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HYPERGEOMETRIC ${}_3F_2(1/4)$ EVALUATIONS OVER FINITE FIELDS AND HECKE EIGENFORMS

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Abstract. Let H denote the hypergeometric ${}_3F_2$ function over $\mathbb{F}_p$ whose three numerator parameters are quadratic characters and whose two denominator parameters are trivial characters. In 1992, Koike posed the problem of evaluating H at the argument $1/4$. This problem was solved by Ono in 1998. Ten years later, Evans and Greene extended Ono's result by evaluating an infinite family of ${}_3F_2(1/4)$ over $\mathbb{F}_q$ in terms of Jacobi sums. Here we present five new ${}_3F_2(1/4)$ over $\mathbb{F}_q$ (involving characters of orders 3, 4, 6, and 8) which are conjecturally evaluable in terms of eigenvalues for Hecke eigenforms of weights 2 and 3. There is ample numerical evidence for these evaluations. We motivate our conjectures by proving a connection between ${}_3F_2(1/4)$ and twisted sums of traces of the third symmetric power of twisted Kloosterman sheaves.

1. Introduction

Let $\mathbb{F}_q$ be a field of q elements, where q is a power of an odd prime p. Let A, B, C, D, E, ϕ, 1 denote complex multiplicative characters on $\mathbb{F}_q^*$, where the last two characters have orders 2 and 1, respectively. By convention, these characters map 0 to 0. Recall the definition of the Jacobi sum

$$J(A,B) = \sum_{x\in\mathbb{F}_q} A(x)B(1-x). \tag{1.1}$$

Following Greene [15, Cor. 3.14], we define the hypergeometric function ${}_3F_2(z)$ over $\mathbb{F}_q$ by

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} A,B,C \\ D,E \end{matrix}\middle| z\right) = \\ &\frac{BCDE(-1)}{q^2}\sum_{x,y\in\mathbb{F}_q} B(x)D\overline{B}(1-x)C(y)E\overline{C}(1-y)\overline{A}(1-xyz), \quad z\in\mathbb{F}_q^*. \end{aligned} \tag{1.2}$$

For $z=0$, the sum in (1.2) degenerates into a product of two Jacobi sums, but the convention is to define ${}_3F_2(0)=0$. This paper concerns evaluations of ${}_3F_2(z)$ at the argument $z=1/4$ only. For evaluations at other arguments, see [6], [7], [8], [25], [26].

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For the moment, let $q = p$. In the special case where the three numerator parameters are quadratic and the two denominator parameters are trivial, the hypergeometric function ${}_3F_2(z)$ over $\mathbb{F}_p$ can be expressed as the following elegant double sum of Legendre symbols:

$$H := {}_3F_2\left(\begin{matrix}\phi,\phi,\phi\\ 1,1\end{matrix}\,\middle|\, z\right) = \frac{1}{p^2}\sum_{x,y\in\mathbb{F}_p}\left(\frac{xy(1-x)(1-y)(1-xyz)}{p}\right). \tag{1.3}$$

In 1992, Koike [20] posed the problem of evaluating H at the argument $z = 1/4$. Ono [25] solved this problem in 1998 by proving that for $p > 3$,

$${}_3F_2\left(\begin{matrix}\phi,\phi,\phi\\ 1,1\end{matrix}\,\middle|\, \frac{1}{4}\right) = \phi(3)(4x^2 - p)/p^2, \tag{1.4}$$

where $x = 0$ when $p \equiv 2 \pmod 3$, and $p = x^2 + 3y^2$ when $p \equiv 1 \pmod 3$. In 2009, Evans and Greene [7] extended (1.4) for general q relatively prime to 3 by proving that if S is any character on $\mathbb{F}_q$ which is not trivial, cubic, or quartic, then

$${}_3F_2\left(\begin{matrix}\overline{S},S^3,S\\ S^2,S\phi\end{matrix}\,\middle|\, \frac{1}{4}\right) = \begin{cases}\phi(-1)S(4)q^{-1}, & \text{if } q\equiv 2 \pmod 3\\ \phi(-1)S(4)q^{-1}(1+u+u^{-1}), & \text{if } q\equiv 1 \pmod 3,\end{cases} \tag{1.5}$$

where $u = J(S,\psi)/J(S,\overline{\psi})$ and ψ denotes a cubic character on $\mathbb{F}_q$ when $q \equiv 1 \pmod 3$. Ono's result (1.4) is the special case $q = p$, $S = \phi$ of (1.5). As was pointed out in [7], there is an analogue of (1.5) for *classical* hypergeometric functions which is much easier to prove.

The primary purpose of this paper is to address Ono's "Open Problem 11.38" [26, p. 204] by offering conjectural evaluations for five new ${}_3F_2(1/4)$ over $\mathbb{F}_q$. These evaluations appear to have no known classical analogues. The first three conjectures, presented in Section 2, give evaluations in terms of eigenvalues $a(p)$ for Hecke eigenforms of weight 3 with quadratic nebentypus. The last two conjectures, presented in Section 3, give evaluations in terms of eigenvalues $a(p)$ for Hecke eigenforms of weight 2 with trivial nebentypus. Numerical evidence provided in Tables 1–5 leaves little room for doubt that the conjectures are correct. In Section 4, we discuss the motivation and theoretical basis for the conjectures.

There are a number of papers that discuss relations between hypergeometric functions over finite fields and modular forms; see for example [2], [9], [14], [24], [25], [26], [27] and the references therein. To our knowledge, the five eigenforms in Sections 2–3 have not arisen in previous work on this topic.

