K[S] = r(S)$.

Proposition 3.3. *Assume that a monomial curve is defined by $0 < 1 < \cdots < p < q := a_{p+1} < \cdots < a_c = \alpha - 1 < \alpha$ with $p \geq \alpha - q$. Then, for this curve, $\operatorname{reg} K[S] = r(S)$.*

Proof. It suffices to show that $\mathcal{A}$ has the following property (P_2):

(P_2) $\quad m\mathcal{A}$ not full $\Rightarrow m\mathcal{A}_2$ does not contain all elements from the list $0, 1, \ldots, \alpha, (m-1)\alpha, (m-1)\alpha + 1, \ldots, m\alpha$.

Indeed, if $m\mathcal{A}$ does not contain $(m\alpha - i, i)$ from the list $(m\alpha, 0), (m\alpha - 1, 1), \ldots, (m\alpha - \alpha, \alpha)$, then $m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\}$ does not contain $((m+1)\alpha - i, i)$, either. If $m\mathcal{A}$ does not contain $(i, m\alpha - i)$ from the list $(m\alpha - (m-1)\alpha, (m-1)\alpha), (m\alpha - ((m-1)\alpha + 1), (m-1)\alpha + 1), \ldots, (0, m\alpha)$, then $m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\}$ does not contain $((m+1)\alpha - (i + \alpha), i + \alpha)$. In any case, $m\mathcal{A} + \{\mathbf{e}_1, \mathbf{e}_2\}$ is not full. Thus, if $\mathcal{A}$ has property (P_2), then it has property (P_1). By Lemma 3.1, $\operatorname{reg} K[S] = r(S)$ for this curve.

Now we show that $\mathcal{A}$ satisfies (P_2). Note that $\mathcal{A}_2 = \{0, 1, \ldots, p, q = a_{p+1}, \ldots, a_c, \alpha\} = [0, p] \cup (\{q\} + \mathcal{G})$, where $[0, l] := \{0, 1, \ldots, l\}$ and $\mathcal{G} \subseteq \mathbf{N}$ is a subset with $0 \in \mathcal{G}$, $\max \mathcal{G} = \alpha - q \leq p$.

For $m \in \mathbf{Z}^+$ one has

$$\begin{aligned} m\mathcal{A}_2 &= m[0,p] \cup (\{q\} + (m-1)[0,p] + \mathcal{G}) \cup \cdots \cup (\{mq\} + m\mathcal{G}) \\ &= [0,mp] \cup (\{q\} + [0,(m-1)p] + \mathcal{G}) \cup \cdots \cup (\{mq\} + m\mathcal{G}). \end{aligned}$$

Let $i \in [0, m-1]$. As in the proof of Theorem 2.7, for a finite subset $B \subset \mathbb{N}$ we denote by $\lambda(B)$ the longest length of its gaps. Then

$$\lambda(\{iq\} + [0,(m-i)p] + i \cdot \mathcal{G}) = \lambda([0,(m-i)p] + i \cdot \mathcal{G}),$$

and the latter number is $\max\{0, \lambda(i\mathcal{G}) - (m-i)p\}$. Since $i < m$, this is bounded above by $\max\{0, \lambda(\mathcal{G}) - p\} \leq \max\{0, \max(\mathcal{G}) - p\} = 0$. Therefore,

$$\{iq\} + [0,(m-i)p] + i \cdot \mathcal{G} = \{iq\} + [0, mp - i(p+q-\alpha)].$$

We have

$$\begin{aligned} &\min\{\{(i+1)q\} + [0,(m-i-1)p] + (i+1)\mathcal{G}\} \\ &\qquad - \max\{\{iq\} + [0,(m-i)p] + i\mathcal{G}\} \\ &\qquad\qquad = (i+1)q - iq - \beta_i = i(p+q-\alpha) + q - mp. \end{aligned}$$

Assume that $m\mathcal{A}$ is not full, i.e. $\lambda(m\mathcal{A}) > 0$. From what was shown above it follows that $i(p+q-\alpha) + q - mp > 1$ for some $i \in \{0, \ldots, m-1\}$ or $\lambda(m\mathcal{G}) > 0$; i.e. precisely one of the two following conditions holds:

(A) $(m-1)(p+q-\alpha) + q - mp > 1$,
(B) $(m-1)(p+q-\alpha) + q - mp \leq 1$ and $\lambda(m\mathcal{G}) > 0$.

