NUMBER THEORETIC PROPERTIES OF
GENERATING FUNCTIONS RELATED TO DYSON’S RANK
FOR PARTITIONS INTO DISTINCT PARTS

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Abstract. Let \( Q(n) \) denote the number of partitions of \( n \) into distinct parts. We show that Dyson’s rank provides a combinatorial interpretation of the well-known fact that \( Q(n) \) is almost always divisible by 4. This interpretation gives rise to a new false theta function identity that reveals surprising analytic properties of one of Ramanujan’s mock theta functions, which in turn gives generating functions for values of certain Dirichlet \( L \)-functions at nonpositive integers.

1. Introduction and statement of results

A partition \( \lambda \) of a positive integer \( n \) is a sequence \( (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of positive integers, written in nonincreasing order, whose sum is \( n \). Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \), we say that \( \lambda_i \) is the \( i \)th part of the partition, and we write \( \ell(\lambda) \) to denote the number of parts of \( \lambda \). The rank of \( \lambda \) is \( \lambda_1 - \ell(\lambda) \). For instance, the rank of \((5, 3, 1, 1)\) is \( 5 - 4 = 1 \). The Young diagram of \( \lambda \) is the partial grid of squares consisting of \( k \) rows, aligned at the left, with the \( i \)th row containing \( \lambda_i \) squares for each \( i \leq k \). (See Figure 1.)

Let \( p(n) \) denote the number of partitions of \( n \). Ramanujan proved the following famous congruence identities for \( p(n) \):

\[
\begin{align*}
(1.1) \quad p(5n + 4) & \equiv 0 \pmod{5}, \\
(1.2) \quad p(7n + 5) & \equiv 0 \pmod{7}, \\
(1.3) \quad p(11n + 6) & \equiv 0 \pmod{11}.
\end{align*}
\]

Several infinite families of arithmetic congruences for \( p(n) \) have been discovered since Ramanujan’s time, producing identities such as

\[
p(157525693n + 111247) \equiv 0 \pmod{13}.
\]

(See [15] for a detailed account of congruences for \( p(n) \).)

Identities (1.1)-(1.3) simply begged for a combinatorial explanation. Dyson [10] conjectured that for any \( m \), the number of partitions of \( 5n + 4 \) having rank congruent to \( m \) (mod 5) is equal to \( \frac{1}{5}p(5n + 4) \), and the number of partitions of \( 7n + 5 \) having rank congruent to \( m \) (mod 7) is equal to \( \frac{1}{7}p(7n + 5) \), thereby providing

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a combinatorial interpretation of (1.1) and (1.2) if true. Atkin and Swinnerton-Dyer proved these conjectures in [7]. Interestingly, equation (1.3) does not have a similar combinatorial interpretation given by the rank. Andrews and Garvan later discovered another combinatorial statistic, the crank, that classifies the partitions of $11n + 6$ into 11 equal classes determined by the crank modulo 11. (See [5], [11].)

Let $Q(n)$ denote the number of partitions of $n$ into distinct parts. We call such partitions strict partitions. For instance, $(5, 3, 2)$ is a strict partition of 10, but $(5, 3, 1, 1)$ is not. Several infinite families of congruence identities have also been shown for $Q$. (See [13], [14], [17], [18].) In fact, it was shown in [14] that for any prime $p$, there exist positive integers $a$ and $b$ such that $Q(an + b) \equiv 0 \pmod{p}$ for all positive integers $n$.

The nearly arithmetic congruence identities modulo 4, first discovered by Rødseth [17], rival (1.1)-(1.3) in their simplicity. The first few such identities are:

\[(1.4) \quad Q(5n + 1) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{5}, \]
\[(1.5) \quad Q(7n + 2) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{7}, \]
\[(1.6) \quad Q(11n + 5) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{11}, \]
\[(1.7) \quad Q(13n + 7) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{13}. \]

It turns out that $Q(n)$ is also highly divisible by arbitrary powers of 2. Gordon and Ono [12] have shown that for any positive integer $j$,

\[(1.8) \quad \lim_{N \to \infty} \frac{\# \{ n < N \mid Q(n) \equiv 0 \pmod{2^j} \}}{N} = 1. \]

The proof of this fact depends on the theory of Galois representations and is not combinatorial. A simple combinatorial argument due to Franklin [19] shows that $Q(n)$ is divisible by 2 if and only if $n \not\equiv k(3k \pm 1)/2$ for any integer $k$, thus proving equation (1.8) in the case $j = 1$. Alladi [11] has provided combinatorial interpretations of (1.8) for $j \leq 4$.