2. Evaluations of ${}_3F_2(1/4)$ in terms of weight 3 newforms

Conjectures 1–3 below evaluate the hypergeometric function over $\mathbb{F}_q$ defined by

$$F(C,q) := {}_3F_2\left(\begin{matrix}\overline{C},\overline{C},\overline{C}\\ 1,\overline{C}\phi\end{matrix}\,\middle|\, \frac{1}{4}\right)$$

when C is cubic, quartic, and sextic, respectively. The evaluations are in terms of Hecke eigenvalues $a(p)$ for three different Hecke eigenforms of weight 3 with quadratic nebentypus. For simplicity, Conjecture 1 is stated only for $q = p$, $p \equiv 1 \pmod 3$ and $q = p^2$, $p \equiv 2 \pmod 3$, but it can be canonically extended for all q; see (2.4). Similar remarks apply to Conjectures 2 and 3.

Conjecture 1. *Let* $f(u) = \sum_{m=1}^{\infty} a(m)e^{2\pi i u m}$ *be the weight* 3 *newform on* $\Gamma_0(243)$ *with nebentypus* $\chi(d) = \left(\frac{d}{3}\right)$ *and eigenfield* $\mathbb{Q}(\sqrt{-1})$. *Suppose that* $q \equiv 1 \pmod 3$, *and let* C *be any cubic character on* $\mathbb{F}_q$.

Define $H_3(q) := -q\overline{C}(3) \ + \ q^2\phi(-1)\overline{C}(48)F(C,q)$. *Then*

$$H_3(q) = \begin{cases} a(p), & \text{if } \ q = p, \ \ p \equiv 1 \pmod 3 \\ a(p)^2 + 2p^2, & \text{if } \ q = p^2, \ \ p \equiv 2 \pmod 3. \end{cases} \tag{2.1}$$

Table 1. Evidence for Conjecture 1

$p \equiv 1 \pmod 3$				$p \equiv 2 \pmod 3$			
p	**a(p)**	$\mathbf{H_3(p)}$		**p**	**a(p)**	$\mathbf{a(p)^2 + 2p^2}$	$\mathbf{H_3(p^2)}$
7	11	11		5	$6i$	14	14
13	5	5		11	$12i$	98	98
19	-19	-19		17	$-18i$	254	254
31	-13	-13		23	$-30i$	158	158
37	17	17		29	$48i$	-622	-622
43	29	29		41	$-30i$	2462	2462
61	-22	-22		47	$-24i$	3842	3842
67	98	98		53	$36i$	4322	4322
73	38	38		59	$6i$	6926	6926

Conjecture 2. *Let* $f(u) = \sum_{m=1}^{\infty} a(m)e^{2\pi i u m}$ *be the weight* 3 *newform on* $\Gamma_0(12)$ *with nebentypus* $\chi(d) = \left(\frac{-1}{d}\right)$ *and eigenfield* $\mathbb{Q}(\sqrt{-3})$. *Suppose that* $q \equiv 1 \pmod 4$, *and let* C *be any quartic character on* $\mathbb{F}_q$. *Define* $H_4(q) := -q + q^2C(-1)F(C,q)$. *Then*

$$H_4(q) = \begin{cases} a(p), & \text{if } \ q = p, \ \ p \equiv 1 \pmod 4 \\ a(p)^2 + 2p^2, & \text{if } \ q = p^2, \ \ p \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

Table 2. Evidence for Conjecture 2

$p \equiv 1 \pmod 4$				$p \equiv 3 \pmod 4$			
p	**a(p)**	$\mathbf{H_4(p)}$		**p**	**a(p)**	$\mathbf{a(p)^2 + 2p^2}$	$\mathbf{H_4(p^2)}$
5	-2	-2		7	$-4\sqrt{-3}$	50	50
13	2	2		11	$4\sqrt{-3}$	194	194
17	10	10		19	$12\sqrt{-3}$	290	290
29	-26	-26		23	$-16\sqrt{-3}$	290	290
37	26	26		31	$4\sqrt{-3}$	1874	1874
41	58	58		43	$-28\sqrt{-3}$	1346	1346
53	-74	-74		47	$40\sqrt{-3}$	-382	-382
61	26	26		59	$-52\sqrt{-3}$	-1150	-1150
73	-46	-46		67	$4\sqrt{-3}$	8930	8930

Conjecture 3. *Let* $f(u) = \sum_{m=1}^{\infty} a(m)e^{2\pi i u m}$ *be the weight* 3 *newform on* $\Gamma_0(972)$ *with nebentypus* $\chi(d) = \left(\frac{d}{3}\right)$ *and eigenfield* $\mathbb{Q}(\sqrt{-1})$, *whose first few eigenvalues are as given in Table 3. Suppose that* $q \equiv 1 \pmod 6$, *and let* C *be any sextic character on* $\mathbb{F}_q$. *Define* $H_6(q) := -q\overline{C}(-3) \ + \ q^2\overline{C}(48)F(C,q)$. *Then*

$$H_6(q) = \begin{cases} a(p), & \text{if } \ q = p, \ \ p \equiv 1 \pmod 6 \\ a(p)^2 + 2p^2, & \text{if } \ q = p^2, \ \ p \equiv 5 \pmod 6. \end{cases} \tag{2.3}$$

Table 3. Evidence for Conjecture 3

$p \equiv 1 \pmod 6$				$p \equiv 5 \pmod 6$			
p	**a(p)**	$\mathbf{H_6(p)}$		**p**	**a(p)**	$\mathbf{a(p)^2 + 2p^2}$	$\mathbf{H_6(p^2)}$
7	-1	-1		5	$6i$	14	14
13	-7	-7		11	$-12i$	98	98
19	5	5		17	$30i$	-322	-322
31	-49	-49		23	$-6i$	1022	1022
37	5	5		29	$24i$	1106	1106
43	-43	-43		41	$42i$	1598	1598
61	-70	-70		47	0	4418	4418
67	-94	-94		53	$-84i$	-1438	-1438
73	-10	-10		59	$-114i$	-6034	-6034

Tables 1–3 provide numerical evidence for Conjectures 1–3, respectively. The eigenvalues $a(p)$ in Table 1, found in William Stein's online Modular Forms Explorer (MFE) database [30], correspond to MFE eigenform #396295. The eigenvalues in Table 2 correspond to MFE eigenform #391417. Since MFE does not include weight 3 eigenforms with levels as large as 972, we obtained the eigenvalues $a(p)$ for Table 3 using a `Sage` program largely developed by William Stein [29].