Case (A): Then $(m\alpha - (mq-1), mq-1) \notin m\mathcal{A}$. Since $mq - (m-1)\alpha - p = (m-1)(p+q-\alpha) + q - mp > 1$, we have $mq - 1 > (m-1)\alpha + p \geq (m-1)\alpha$ and, therefore, (P_2) holds.

Case (B): There exists $j \in \mathbf{N}$ such that $(m-1)\alpha + p = \max\{\{(m-1)q\} + [0,p] + (m-1)\mathcal{G}\} < j \leq \max\{m\mathcal{A}\}$ and $(m\alpha - j, j) \notin m\mathcal{A}$. Because of $j > (m-1)\alpha + p \geq (m-1)\alpha$, property (P_2) holds in this case, too. □

The next result provides a family for which the bound in Theorem 2.7 is sharp and where $\operatorname{reg} K[S] = r(S)$ holds. Note that for the lower bound we only need to assume that the curve has at most one singular point.

Proposition 3.4. *Assume that the curve is defined by* $0 < 1 < \cdots < \varepsilon < p := a_{\varepsilon+1} < \cdots < a_c < \alpha$, *where* $p \geq \varepsilon + 2$. *Then*

$$\operatorname{reg} K[S] \geq r(S) \geq \left[\frac{p-2}{\varepsilon}\right] + 1.$$

Moreover, if also $(\alpha - i, i) \in \mathcal{A}$ *for all* $0 \leq i \leq \varepsilon$ *(in particular, the curve is smooth) and* $p - \varepsilon - 1 = \lambda(\mathcal{A})$, *then*

$$\operatorname{reg} K[S] = r(S) = \left[\frac{\lambda(\mathcal{A}) - 1}{\varepsilon}\right] + 2.$$

Proof. Let $\delta = \left[\frac{p-\varepsilon-2}{\varepsilon}\right]$ and $\mathbf{u} := (u_1, u_2) = (\alpha - p + 1, p - 1) + \delta\mathbf{e}_1$. If $\mathbf{u} \in S$, then $\mathbf{u} = \sum_{0 \leq i \leq \alpha} q_i(\alpha - i, i)$, where $\sum q_i = \delta + 1$. Comparing the second coordinates we get $q_i = 0$ for all $i > \varepsilon$. Then

$$u_2 = \sum_{i=0}^{\varepsilon} iq_i \leq \varepsilon \sum_{i=0}^{\varepsilon} q_i = \varepsilon(\delta + 1) \leq p - 2 < p - 1,$$

a contradiction. Hence $\mathbf{u} \notin S$. On the other hand, letting $p-\varepsilon-1 = \delta\varepsilon+\gamma$, where $0 \le \gamma \le \varepsilon$, we get

$$\begin{aligned}\mathbf{u}+\mathbf{e}_1 &= (\alpha-p-1, p-1) + (\delta+1)\mathbf{e}_1 \\ &= (\alpha+(\delta+1)(\alpha-\varepsilon)-\gamma, (\delta+1)\varepsilon+\gamma) \\ &= (\delta+1)(\alpha-\varepsilon,\varepsilon) + (\alpha-\gamma,\gamma) \in S.\end{aligned}$$

Since $\mathbf{u} \notin S$, $\mathbf{u}+\mathbf{e}_1 \notin \mathbf{e}_1+S$. Comparing the second coordinate we also get $\mathbf{u}+\mathbf{e}_1 \notin \mathbf{e}_2+S$. Thus $\mathbf{u}+\mathbf{e}_1 \notin \{\mathbf{e}_1,\mathbf{e}_2\}+S$ and $r(S) \ge \deg(\mathbf{u}+\mathbf{e}_2) = \delta+2$. Since we always have $r(S) \le \operatorname{reg} K[S]$, this proves the first statement.