We show that Dyson’s rank also provides a combinatorial interpretation of (1.3)-(1.7), and more generally of (1.8) for $j = 2$, as follows. Define $T(m, k; n)$ to be the number of strict partitions of $n$ having rank congruent to $m \pmod{k}$.

**Theorem 1.1.** Let $n$ be a positive integer. We have

$$T(0, 4; n) = T(1, 4; n) = T(2, 4; n) = T(3, 4; n)$$

if and only if $24n + 1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$, and the largest power of $p$ dividing $24n + 1$ is $p^e$ where $e$ is odd.
Table 1. The strict partitions of 12 and of 16 sorted by rank.

<table>
<thead>
<tr>
<th>Rank (mod 4)</th>
<th>Partitions of 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(10, 2), (7, 4, 1), (7, 3, 2)</td>
</tr>
<tr>
<td>1</td>
<td>(11, 1), (8, 3, 1), (7, 5), (5, 4, 2, 1)</td>
</tr>
<tr>
<td>2</td>
<td>(9, 2, 1), (8, 4), (6, 3, 2, 1), (5, 4, 3)</td>
</tr>
<tr>
<td>3</td>
<td>(12), (9, 3), (6, 5, 1), (6, 4, 2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rank (mod 4)</th>
<th>Partitions of 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(14, 2), (11, 4, 1), (11, 3, 2), (10, 6) (8, 5, 2, 1), (8, 4, 3, 1), (7, 6, 3), (7, 5, 4)</td>
</tr>
<tr>
<td>1</td>
<td>(15, 1), (12, 3, 1), (11, 5), (9, 4, 2, 1) (8, 7, 1), (8, 6, 2), (8, 5, 3), (6, 4, 3, 2, 1)</td>
</tr>
<tr>
<td>2</td>
<td>(13, 2, 1), (12, 4), (10, 3, 2, 1), (9, 6, 1) (9, 5, 2), (9, 4, 3), (6, 5, 4, 1), (6, 5, 3, 2)</td>
</tr>
<tr>
<td>3</td>
<td>(16), (13, 3), (10, 5, 1), (10, 4, 2) (9, 7), (7, 6, 2, 1), (7, 5, 3, 1), (7, 4, 3, 2)</td>
</tr>
</tbody>
</table>

To illustrate Theorem 1.1, we sort the partitions of 12 and of 16 by rank in Table 1. Notice that 24 · 12 + 1 = 289 = 17², and so n = 12 does not satisfy the conditions of Theorem 1.1. On the other hand, 24 · 16 + 1 = 385 = 5 · 77, so n = 16 satisfies the conditions with p = 5.

Notice that if n satisfies the conditions of Theorem 1.1 then Q(n) ≡ 0 (mod 4). It is easily shown that this set of integers contains those of the form pn + \( \frac{p^2 - 1}{24} \), n ≠ 0 (mod p), for all primes p > 3 not congruent to ±1 (mod 24), thus proving (1.4)-(1.7) combinatorially via the rank.

Theorem 1.1 reveals fascinating properties of the generating functions related to the ranks of strict partitions. For |z| ≤ 1 and |q| < 1, define

\[
G(z, q) := 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(1 - zq)(1 - zq^2) \cdots (1 - zq^s)}.
\]

Let Q(n, r) denote the number of partitions of n into distinct parts with rank r. A combinatorial argument shows that

\[
G(z, q) = \sum_{n, r} Q(n, r) z^r q^n,
\]

where n and r range from 0 to ∞.

The next theorem shows that the specializations of this series at fourth roots of unity z have elegant and useful q-series expansions.

**Theorem 1.2.** Let z, q ∈ ℂ with |z| ≤ 1, |q| < 1. Then

\[
G(i, q) = \sum_{k=0}^{\infty} i^k q^{(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{(3k-1)/2},
\]

\[
G(-i, q) = \sum_{k=0}^{\infty} (-i)^k q^{(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{(3k-1)/2},
\]

where we define Q(0) = 1.
The functions $G(1, q)$ and $G(-1, q)$ are both related to automorphic forms in the variable $\tau$ where $q = e^{2\pi i \tau}$ (we use this notation throughout). Since we have that
\[ qG(1, q^{24}) = \frac{\eta(48\tau)}{\eta(24\tau)}, \]
where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the usual classical weight $1/2$ modular form of Dedekind, it follows that $G(1, q)$ is essentially a weight $0$ modular form. The work of Andrews, Dyson, and Hickerson shows that $G(-1, q)$ is related to the Fourier expansion of a Maass cusp form that has eigenvalue $1/4$ with respect to the hyperbolic Laplacian operator $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$, where $\tau = x + iy$. (See [9].)