Conjecture 1 suggests the existence of a 2-dimensional cohomology group on which a Frobenius σ acts, such that $a(p) = \text{trace}(\sigma)$ is the p-th Fourier coefficient of the given weight 3 newform of level 243 with nebentypus χ and such that for each $t \geq 1$ for which $q = p^t \equiv 1 \pmod 3$,

$$\text{trace}(\sigma^t) = H_3(q).$$

Such σ has eigenvalues α and $\chi(p)\overline{\alpha}$ (see [17, eq. (6.57)]) with

$$a(p) = \text{trace}(\sigma) = \alpha + \chi(p)\overline{\alpha}, \quad \alpha\overline{\alpha} = p^2.$$

Thus for $p \equiv 2 \pmod 3$, we have

$$\text{trace}(\sigma^2) = \alpha^2 + \overline{\alpha}^2 = a(p)^2 + 2p^2,$$

in agreement with (2.1). More generally, for each $t \geq 1$ such that $q = p^t \equiv 1 \pmod 3$,

$$\text{trace}(\sigma^t) = \alpha^t + \overline{\alpha}^t = q(\beta^t + \beta^{-t}) = 2qP_t((\beta + \beta^{-1})/2),$$

where $\beta = \alpha/p$ and $P_t(x)$ is the t-th Chebyshev polynomial of the first kind defined by

$$P_t\left(\frac{y + y^{-1}}{2}\right) = \frac{y^t + y^{-t}}{2}.$$

Thus we could extend Conjecture 1 to hold for all q by replacing (2.1) by

$$H_3(q) = \begin{cases} 2q\ P_r\left(\ a(p)/(2p)\ \right), & \text{if } \ q = p^r,\ p \equiv 1 \pmod 3 \\ 2q\ P_r\left(\ (a(p)^2 + 2p^2)/(2p^2)\ \right), & \text{if } \ q = p^{2r},\ p \equiv 2 \pmod 3. \end{cases} \tag{2.4}$$

Similar remarks apply to Conjectures 2 and 3.

3. Evaluations of ${}_3F_2(1/4)$ in terms of weight 2 newforms

Conjectures 4 and 5 below evaluate the hypergeometric function over $\mathbb{F}_q$ defined by

$$f(C,q) := \ {}_3F_2\left(\begin{matrix} \overline{C}, C^3, C \\ \overline{C}^2, C\phi \end{matrix} \middle| \frac{1}{4} \right)$$

when C is sextic and octic, respectively. The evaluations are in terms of Hecke eigenvalues $a(p)$ for two different Hecke eigenforms of weight 2 with trivial nebentypus. As before, the conjectures may be extended for all q but with a formulation slightly simpler than (2.4); see (3.3).

Conjecture 4. *Let* $f(u) = \sum_{m=1}^{\infty} a(m)e^{2\pi i u m}$ *be the weight* 2 *newform on* $\Gamma_0(972)$ *with trivial nebentypus and eigenfield* $\mathbb{Q}(\sqrt{2})$. *Suppose that* $q \equiv 1 \pmod 6$, *and let* C *be any sextic character on* $\mathbb{F}_q$.

Define $h_6(q) := \overline{C}(12)J(C,C) \ - \ q\overline{C}(-3)J(\overline{C},\overline{C})f(C,q)$. *Then*

$$h_6(q) = \begin{cases} a(p), & \text{if } \ q = p \\ a(p)^2 \ - 2p, & \text{if } \ q = p^2. \end{cases} \tag{3.1}$$

Table 4. Evidence for Conjecture 4

$p \equiv 1 \pmod 6$				$p \equiv 5 \pmod 6$			
p	**a(p)**	**h_6(p)**		**p**	**a(p)**	**$a(p)^2 - 2p$**	**$h_6(p^2)$**
7	2	2		5	$3\sqrt{2}$	8	8
13	-1	-1		11	$3\sqrt{2}$	-4	-4
19	5	5		17	$-3\sqrt{2}$	-16	-16
31	-7	-7		23	$-3\sqrt{2}$	-28	-28
37	-4	-4		29	$-6\sqrt{2}$	14	14
43	5	5		41	$-6\sqrt{2}$	-10	-10
61	5	5		47	0	-94	-94
67	11	11		53	$3\sqrt{2}$	-88	-88
73	-1	-1		59	$-3\sqrt{2}$	-100	-100

Conjecture 5. *Let* $f(u) = \sum_{m=1}^{\infty} a(m)e^{2\pi i u m}$ *be the weight* 2 *newform on* $\Gamma_0(768)$ *with trivial nebentypus and eigenfield* $\mathbb{Q}(\sqrt{3})$, *whose first few eigenvalues are as given in Table 5. Suppose that* $q \equiv 1 \pmod 8$, *and let* C *be any octic character on* $\mathbb{F}_q$. *Define* $h_8(q) := C(-4)J(C,C) \ - \ qC(-1)J(\overline{C},\overline{C})f(C,q)$. *Then*

$$h_8(q) = \begin{cases} a(p), & \text{if } \ q = p \\ a(p)^2 \ - 2p, & \text{if } \ q = p^2. \end{cases} \tag{3.2}$$

TABLE 5. Evidence for Conjecture 5

$p \equiv 1 \pmod 8$				$p \not\equiv 1 \pmod 8$			
$\mathbf{p}$	$\mathbf{a(p)}$	$\mathbf{h_8(p)}$		$\mathbf{p}$	$\mathbf{a(p)}$	$\mathbf{a(p)^2 - 2p}$	$\mathbf{h_8(p^2)}$
17	6	6		5	$2\sqrt{3}$	2	2
41	6	6		7	$2\sqrt{3}$	-2	-2
73	-2	-2		11	0	-22	-22
89	-6	-6		13	0	-26	-26
97	-2	-2		19	-4	-22	-22
113	-6	-6		23	$-4\sqrt{3}$	2	2
137	6	6		29	$-2\sqrt{3}$	-46	-46
193	2	2		31	$2\sqrt{3}$	-50	-50
233	18	18		37	$-4\sqrt{3}$	-26	-26

Tables 4–5 provide numerical evidence for Conjectures 4–5, respectively. The eigenvalues $a(p)$ in Table 4 correspond to MFE eigenform #53240, while the eigenvalues in Table 5 correspond to MFE eigenform #51561.