If $p-\varepsilon-1 = \lambda(\mathcal{A})$, then combining with Theorem 2.7, we finally get

$$\delta+2 \le r(S) \le \operatorname{reg} K[S] \le \left[\frac{\lambda(\mathcal{A})-1}{\varepsilon}\right]+2 = \left[\frac{p-\varepsilon-2}{\varepsilon}\right]+2 = \delta+2,$$

which yields $r(S) = \operatorname{reg} K[S] = [(\lambda(\mathcal{A})-1)/\varepsilon]+2$. □

By (3), if $r(S) \le 2$ or $\operatorname{reg} K[S] \le 3$, then $r(S) = \operatorname{reg} K[S]$. In this sense, Example 3.2 is the "smallest" possible example for $\operatorname{reg} K[S] \ne r(S)$. One might also ask for a smallest example in the following sense: What is the smallest n such that there exists a monomial curve in $\mathbb{P}^n$ satisfying $r(S) \ne \operatorname{reg} K[S]$? Restricting to smooth monomial curves, from Proposition 3.4 we see that $r(S) = \operatorname{reg} K[S]$ for $n = 3, 4$. We do not know if there exists any monomial curve in $\mathbb{P}^5$ or in $\mathbb{P}^6$ such that $\operatorname{reg} K[S] \ne r(S)$. However, the next result, in which the Castelnuovo-Mumford regularity of certain - namely "symmetric" - smooth curves in $\mathbb{P}^5$ is computed, will show that $\operatorname{reg} K[S] = r(S)$ holds for those curves.

Proposition 3.5. *Assume that the curve is defined by* $0 < 1 < l < \alpha-l < \alpha-1 < \alpha$ *($l \ge 2$). Then reg* $K[S] = r(S) = [\alpha/l]+l-3$.

Proof. The case $l = 2$ or $\alpha-2l-1 \le l-2$ (i.e. the middle gap of $\mathcal{A}$ is shorter than the other two) follows from Proposition 3.4. Assume $l \ge 3$ and $\alpha \ge 3l$. As in the proof of Theorem 2.7, let $\mathbf{u} = \mathbf{y}+p\mathbf{e}_1+q\mathbf{e}_2 \in S' \setminus S$ such that $\deg u = \operatorname{reg} K[S]-1$. We first show that

$$\deg(\mathbf{u}) \le [\alpha/l]+l-4. \tag{4}$$

If $\mathbf{y}$ belongs to the first or the last gap of $\mathcal{A}$, then by (2) we get $\deg(\mathbf{u}) \le l-2$ and (4) holds. Let us consider the case when $\mathbf{y} := (y_1, y_2)$ belongs to the middle gap of $\mathcal{A}$, i.e. $l < y_2 < \alpha-l$. Since $[\alpha/l] \ge [y_1/l]+[y_2/l]$, by symmetry it suffices to consider the case $p \ge [y_2/l]$. Let $a = [y_2/l]$ and $y_2 = al+b$, where $1 < b \le l-1$. Then

$$\begin{aligned}\mathbf{u} &= \mathbf{y}+p\mathbf{e}_1+q\mathbf{e}_2 = \mathbf{y}+a\mathbf{e}_1+(p-a)\mathbf{e}_1+q\mathbf{e}_2 \\ &= a\mathbf{g}_l+\mathbf{g}_b+(p-a)\mathbf{e}_1+q\mathbf{e}_2.\end{aligned}$$

Note that $\mathbf{g}_b$ lies in the first gap of $\mathcal{A}$. Since $\mathbf{g}_b+(p-a)\mathbf{e}_1+q\mathbf{e}_2 \notin S$, by (2) we must have $\deg(\mathbf{g}_b+(p-a)\mathbf{e}_1+q\mathbf{e}_2) = 1+(p-a)+q \le l-2$. Hence

$$\deg(\mathbf{u}) \le a+l-2 = [y_2/l]+l-2. \tag{5}$$

Note that $a \le [\alpha/l]-1$, and it suffices to consider the case $a = [\alpha/l]-1$. Since $y_2 \le \alpha-l-1$, i.e. $([\alpha/l]-1)l+b \le \alpha-l-1$, we have $b \le \alpha-[\alpha/l]l-1 \le l-2$. There are two cases:

$q = 0$. Let $\mathbf{u} = (u_1, u_2)$. Since $y_2 < \alpha-l$, it is easy to check that the degree $\delta_2(u_2) = \delta_2(y_2) = a+b \le a+l-2$. By Lemma 2.6 we must have $\deg(\mathbf{u}) < a+l-2 = [\alpha/l]+l-3$.

$q \geq 1$. In this case $[y_1/l] = [(\alpha - [\alpha/l]l - b + l)/l] = 1 \leq q$. Hence we get an inequality similar to (5), namely, $\deg(\mathbf{u}) \leq [y_1/l] + l - 2$. Adding this inequality and (5), we get

$$2\deg(\mathbf{u}) \leq [y_1/l] + [y_2/2] + 2(l-2) \leq [\alpha/l] + 2(l-2).$$

Since $[\alpha/l] \geq 2$, it follows that $\deg(\mathbf{u}) \leq [\alpha/l] + l - 4$.