This prompts one to ask if the functions $G(z, q)$ have interesting analytic properties when $z$ is an arbitrary root of unity. Theorem 1.2 shows that these series have a simple form when $z = i$ and when $z = -i$. In fact, they are examples of false theta functions. To demonstrate this, we first recall some necessary background and notation. A Dirichlet character of order $a$ is a map $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying
\begin{itemize}
  \item $\chi(n + a) = \chi(n)$ for any integer $n$,
  \item $\chi(mn) = \chi(m)\chi(n)$ for any integers $m, n$, and
  \item $\chi(n) = 0$ for any $n$ such that $\gcd(a, n) > 1$.
\end{itemize}

The eight Dirichlet characters of order $24$ are shown in Table 2.

A theta function is a function $\theta(z; \tau)$, where $z$ is a fixed complex number and the domain of $\tau$ is the complex upper half-plane $H$, of the form
\[ \theta(z; \tau) = \sum_{n=-\infty}^{\infty} e^{2\pi inz} e^{2\pi in^2 \tau} = \sum_{n=-\infty}^{\infty} e^{2\pi inz} q^{n^2}. \]

Several variants of these functions are also called theta functions if they satisfy certain modular transformation laws. In particular, if $\chi$ is an even Dirichlet character of order $a$, then
\[ \sum_{n=-\infty}^{\infty} \chi(n) q^{n^2} \]
is a modular form of weight $1/2$ over the congruence subgroup $\Gamma_0(4a^2)$ of the full modular group $\text{PSL}_2(\mathbb{Z})$. Moreover, these theta functions essentially form a basis of all modular forms of weight $1/2$ by a classical theorem due to Serre and Stark [16].

**Table 2.** The nonzero values of the 8 Dirichlet characters of order 24.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
<th>17</th>
<th>19</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0(n)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1(n)$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_2(n)$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3(n)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4(n)$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_5(n)$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_6(n)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_7(n)$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>
Notice that, by Theorem 1.2, 
\[ qG(i, q^{24}) = \sum_{k=0}^{\infty} i^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} i^{k-1} q^{(6k-1)^2}. \]

This closely resembles the theta functions described above, but the coefficients cannot be written as a linear combination of even Dirichlet characters. Thus, we have encountered a false theta function.

False theta function identities can be used to obtain generating functions for the values of Dirichlet L-functions at nonpositive integers. This was first observed by Andrews, Ono and Jiménez-Urroz [4], and by Zagier [20]. Here we show that the identities in Theorem 1.2 also may be used in this way. We first recall the definition of L-functions. Given a Dirichlet character \( \chi \), the corresponding Dirichlet L-function is a generalization of the Riemann \( \zeta \)-function defined by

\[ L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \]

Each L-function has a meromorphic continuation to the entire complex plane. In Section 2.3, we use our expressions for \( G(\pm i, z) \) to obtain the following.

**Theorem 1.3.** We have

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_6, -2n)t^n = e^{-t} + \sum_{n=1}^{\infty} \frac{(1-i)e^{-(12n^2+12n+1)t}}{2 \prod_{r=1}^{n}(1 - ie^{-24rt})} + \frac{(1+i)e^{-(12n^2+12n+1)t}}{2 \prod_{r=1}^{n}(1 + ie^{-24rt})} \]

(1.10)

and

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_7, -2n)t^n = e^{-t} + \sum_{n=1}^{\infty} \frac{(1+i)e^{-(12n^2+12n+1)t}}{2 \prod_{r=1}^{n}(1 - ie^{-24rt})} + \frac{(1-i)e^{-(12n^2+12n+1)t}}{2 \prod_{r=1}^{n}(1 + ie^{-24rt})}. \]

(1.11)

The L-values at negative integers can also be obtained using generalized Bernoulli numbers. The Bernoulli numbers \( B_{n, \chi} \) associated with the Dirichlet character \( \chi \) of order \( a \) are defined by the generating function

\[ \sum_{m=1}^{a} \chi(m) \frac{te^{nt}}{e^{nt} - 1} = \sum_{t=0}^{\infty} B_{n, \chi} \frac{t^n}{n!}. \]