Conjecture 5 suggests the existence of a 2-dimensional cohomology group on which a Frobenius σ acts such that $a(p) = \mathrm{trace}(\sigma)$ is the p-th Fourier coefficient of the given weight 2 newform of level 768 and such that for each $t \geq 1$ for which $q = p^t \equiv 1 \pmod 8$,

$$\mathrm{trace}(\sigma^t) = h_8(q).$$

Such σ has eigenvalues α and $\overline{\alpha}$ with

$$a(p) = \mathrm{trace}(\sigma) = \alpha + \overline{\alpha}, \quad \alpha\overline{\alpha} = p.$$

Thus

$$\mathrm{trace}(\sigma^2) = \alpha^2 + \overline{\alpha}^2 = a(p)^2 - 2p,$$

in agreement with (3.2). More generally, for each $t \geq 1$ such that $q = p^t \equiv 1 \pmod 8$,

$$\mathrm{trace}(\sigma^t) = \alpha^t + \overline{\alpha}^t = q^{1/2}(\beta^t + \beta^{-t}) = 2q^{1/2}P_t((\beta + \beta^{-1})/2),$$

where $\beta = \alpha/p^{1/2}$ and $P_t(x)$ is the t-th Chebyshev polynomial of the first kind. Thus we could extend Conjecture 5 to hold for all q by replacing (3.2) by

$$h_8(q) = 2q^{1/2}\ P_t\left(\frac{a(p)}{2p^{1/2}}\right). \tag{3.3}$$

A similar remark applies to Conjecture 4.

4. Motivation and theoretical basis for the conjectures

For $u \in \mathbb{F}_q$ and the map $\mathrm{Trace} : \mathbb{F}_q \to \mathbb{F}_p$, define the additive character

$$\zeta^u = \exp\left(\frac{2\pi i \mathrm{Trace}(u)}{p}\right).$$

For $a \in \mathbb{F}_q^*$ and a multiplicative character C on $\mathbb{F}_q^*$, consider the twisted Kloosterman sum

$$K(C^k, a) := \sum_{x \in \mathbb{F}_q^*} C^k(x)\zeta^{x+a/x} = -g_k(a) - h_k(a), \tag{4.1}$$

where $g_k(a)$ and $h_k(a)$ are the Frobenius eigenvalues for the twisted Kloosterman sheaf at a, each with absolute value $q^{1/2}$. The eigenvalues $g_k(a), h_k(a)$ are the zeros of the quadratic polynomial $X^2 + XK(C^k, a) + C^k(-a)q$ [18, (7.4.1.3, 7.4.1.4)]. In the case $a = 0$, $K(C^k, a)$ becomes the Gauss sum $G(C^k)$, which has absolute value $q^{1/2}$ unless C^k is trivial. A twisted sum of traces of the n-th symmetric power of these twisted Kloosterman sheaves is

$$T_n(C, k, \ell) := \sum_{a \in \mathbb{F}_q^*} \overline{C}^{\ell}(a)(g_k(a)^n + g_k(a)^{n-1}h_k(a) + \cdots + h_k(a)^n). \tag{4.2}$$

For brevity, write $T_n(C, k, 1) = T_n(C, k)$. The following theorem converts the problem of evaluating the ${}_3F_2(1/4)$ in Conjectures 1–5 to the problem of evaluating certain $T_3(C, 0)$ and $T_3(C, 2)$ in terms of Hecke eigenvalues.

Theorem. *In the notation of Conjectures 1–5,*

$$H_3(q) = -T_3(C, 0)\overline{C}(3)/G^2(\overline{C}), \quad \textit{for cubic } C; \tag{4.3}$$

$$H_4(q) = -T_3(C, 0)/G^2(\overline{C}), \quad \textit{for quartic } C; \tag{4.4}$$

$$H_6(q) = -T_3(C, 0)\overline{C}(-3)/G^2(\overline{C}), \quad \textit{for sextic } C; \tag{4.5}$$

$$h_6(q) = T_3(C, 2)\overline{C}(48)G(C^2)/q^2, \quad \textit{for sextic } C; \tag{4.6}$$

$$h_8(q) = T_3(C, 2)G(\phi)/q^2, \quad \textit{for octic } C. \tag{4.7}$$

Before proving the Theorem, we provide some motivation, first by pointing out several previously known or conjectured relationships between $T_n(C, k, \ell)$ and Hecke eigenforms for trivial and quadratic characters C on $\mathbb{F}_p$. For this discussion, let $q = p$.

We first look at some untwisted T_n, i.e., $T_n(1, 0)$. It is known [23], [28] that

$$(-T_5(1, 0) - 1)/p^2$$

is the p-th Fourier coefficient of a weight 3 eigenform on $\Gamma_0(15)$ with quadratic nebentypus of conductor 15, namely MFE#391476. It is also known [16] that

$$(-T_6(1, 0) - 1)/p^2$$

is the p-th Fourier coefficient of a weight 4 eigenform on $\Gamma_0(6)$ with trivial nebentypus, namely MFE#76747. We have conjectured [5] that

$$\left(\frac{p}{105}\right)(-T_7(1, 0) - 1)/p^2 = |a(p)|^2 - p^2,$$

where $a(p)$ is the p-th Fourier coefficient of a weight 3 eigenform on $\Gamma_0(525)$ with a quartic nebentypus of conductor 105. We also conjecture that

$$(-T_8(1, 0) - 1)/p^2 = a(p) + p^2,$$

where $a(p)$ is the p-th Fourier coefficient of a weight 6 newform on $\Gamma_0(6)$ with trivial nebentypus, namely MFE#72259.