Thus (4) is proved, which implies that $\operatorname{reg} K[S] \leq [\alpha/l] + l - 3$. In order to complete the proof, by (3) one only has to show $r(S) \geq [\alpha/l] + l - 3$. Let $p = [\alpha/l]$. Since $(p-1)l - 1 < \alpha - l$, it is clear that $\delta_2((p-1)l-1)) = (p-2) + (l-1) = p + l - 3$. Let $\mathbf{u} = ((p+l-3)\alpha - (p-1)l + 1, (p-1)l - 1)$. By Lemma 2.6 $\mathbf{u} \in S$, but $\mathbf{u} - \mathbf{e}_1 \notin S$. Since $(p-1)l - 1 < \alpha - l$, also $\mathbf{u} - \mathbf{e}_2 \notin S$. Hence $\mathbf{u} \notin (p+l-3)\mathcal{A} + \{\mathbf{e}_1 + \mathbf{e}_2\}$, which implies that $r(S) \geq p + l - 3$, as required. □

It would be nice if one could find all smooth monomial curves for which the bound in Theorem 2.7 is attained. However this is probably impossible. From the above proposition one can deduce that all smooth "symmetric" curves in $\mathbb{P}^5$ attaining that bound are defined by $0, 1, l, \alpha - l, \alpha - 1, \alpha$ with $2l < \alpha \leq 3l$.

In [2] one can find a classification of all simplicial toric rings of codimension two where the Eisenbud-Goto bound is attained. It is interesting that the monomial curves in $\mathbb{P}^3$ that reach the Eisenbud-Goto bound are precisely the smooth ones. By Proposition 3.4 smooth monomial curves in $\mathbb{P}^3$ and in $\mathbb{P}^4$ always reach the bound in Theorem 2.7.

In the last result the curve should have at most one singular point.

Proposition 3.6. *Assume that the monomial curve is defined by* $0 < 1 < 2 < \cdots < \epsilon < a_{\epsilon+1} < \cdots < a_c < \alpha$, *where* $\epsilon \geq 1$. *Then*

$$\operatorname{reg} K[S] \leq \left[\frac{\lambda(\mathcal{A}) - 1}{\epsilon}\right] + r(S).$$

In particular, if $\lambda(\mathcal{A}) \leq \epsilon$, *then* $\operatorname{reg} K[S] = r(S)$.

Proof. The second statement follows from the first one and the inequality $\operatorname{reg} K[S] \geq r(S)$. To prove the first one, we may assume that $\operatorname{reg} K[S] \geq r(S) + 1$. As in the proof of Theorem 2.7 take $\mathbf{u} = \mathbf{y} + p\mathbf{e}_1 + \mathbf{e}_2 \in S' \setminus S$ such that $\deg(\mathbf{u}) = \operatorname{reg} K[S] - 1$. Assume that $\mathbf{y}$ is lying in the gap L. By (1) we have

$$p \leq [(\sharp L - 1)/\epsilon] \leq [\lambda(\mathcal{A}) - 1)/\epsilon].$$

Since $\mathbf{u} + \mathbf{e}_2 \in S$ and $\operatorname{reg} K[S] \geq r(S) + 1$ but $\mathbf{u} \notin S$, we must have $\mathbf{u} + \mathbf{e}_2 = a\mathbf{e}_1 + \mathbf{z}$, where $\mathbf{z} \in r(S) \cdot \mathcal{A}$. Comparing the second coordinates of both sides we then get $y_2 + (q+1)\alpha = z_2 \leq \alpha r(S)$, which implies that $q \leq r(S) - 2$ (noticing that $y_2 > 0$). This gives $\operatorname{reg} K[S] = \deg(\mathbf{u}) + 1 = 2 + p + q \leq [\lambda(\mathcal{A}) - 1)/\epsilon] + r(S) - 1$, as required. □

Remark. The main point of our investigation is to decide whether an element $\mathbf{u} \in S'$ belongs to S. The method proposed in this paper is to express $\mathbf{u}$ as sums of elements $\mathbf{g}_0, ..., \mathbf{g}_\alpha$ in different ways. The more expressions we can find, the higher the possibility that $\mathbf{u}$ belongs to S. This explains why we are mainly concerned with smooth monomial curves.

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