It is well-known that

\[ L(\chi, 1 - k) = -\frac{B_{k, \chi}}{k} \]

for any positive integer \( k \). The right hand side of (1.10) is, as a power series in \( t \),

\[ 2 + 46t + 3985t^2 + \frac{1743623}{3} t^3 + \cdots, \]

which matches the values given by the Bernoulli numbers for \( \chi_6 \). As another illustration, the right hand side of (1.12) is

\[ -48t - 3984t^2 - 581208t^3 - \cdots. \]
In addition to being a false theta function, \( G(i, q) \) is related to the famous mock theta functions of Ramanujan, which Bringmann and Ono [8] recently have established are the holomorphic parts of certain weight \( 1/2 \) harmonic Maass forms. One famous such function is

\[
R(z, q) = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{q^{n^2}}{(1 - zq^k)(1 - z^{-1}q^k)}.
\]

The coefficient of \( z^n q^n \) in \( R(z, q) \) is the number of partitions of \( n \) having rank \( m \). Thus, evaluating \( R(z, q) \) at roots of unity \( z \) is useful in obtaining congruence relations for \( p(n) \) via the rank.

Replacing \( q \) by \( 1/q \) in (1.13), we obtain the following theorem.

**Theorem 1.4.** We have

\[
R(i, 1/q) = R(-i, 1/q) = \frac{1 - i}{2} G(i, q) + \frac{1 + i}{2} G(-i, q)
\]

or alternatively,

\[
qR(i, 1/q^{24}) = \sum_{n=0}^{\infty} (-1)^n \left( q^{(12n+1)^2} + q^{(12n+5)^2} + q^{(12n+7)^2} + q^{(12n+11)^2} \right)
\]

\[
= q + q^{25} + q^{49} + q^{121} - q^{169} - q^{289} - q^{361} - q^{529} + q^{625} + \cdots.
\]

Thus, the analytic behavior of the false theta functions \( G(\pm i, q) \) gives the behavior of \( R(\pm i; q) \) for \( q \) outside the unit disk! This is a remarkable connection between the rank generating functions of strict and unrestricted partitions.

2. **Proofs**

We now present the proofs of the main results.

2.1. **\( Q(n) \mod 4 \) via the rank.** Let \( \mathcal{D} \) denote the set of all strict partitions (partitions having distinct parts), and let \( \mathcal{P} \) denote the set of all (unrestricted) partitions. Let \( \mathcal{D}_n \) denote the set of all strict partitions of \( n \).

Define a **pentagonal partition** to be a partition of the form \((2k, 2k - 1, \ldots, k + 1)\) or \((2k - 1, 2k - 2, \ldots, k)\) for some positive integer \( k \). The former is a partition of \( k(3k + 1)/2 \), and the latter is a partition of \( k(3k - 1)/2 \). Numbers of the form \( k(3k \pm 1)/2 \) are called **pentagonal numbers.** An example of each type of pentagonal partition is shown in Figure 2.

![Figure 2](image-url)
Let $D'_n$ denote the set of all strict partitions of $n$ that are not pentagonal partitions. For any partition $\lambda$, let $m(\lambda)$ be the largest index $m$ such that $\lambda_1 = \lambda_2 + 1 = \lambda_3 + 2 = \cdots = \lambda_m + m - 1$. Also let $s(\lambda)$ denote the smallest part of $\lambda$.

Given a partition $\lambda$, the conjugate partition of $\lambda$, denoted $\lambda'$, is the partition formed by interchanging the rows and columns of its Young diagram.

To prove Theorem 1.4, we first provide a necessary and sufficient condition for the equalities $T(0, 4; n) = T(2, 4; n)$ and $T(1, 4; n) = T(3, 4; n)$ to hold.

**Lemma 2.1.** If $n \neq k(3k \pm 1)/2$ for any $k$, we have

$$T(1, 4; n) = T(3, 4; n) \text{ and } T(0, 4; n) = T(2, 4; n).$$

Otherwise, if $n = k(3k + 1)/2$, then

$$T(k, 4; n) = T(k + 2, 4; n) + 1 \text{ and } T(k + 1, 4; n) = T(k + 3, 4; n).$$

If $n = k(3k - 1)/2$, then

$$T(k - 1, 4; n) = T(k + 1, 4; n) + 1 \text{ and } T(k, 4; n) = T(k + 2, 4; n).$$

**Proof.** We require an involution $\phi : D'_n \to D'_n$, commonly known as Franklin's Involution, defined as follows. Let $\lambda \in D'_n$, and let $m = m(\lambda)$ and $s = s(\lambda)$. If $s \leq m$, define $\phi(\lambda)$ to be the partition formed by removing the part $s$ from the partition and increasing each of the first $s$ parts by 1. If $s > m$, define $\phi(\lambda)$ to be the partition formed by decreasing each of the first $m$ parts of $\lambda$ by 1 and inserting a part of size $m$ into the partition. (See Figure 3.) Notice that these operations are well defined on pentagonal partitions.