We now put in a quadratic twist and look at some $T_n(\phi, 0)$. In 1960, the Lehmers [21] essentially showed that

$$-\phi(-1)T_3(\phi, 0)/p$$

is the p-th Fourier coefficient of a weight 3 eigenform on $\Gamma_0(12)$ with nebentypus $\chi(d) = \left(\frac{d}{3}\right)$, namely, the eigenform $\eta(2z)^3\eta(6z)^3$. This p-th Fourier coefficient

vanishes when $p \equiv 2 \pmod 3$, and it equals $4x^2 - 2p$ when $p = x^2 + 3y^2$. (There is a misprint in the value reported in [22, p. 1888].) We conjecture that

$$-T_4(\phi, 0)/p$$

is the p-th Fourier coefficient of a weight 4 eigenform on $\Gamma_0(8)$ with trivial nebentypus, namely, the eigenform $\eta(2z)^4\eta(4z)^4$. (This eigenform has been examined by Ahlgren and Ono [1].) For $n = 5$, we conjecture that

$$-\left(\frac{p}{15}\right)\phi(-1)T_5(\phi, 0)/p = |a(p)|^2 - p^2,$$

where $a(p)$ is the p-th Fourier coefficient of a weight 3 eigenform on $\Gamma_0(300)$ with nebentypus $\chi(d) = \left(\frac{d}{3}\right)$, namely MFE#397060. For $n = 6$, we conjecture that

$$-T_6(\phi, 0)/p = a(p) + p^2,$$

where $a(p)$ is the p-th Fourier coefficient of a weight 6 eigenform on $\Gamma_0(24)$ with trivial nebentypus, namely MFE#72299.

There are also conjectural formulas for certain $T_n(C, k, \ell)$ when C has order exceeding 2. We list several of these in the Appendix; most are due to Katz [19].

We now indicate how the Theorem led to the formulation of Conjectures 1–5. We conjecture that no prime $p > n$ divides the putative newform level corresponding to T_n, so accordingly, no prime $p > 3$ should divide the levels of eigenforms corresponding to (4.3)–(4.7). We found the corresponding levels 243, 12, 972, 972, and 768 by searching systematically through levels of the form 2^a3^b. Our decision on which weights to employ was based on the generic bound [4, Cor. 3.3.4, p. 206]

$$T_n(C, k, \ell) = O(q^{(n+1)/2}), \quad q \to \infty, \tag{4.8}$$

where the implied constant depends only on n. This bound suggested searching weight 3 newforms for (4.3)–(4.5) and weight 2 newforms for (4.6)–(4.7), because Deligne's bound for the p-th Fourier coefficient of a weight w newform is $O(p^{(w-1)/2})$.

We do not know if the generic bound in (4.8) holds in general for all C, k, ℓ, p. It is an open problem to classify the exceptional cases. D. Wan and N. Katz have kindly pointed out that (4.8) holds without exception when $k = 0$. By a theorem of Katz [18, p. 8], the classical Kloosterman sheaf KL (with $k = 0$) has geometric monodromy group Sp(2), and thus $\mathrm{Sym}^n = \mathrm{Sym}^n(\mathrm{KL})$ is irreducible. As Sym^n has no geometrically trivial component, its second cohomology group vanishes, from which (4.8) follows if C^ℓ is trivial. For general C^ℓ with $k = 0$, we have a sheaf which is the tensor product of the rank 1 character $\overline{C}^\ell$ with Sym^n. Its geometric monodromy group is the same as that for Sym^n, so (4.8) follows for general C^ℓ with $k = 0$. For a different approach in the case $k = 0$ which works for all characters C^ℓ except ϕ, see [22, Theorem 1]. For general k, the second cohomology group for Sym^n vanishes if and only if the (rational) Hasse-Weil zeta function $L(\mathrm{Sym}^n, t)$ is actually a polynomial in t. For $k = 0$, this polynomial has been studied in detail by Fu and Wan [10, 11, 12, 13].

Proof of Theorem. By (4.1)–(4.2),

$$T_3(C, k) = -\sum_{a \neq 0} \overline{C}(a)K(C^k, a)^3 \; + 2\sum_{a \neq 0} \overline{C}(a)K(C^k, a)g_k(a)h_k(a). \tag{4.9}$$

Replacing x by $-a/x$ in (4.1), we see that

$$K(C^k, a) = C^k(-a)\overline{K(C^k, a)}. \tag{4.10}$$

Multiplying (4.10) by $g_k(a)h_k(a)$ and using (4.1), we obtain

$$K(C^k, a)g_k(a)h_k(a) = C^k(-a)qK(C^k, a). \tag{4.11}$$

Thus the rightmost term in (4.9) equals

$$2qC^k(-1)\sum_{a\neq 0} C^{k-1}(a)K(C^k, a) = 2qC^k(-1)G(C^{k-1})G(C^{2k-1}). \tag{4.12}$$

In accordance with the hypotheses in (4.3)–(4.7), assume from now on that $k \in \{0, 2\}$, that C has order 6 or 8 when $k = 2$, and that C has order 3, 4, or 6 when $k = 0$. From (4.9) and (4.12),

$$T_3(C, 2) = -S_3(2) + 2qG(C)G(C^3), \quad T_3(C, 0) = -S_3(0) + 2qG(\overline{C})^2, \tag{4.13}$$

where

$$S_3(k) := \sum_{a\neq 0} \overline{C}(a)K(C^k, a)^3. \tag{4.14}$$

By (4.1),

$$S_3(k) = \sum_{a\neq 0} \overline{C}(a) \sum_{x,y,z\neq 0} C^k(xyz)\zeta^{x+y+z+a(1/x+1/y+1/z)}.$$