It is easily verified that $\phi$ is an involution. Furthermore, for any nonpentagonal partition $\lambda$, the rank of $\phi(\lambda)$ differs from the rank of $\lambda$ by $\pm 2$. Thus, if $n \neq k(3k \pm 1)/2$, we have $T(1, 4; n) = T(3, 4; n)$ and $T(0, 4; n) = T(2, 4; n)$.

If $n = k(3k + 1)/2$, there is one pentagonal partition of $n$, namely $(2k, 2k - 1, \ldots, k + 1)$, and the rank of this partition is $k$. Thus $T(k, 4; n) = T(k + 2, 4; n) + 1$ and $T(k + 1, 4; n) = T(k + 3, 4; n)$.

If $n = k(3k - 1)/2$, there is one pentagonal partition of $n$, namely $(2k - 1, 2k - 2, \ldots, k)$, and the rank of this partition is $k - 1$. Thus $T(k - 1, 4; n) = T(k + 1, 4; n) + 1$ and $T(k, 4; n) = T(k + 2, 4; n)$. This completes the proof. \qed

Now, notice that $T(0, 4; n) + T(2, 4; n) = T(0, 2; n)$ and $T(1, 4; n) + T(3, 4; n) = T(1, 2; n)$. Thus, by Lemma 2.1, in order to find exactly when $T(m, 4; n) = \frac{4}{Q}(n)$ for $m = 0, 1, 2, 3$ it suffices to find the difference between the number of partitions of $n$ having even rank, $T(0, 2; n)$, and the number having odd rank, $T(1, 2; n)$. Let $S(n) = T(0, 2; n) - T(1, 2; n)$ be this difference. An explicit formula for the function $S(n)$ has already been obtained [3], and we state this result below.

![Figure 3. Franklin's Involution $\phi : D'_n \to D'_n$.](image)
Theorem 2.1. We have $S(n) = T(24n + 1)$, where the function $T(m)$ is defined on the set of integers $m \neq 1$ congruent to 1 (mod 6) as follows. Write $m$ in the form $p_1^{e_1}p_2^{e_2} \cdots p_k^{e_k}$ where each $p_i$ is either a prime congruent to 1 (mod 6) or the negative of a prime congruent to 5 (mod 6). Then

$$T(p^e) = \begin{cases} 
0 & \text{if } p \equiv 1 \pmod{24} \text{ and } e \text{ is odd}, \\
1 & \text{if } p \equiv 13 \text{ or } 19 \pmod{24} \text{ and } e \text{ is even}, \\
(-1)^{e/2} & \text{if } p \equiv 1 \pmod{24} \text{ and } e \text{ is even}, \\
e + 1 & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = 2, \\
(-1)^{e}(e + 1) & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = -2.
\end{cases}$$

It follows that $S(n) = 0$ if and only if $24n + 1$ has a prime divisor $p \equiv \pm 1 \pmod{24}$, and the largest power of $p$ dividing $24n + 1$ is $p^e$ for some odd positive integer $e$. Suppose $n$ is a pentagonal number. Then $24n + 1 = 24(k(3k \pm 1)/2) + 1 = (6k \pm 1)^2$ for some $k$, which cannot have a prime raised to an odd power in its prime factorization since it is a perfect square. Thus, if $S(n) = 0$, then $n$ is not a pentagonal number, and so by Lemma 2.1, $T(0, 4; n) = T(2, 4; n)$ and $T(1, 4; n) = T(3, 4; n)$. Furthermore, if $S(n) = 0$, then $T(0, 4; n) + T(2, 4; n) = T(0, 2; n) = T(1, 2; n) = T(1, 4; n) + T(3, 4; n)$ by the definition of $S(n)$. Thus $S(n) = 0$ if and only if $T(0, 4; n) = T(1, 4; n) = T(2, 4; n) = T(3, 4; n)$. This proves Theorem 2.1.

To analyze the generating functions that arise in studying $S(n)$ and other functions related to the rank, we first recall some standard notation. For any positive integer $n$, we define

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

and

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k).$$

The Sylvester triangle of a partition $\lambda$ is the largest partition of the form $(s, s-1, \ldots, 3, 2, 1)$ such that $s - i + 1 \leq \lambda_i$ for $i = 1, 2, \ldots, s$. An example is shown in Figure 4. Notice that if $\lambda$ is a strict partition, then $\lambda$ has the same number of parts as its Sylvester triangle.

We proceed to prove Theorem 2.2 below. Recall that

$$G(z, q) = 1 + \sum_{s=1}^{\infty} \frac{q^{(s+1)/2}}{(zq; q)_s}$$

for $|z| \leq 1$ and $|q| < 1$.

Theorem 2.2. Let $z, q \in \mathbb{C}$ with $|z| \leq 1$, $|q| < 1$. Then

$$G(i, q) = \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2},$$

$$G(-i, q) = \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2},$$

where we define $Q(0) = 1$. 
Proof. We first provide an elementary combinatorial proof of the identity
\[ G(z, q) = \sum_{n, r} Q(n, r) q^n z^r. \]

Similar identities have already appeared in the literature (see, for instance, [3]), but we state the proof here for the reader’s enjoyment.

Let \( Q(n, r, s) \) denote the number of strict partitions with rank \( r \) and exactly \( s \) parts, and let \( p(n, r, s) \) denote the number of partitions of \( n \) with largest part at most \( s \) and exactly \( r \) parts. It is easily verified combinatorially that for any positive integer \( s \),

\[ \sum_{n, r} p(n, r, s) q^n z^r = \frac{1}{(zq; q)_s}. \]

We now define a map \( \varphi : D \rightarrow P \) as follows. Suppose \( \lambda \) is a partition of \( n \) into \( s \) distinct parts.

By removing the Sylvester triangle from \( \lambda \), we are left with a partition \( \nu = (\lambda_1 - s, \lambda_2 - (s - 1), \ldots, \lambda_s - 1) \) of \( n - s(s + 1)/2 \). We define \( \varphi(\lambda) \) to be the conjugate partition \( \nu' \) of \( \nu \). Notice that \( \nu' \) has at most \( s \) parts, and the number of parts of \( \nu' \) is equal to the rank of \( \lambda \).

For each nonnegative integer \( s \), \( \varphi \) is a bijection from the set of partitions of \( n \) into exactly \( s \) distinct parts to the set of partitions \( \nu' \) having largest part at most \( s \). Hence

\[ \sum_{n, r, s} Q(n, r, s) q^n z^r = \sum_s q^{s(s+1)/2} \sum_{n, r} p(n, r, s) q^n z^r, \]

where the variables range over all nonnegative integers. Note that

\[ \sum_s Q(n, r, s) = Q(n, r). \]

By (2.1), (2.2), and (2.3), we have

\[ \sum_{n, r} Q(n, r) q^n z^r = 1 + \sum_{s=1}^{\infty} q^{s(s+1)/2} \frac{1}{(zq; q)_s} = G(z, q). \]
Setting $z = i$, we have
\[
G(i, q) = \sum_{n,r} Q(n,r) i^n q^n
\]
\[
= \sum_n [T(0, 4; n) + iT(1, 4; n) - T(2, 4; n) - iT(3, 4; n)] q^n
\]
\[
= \sum_n [(T(0, 4; n) - T(2, 4; n)) + iT(1, 4; n) - T(3, 4; n)] q^n
\]
\[
= \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2},
\]
where the last equality follows from Lemma 2.1. Analogously,
\[
G(-i, q) = \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}.
\]
This completes the proof. \(\square\)