Since C is nontrivial, the sum on a vanishes when $1/x + 1/y + 1/z = 0$. Therefore

$$S_3(k) = G(\overline{C}) \sum_{x,y,z\neq 0} C^k(xyz)C(1/x + 1/y + 1/z)\zeta^{x+y+z}.$$

Replace x by xz and y by yz to obtain

$$S_3(k) = G(\overline{C}) \sum_{z\neq 0} C^{3k-1}(z) \sum_{x,y\neq 0} C(1/x + 1/y + 1)C^k(xy)\zeta^{z(x+y+1)}.$$

Since C^{3k-1} is nontrivial, the sum on z vanishes if $x + y + 1 = 0$. Thus

$$\frac{S_3(k)}{G(\overline{C})G(C^{3k-1})} = \sum_{x,y\neq 0} \overline{C}^{3k-1}(x + y + 1)C(x + y + xy)C^{k-1}(xy).$$

With the change of variables $r = x + y$ and $s = xy$,

$$\frac{S_3(k)}{G(\overline{C})G(C^{3k-1})} = \sum_r \sum_{s\neq 0} C(r + s)\overline{C}^{3k-1}(1 + r)C^{k-1}(s)\{1 + \phi(r^2 - 4s)\}.$$

When $r = 0$, this equals $(q - 1)\delta_{0k}$, where $\delta_{00} = 1$ and $\delta_{02} = 0$. After replacing s by sr^2 for nonzero r, we get

$$\begin{aligned}\frac{S_3(k)}{G(\overline{C})G(C^{3k-1})} &- (q - 1)\delta_{0k} \\ &= \sum_{r,s\neq 0} C^{2k-1}(r)C^{k-1}(s)\overline{C}^{3k-1}(1 + r)C(1 + rs)\{1 + \phi(1 - 4s)\}.\end{aligned}$$

Replace r by $-r$ and s by $s/4$ to see that

$$\frac{C^{k-1}(-4)S_3(k)}{G(\overline{C})G(C^{3k-1})} - C^{k-1}(-4)(q - 1)\delta_{0k} = A_1(k) + A_2(k), \tag{4.15}$$

where

$$A_1(k) := \sum_{r,s\neq 0} C^{2k-1}(r)C^{k-1}(s)\overline{C}^{3k-1}(1-r)C(1-rs/4) \tag{4.16}$$

and

$$A_2(k) := \sum_{r,s\neq 0} C^{2k-1}(r)C^{k-1}(s)\overline{C}^{3k-1}(1-r)\phi(1-s)C(1-rs/4). \tag{4.17}$$

Replace s by $4s/r$ in (4.16) to get

$$\begin{aligned} A_1(k) &= C^{k-1}(4)\sum_{r,s\neq 0} C^k(r)\overline{C}^{3k-1}(1-r)C^{k-1}(s)C(1-s) \\ &= C^{k-1}(4)J(C^k,\overline{C}^{3k-1})J(C^{k-1},C). \end{aligned}$$

After some manipulation using the factorization of Jacobi sums into Gauss sums, this yields

$$A_1(2) = qC(-4)\frac{G(C)G(C^3)}{G(\overline{C})G(C^5)}, \quad A_1(0) = \overline{C}(-4). \tag{4.18}$$

By (1.2) and (4.17),

$$A_2(2) = C\phi(-1)q^2 f(C,q), \quad A_2(0) = C\phi(-1)q^2 F(C,q), \tag{4.19}$$

where $F(C,q)$ and $f(C,q)$ are the hypergeometric functions defined at the beginning of Sections 2 and 3, respectively. Combining (4.13), (4.15), (4.18), and (4.19), we obtain

$$T_3(C,2) = qG(C)G(C^3) - \phi(-1)\overline{C}(4)q^2 G(\overline{C})G(C^5)f(C,q) \tag{4.20}$$

and

$$-T_3(C,0)/G(\overline{C})^2 = -q + \phi(-1)C(4)q^2 F(C,q). \tag{4.21}$$

Formulas (4.3)–(4.5) now follow immediately from (4.21).

It remains to consider the case $k=2$. Suppose first that C is sextic. By (4.20),

$$\frac{T_3(C,2)\overline{C}(48)G(C^2)}{q^2} = \frac{\overline{C}(48)G(C)G(\phi)}{G(\overline{C}^2)} - \overline{C}(-3)qJ(\overline{C},\overline{C})f(C,q). \tag{4.22}$$

By the Hasse-Davenport product formula [3, Theorem 2.1.4], the first term on the right of (4.22) equals $\overline{C}(12)J(C,C)$, and so (4.6) follows in view of the definition of $h_6(q)$ in Conjecture 4. Finally, suppose that C is octic. By (4.20),

$$T_3(C,2)G(\phi)q^{-2} = J(C,C^3) - \overline{C}(4)qJ(\overline{C},\overline{C}^3)f(C,q). \tag{4.23}$$

By [3, eq. (2.1.2)], $J(C,C^3) = C(-4)J(C,C)$. Thus (4.7) follows in view of the definition of $h_8(q)$ in Conjecture 5. □

5. Appendix

We list here some conjectured formulas for various $T_n(C,k,\ell)$ when $q=p>n$, where C is a character on $\mathbb{F}_p$ of order 3, 4, 6, or 12. There is substantial numerical evidence supporting those formulas involving explicit levels. The formulas involving newforms of undetermined level are speculative and may require tweaking. As we mentioned earlier, most of these conjectures are due to Katz [19].