2.2. The relation between $R(\pm i, q)$ and $G(\pm i, q)$. To prove Theorem 1.4 we require the following identity given in Ramanujan’s “lost” notebook:
\[
1 + \sum_{n=1}^{\infty} \frac{q^n}{(-aq; q)_n(-a^{-1}q; q)_n} = (1 + a) \sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)} (1 - a^2 q^{2n+1})
\]
(2.4) \[
- a \sum_{n=0}^{\infty} (-1)^n a^{2n} q^{n(n+1)} (-aq; q)_{\infty} \left( -a^{-1}q; q \right)_{\infty}
\]
Andrews [2] noted that by substituting $a = i$ in (2.3) and taking the real part of both sides, we obtain the identity
\[
1 + \sum_{n=1}^{\infty} \frac{q^n}{(-q^2; q^2)^n} = \sum_{n=0}^{\infty} (-1)^n q^{n(6n+1)} (1+q^{4n+1}) + \sum_{n=0}^{\infty} (-1)^n q^{2n+1)(3n+2)} (1+q^{4n+3}).
\]
(2.5) Notice that the left hand side of (2.3) is equal to $R(i, 1/q)$. Replacing $q$ by $q^{24}$ in (2.5) and multiplying by $q$, we obtain, by Theorem 1.2,
\[
qR(i, 1/q^{24}) = \sum_{n=0}^{\infty} (-1)^n \left( q^{12n+1} + q^{12n+5} + q^{12n+7} + q^{12n+11} \right)
\]
\[
= \frac{1 - i}{2} qG(i, q^{24}) + \frac{1 + i}{2} qG(-i, q^{24}),
\]
and the result follows.

2.3. Exponential generating functions for Dirichlet $L$-values. In order to prove Theorem 1.3 we first prove the following.

Lemma 2.2. Let $\chi_6$ and $\chi_7$ denote the Dirichlet characters of order 24 given in Table 2 and let $0 \leq t < 1$. We have
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{1+i}{2} L(\chi_6, -2n) + \frac{1-i}{2} L(\chi_7, -2n) \right) t^n = e^{-t} + \sum_{n=1}^{\infty} \frac{e^{-(12n^2+12n+1)t}}{n!} \prod_{r=1}^{n}(1 - ie^{-24rt})
\]
(2.6)
and
\[
(2.7) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{1-i}{2} L(\chi_6, -2n) + \frac{1+i}{2} L(\chi_7, -2n) \right) t^n = e^{-t} + \sum_{n=1}^{\infty} e^{-(12n^2+12n+1)t} \prod_{r=1}^{n} (1 + (1 + i) 24r^{-1}).
\]

Proof. Define \( H : \mathbb{R} \to \mathbb{C} \) by
\[
H(t) = e^{-t} + \sum_{n=1}^{\infty} e^{-(12n^2+12n+1)t} \prod_{r=1}^{n} (1 + (1 + i) 24r^{-1}).
\]
By Theorem 1.2 for \( t > 0 \),
\[
(2.8) \quad H(t) = e^{-t} G(i, e^{-24t}) = \sum_{k=0}^{\infty} \int_{t}^{\infty} e^{-(6k+1)^2t} \sum_{k=1}^{\infty} e^{-(6k-1)^2t}.
\]
Notice that
\[
\frac{1+i}{2} \chi_6(6k+1) + \frac{1-i}{2} \chi_7(6k+1) = i^k
\]
and
\[
\frac{1+i}{2} \chi_6(6k-1) + \frac{1-i}{2} \chi_7(6k-1) = i^{-k},
\]
and that \( \chi_6(n) = \chi_7(n) = 0 \) when \( n \) is not of the form \( 6k+1 \) or \( 6k-1 \). Thus (2.8) becomes
\[
H(t) = \sum_{n=0}^{\infty} \left( \frac{1+i}{2} \chi_6(n) + \frac{1-i}{2} \chi_7(n) \right) e^{-n^2 t}.
\]
Now, let \( F : \{ s \in \mathbb{C} : \text{Re}(s) > 0 \} \to \mathbb{C} \) be defined by \( F(s) = \int_{0}^{\infty} H(t) t^{s-1} \, dt \). For any \( s \) with \( \text{Re}(s) > 0 \), we have
\[
(2.9) \quad F(s) = \int_{0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{1+i}{2} \chi_6(n) + \frac{1-i}{2} \chi_7(n) \right) e^{-n^2 t} t^{s-1} \, dt
\]
and
\[
(2.10) \quad \sum_{n=0}^{\infty} \left( \frac{1+i}{2} \chi_6(n) + \frac{1-i}{2} \chi_7(n) \right) \int_{0}^{\infty} e^{-n^2 t} t^{s-1} \, dt
\]
since the integral and sum are absolutely convergent for \( \text{Re}(s) > 0 \). Recall that the \( \Gamma \) function is commonly defined as \( \Gamma(s) = \int_{0}^{\infty} e^{-t} t^{s-1} \, dt \). Substituting \( u = n^2 t \) in the integral in each summand in (2.10), we find
\[
F(s) = \sum_{n=0}^{\infty} \left( \frac{1+i}{2} \chi_6(n) + \frac{1-i}{2} \chi_7(n) \right) \frac{1}{n^{2s}} \int_{0}^{\infty} e^{-u} u^{s-1} \, du
\]
\[
= \Gamma(s) \left( \frac{1+i}{2} \sum_{n=0}^{\infty} \frac{\chi_6(n)}{n^{2s}} + \frac{1-i}{2} \sum_{n=0}^{\infty} \frac{\chi_7(n)}{n^{2s}} \right)
\]
\[
= \Gamma(s) \left( \frac{1+i}{2} L(\chi_6, 2s) + \frac{1-i}{2} L(\chi_7, 2s) \right).
\]
It is well-known ([8], p. 250) that \( \Gamma \) has an analytic continuation to \( \mathbb{C} \setminus \{ n \in \mathbb{Z} : n \leq 0 \} \), with poles of order 1 at the nonpositive integers, defined as follows. For any negative integer \( n \) and any \( s \) with \( n < \text{Re}(s) \leq n+1 \), \( \Gamma(s) = \frac{1}{n^{(s+n-1)}} \Gamma(s+n) \). It is easily verified that the residue of \( \Gamma \) at the negative integer \( k \) is \( (-1)^{n}/n! \).
Using the analytic continuations of $L(\chi_6, s)$ and $L(\chi_7, s)$, we can extend $F(s)$ to a meromorphic function on $\mathbb{C}$ that has poles of order 1 at the nonpositive integers and is analytic elsewhere. Moreover, the residue at the pole $s = -n$ of $F(s)$ is