C cubic. Let $q = p \equiv 1 \pmod 3$, and write $r_3 = r_3(p)$ for the (unique) integer satisfying

$$4p = r_3^2 + 27t_3^2, \quad r_3 \equiv 1 \pmod 3.$$

Let C be a cubic character on $\mathbb{F}_q$. For $n = 4$,

$$T_4(C,2,0)/G(C)^2 = -r_3 J(\overline{C},\overline{C}), \qquad -T_4(C,0)/G(\overline{C})^2 = a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(54)$ with trivial nebentypus, namely MFE#76862. For $n = 5$,

$$T_5(C,1)/p^2 = r_3 - \left(\frac{p}{5}\right)p, \qquad T_5(C,2)/(pG(C)^2) = r_3 - \left(\frac{p}{5}\right)C(15)J(\overline{C},\overline{C})^2.$$

(We remark that $-2\operatorname{Re} C(15)J(C,C)$ is the p-th Fourier coefficient of a weight 2 eigenform on $\Gamma_0(6075)$ with trivial nebentypus, namely MFE#416779.) For $n = 6$,

$$T_6(C,2,0)/p^2 = p - r_3^2, \qquad T_6(C,2,2)/(pG(C)^2) = pC(3)J(C,C) - p - J(\overline{C},\overline{C})^2.$$

(We remark that $-2\operatorname{Re} C(3)J(C,C)$ is the p-th Fourier coefficient of a weight 2 eigenform on $\Gamma_0(243)$ with trivial nebentypus, namely MFE#43739.) For $n = 7$,

$$\frac{T_7(C,2,0)}{pG(C)^2} = (r_3^2 - p)J(\overline{C},\overline{C}) - \left(\frac{p}{105}\right)C(105)p^2, \qquad T_7(C,2)/p^3 = r_3 - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 3 eigenform with integer coefficients on $\Gamma_0(35)$ with nebentypus $\left(\frac{d}{35}\right)$, namely MFE#391997. (We conjecture further that for $\left(\frac{p}{35}\right) = 1$,

$$a(p) = |c(p)|^2 - 2p,$$

where $c(p)$ is the p-th coefficient of a weight 2 eigenform on $\Gamma_0(175)$ with a quartic nebentypus of conductor 35, with eigenfield $\mathbb{Q}((1+i)\sqrt{14})$.) For $n = 8$,

$$T_8(C,1)/p^3 = -r_3^2, \qquad T_8(C,2)/(p^2G(C)^2) = p - r_3^2 + C(6)J(\overline{C},\overline{C})^3.$$

For $n = 9$,

$$T_9(C,1,0)/p^3 = -\left(\frac{p}{105}\right)p^2 + (r_3^3 - 2r_3p).$$

For $n = 10$,

$$\frac{T_{10}(C,2,0)}{p^2G(C)^2} = p^2 - pr_3^2 + p^2C(30)J(C,C) - J(\overline{C},\overline{C})^4, \qquad T_{10}(C,2)/p^4 = p - r_3^2 - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(10)$, namely MFE#76751. For $n = 11$,

$$T_{11}(C,1)/p^5 = r_3^3/p - 2r_3 - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 3 eigenform with integer coefficients on $\Gamma_0(1155)$ with nebentypus $\left(\frac{d}{1155}\right)$. (We conjecture further that for $\left(\frac{p}{33}\right) = \left(\frac{p}{35}\right) = 1$,

$$a(p) = |c(p)|^2 - 2p,$$

where $c(p)$ is the p-th coefficient of a weight 2 eigenform on $\Gamma_0(5775)$ with a quartic nebentypus of conductor 1155.) For $n = 12$,

$$T_{12}(C,2,0)/p^4 = 3pr_3^2 - 2p^2 - r_3^4, \qquad \frac{T_{12}(C,2,2)}{p^4G(C)^2} = J(\overline{C},\overline{C})(2r_3 - r_3^3/p) - \overline{C}(90)a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform on $\Gamma_0(2700)$ with trivial nebentypus, with eigenfield $\mathbb{Q}(\sqrt{55})$. For $n = 13$,

$$T_{13}(C,2)/p^5 = r_3^3 - 2pr_3 + \left(\frac{p}{15015}\right)(p^2 - |c(p)|^2),$$

where $c(p)$ is the p-th coefficient of a weight 3 eigenform whose level is a multiple of 15015, whose nebentypus has conductor 15015, and whose eigenfield contains $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{11}, \sqrt{13})$. For $n = 14$,

$$T_{14}(C,1)/p^6 = 3r_3^2 - r_3^4/p - p - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(70)$, namely MFE#76919. Also for $n = 14$,

$$T_{14}(C,2)/(p^4 G(C)^2) = 3pr_3^2 - r_3^4 - p^2 - \left(\frac{p}{5}\right)\overline{C}(630)J(\overline{C},\overline{C})^2 a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with trivial nebentypus whose level divides a power of 30030 and whose eigenfield contains $\sqrt{286}$. For $n = 15$,

$$T_{15}(C,1,0)/p^7 = r_3^5/p^2 - 4r_3^3/p + 3r_3 - \left(\frac{p}{105}\right)a(p),$$

where $a(p)$ is the p-th coefficient of a weight 3 eigenform with integer coefficients on $\Gamma_0(3003)$ with nebentypus $\left(\frac{d}{3003}\right)$. (We conjecture further that for $\left(\frac{p}{21}\right) = \left(\frac{p}{143}\right) = \left(\frac{p}{5}\right)$,

$$a(p) = |c(p)|^2 - 2p,$$

where $c(p)$ is the p-th coefficient of a weight 2 eigenform on $\Gamma_0(39039)$ with quartic nebentypus of conductor 3003.) For $n = 16$,

$$T_{16}(C,2)/p^6 = 3pr_3^2 - r_3^4 - 2p^2 - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 6 eigenform with integer coefficients on $\Gamma_0(70)$, namely MFE#72483. Also for $n = 16$,