$$\frac{(-1)^n}{n!} \left( \frac{1 + i}{2} L(\chi_6, -2n) + \frac{1 - i}{2} L(\chi_7, -2n) \right).$$

Define the complex numbers $a(n)$ by $H(t) = \sum_{n=0}^{\infty} a(n) t^n$, since $H$ is analytic. Then, for any positive integer $N$,

$$\int_0^\infty H(t) t^{s-1} dt = \int_0^1 \sum_{n=0}^{\infty} a(n) t^{n+s-1} dt + \int_1^\infty H(t) t^{s-1} dt = \sum_{n=0}^N \frac{a(n)}{n+s} + \sum_{n=N+1}^{\infty} \frac{a(n)}{n+s} + \int_1^\infty H(t) t^{s-1} dt.$$

Since $\sum_{n=0}^\infty a(n) t^n = \int_1^\infty H(t) t^{s-1} dt$ is an analytic function of $s$ in the half-plane $\Re(s) > N$, the residue of the pole at $s = -n$ is $a(n)$. Thus

$$a(n) = \frac{(-1)^n}{n!} \left( \frac{1 + i}{2} L(\chi_6, -2n) + \frac{1 - i}{2} L(\chi_7, -2n) \right)$$

for all $n$, and equation (2.6) follows.

The proof of equation (2.7) is analogous. □

Using (2.6) and (2.7) to solve for $\sum (-1)^n L(\chi_6, -2n) t^n$ and $\sum (-1)^n L(\chi_7, -2n) t^n$, we obtain equations (1.10) and (1.12) of Theorem 1.3.

To prove equality (1.11) of Theorem 1.3, note that by Theorem 1.4,

$$qR(i, 1/q^{24}) = \sum_{n=0}^{\infty} \chi_6(n) q^{n^2}.$$

Replacing $q$ by $e^{-t}$, an argument identical to that for Lemma 2.2 gives the result.

3. Future work

Given the fascinating properties of the functions $G(z, q)$ and $R(z, q)$ when $z$ is a fourth root of unity, it is natural to ask whether Theorems 1.2 and 1.4 are specializations of a more general phenomenon that occurs when $z$ is an arbitrary root of unity. Understanding the behavior of the coefficients of these functions at $m$th roots of unity may also unlock more information about the distribution of the rank function modulo $m$, for both strict and unrestricted partitions.

Dyson’s rank does not provide a combinatorial interpretation of the fact that $Q(n)$ is usually divisible by 8 in the same manner as it does for 2 and 4. Thus, it may also be fruitful to investigate generalizations of Dyson’s rank in order to find a partition statistic that, when taken modulo $2^j$, classifies the partitions of $n$ into $2^j$ equal classes. More generally, perhaps an analog of the crank function that applies to $Q(n)$ is waiting to be discovered.
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