$$T_{16}(C,2,0)/(p^6 G(C)^2) = 3r_3^2 - p - r_3^4/p - J(\overline{C},\overline{C})^6/p^2 - \overline{C}(315)a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with trivial nebentypus whose level divides a power of 30030 and whose eigenfield contains $\sqrt{910}$. For $n = 17$,

$$T_{17}(C,1)/p^7 = r_3^5/p - 4r_3^3 + 3pr_3 + \left(\frac{p}{51051}\right)(p^2 - |c(p)|^2),$$

where $c(p)$ is the p-th coefficient of a weight 3 eigenform whose level is a multiple of 51051, whose nebentypus has conductor 51051, and whose eigenfield contains $\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{7}, \sqrt{13}, \sqrt{17})$. For $n = 18$,

$$T_{18}(C,2,0)/p^8 = p - 6r_3^2 + 5r_3^4/p - r_3^6/p^2 - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(210)$, namely MFE#77669. For $n = 20$,

$$T_{20}(C,1)/p^8 = -6pr_3^2 + 5r_3^4 - r_3^6/p - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 6 eigenform with integer coefficients on $\Gamma_0(210)$. For $n = 21$,

$$T_{21}(C,2,0)/p^9 = r_3^7/p^2 - 6r_3^5/p + 10r_3^3 - 4pr_3 + \left(\frac{p}{138567}\right)(p^2 - |c(p)|^2),$$

where $c(p)$ is the p-th coefficient of a weight 3 eigenform whose level is a multiple of 138567, whose nebentypus has conductor 138567 and whose eigenfield contains $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{17}, \sqrt{19})$.

C quartic. Let $q = p \equiv 1 \pmod 4$, and write $p = a_4^2 + b_4^2$ with a_4 odd. Let C be a quartic character on $\mathbb{F}_q$. For $n = 4$,

$$T_4(C, 0)/G(\overline{C})^2 = -\overline{C}(2)a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(256)$ with trivial nebentypus, namely MFE#77957. For $n = 6$, if $p \equiv 1 \pmod 8$,

$$T_6(C, 1)/p^2 = p - 4a_4^2, \quad T_6(C, 3)/p^2 = -p - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(12)$ with trivial nebentypus, namely MFE#76753. (We remark that $4a_4^2$ is the square of the p-th coefficient of the weight 2 newform MFE#53509 on $\Gamma_0(32)$.) For $n = 8$, if $p \equiv 1 \pmod 8$,

$$T_8(C, 1, 0)/p^2 = -(4a_4^2 - 2p)^2, \quad T_8(C, 1, 2)/p^3 = 2p - 4a_4^2 - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(24)$ with trivial nebentypus, namely MFE#76775. For $n = 10$ and $p \equiv 1 \pmod 8$,

$$T_{10}(C, 3)/p^3 = p^2 - 4pa_4^2 - a(p),$$

where $a(p)$ is the p-th coefficient of a weight 6 eigenform with integer coefficients on $\Gamma_0(120)$ with trivial nebentypus.

C sextic. Let $q = p \equiv 1 \pmod 6$, and let C be a sextic character on $\mathbb{F}_q$. For $n = 3$ and $p \equiv 1 \pmod{12}$,

$$T_3(C, 1, 4)/(G(\overline{C}^2)G(C^3)) = -\overline{C}(36)a(p),$$

where $a(p)$ is the p-th coefficient of a weight 3 eigenform on $\Gamma_0(972)$ with nebentypus $\left(\frac{d}{3}\right)$ and eigenfield $\mathbb{Q}(i\sqrt{6})$. For $n = 4$,

$$T_4(C, 0)/G(\overline{C})^2 = -\left(\frac{-1}{p}\right)a(p),$$

where $a(p)$ is the p-th coefficient of a weight 4 eigenform with integer coefficients on $\Gamma_0(216)$ with trivial nebentypus, namely MFE#77695. Also for $n = 4$,

$$T_4(C, 2)/p^2 = -a(p),$$

where $a(p)$ is the p-th coefficient of a weight 2 eigenform with integer coefficients on $\Gamma_0(24)$ with trivial nebentypus, namely MFE#53504. For $n = 4$ and $n = 6$,

$$\left(\frac{-1}{p}\right)\frac{T_4(C, 2, 3)}{pG(\overline{C})G(C^3)} = \left(\frac{-1}{p}\right)\frac{T_4(C, 4)}{pG(\overline{C})G(C^3)} = 1 + T_6(C, 2, 3)/p^3 = -a(p),$$

where $a(p)$ is the p-th coefficient of a weight 2 eigenform with integer coefficients on $\Gamma_0(216)$ with trivial nebentypus, namely MFE#43627. For $n = 5$,

$$T_5(C, 4)/p^2 = \left(\frac{-5}{p}\right)(p - c(p)^2),$$

where $c(p)$ is the p-th coefficient of a weight 2 eigenform on $\Gamma_0(2700)$ with trivial nebentypus and eigenfield $\mathbb{Q}(\sqrt{10})$, namely MFE#382443. For $n = 6$ and $p \equiv 1 \pmod{12}$,

$$T_6(C,5,3)/p^2 = -p - a(p), \quad T_6(C,1,0)/p = p^2 - c(p)^2,$$

where $a(p)$ is the p-th coefficient of the aforementioned weight 4 eigenform with integer coefficients on $\Gamma_0(54)$ with trivial nebentypus, namely MFE#76862, and where $c(p) \equiv 2 \pmod 3$ is the p-th coefficient of a weight 3 eigenform with integer coefficients on $\Gamma_0(27)$ with nebentypus $\left(\frac{d}{3}\right)$, namely MFE#391774.

C duodecic. Let $q = p \equiv 1 \pmod{12}$, and let C be a duodecic character on $\mathbb{F}_q$. For $n = 3$,

$$T_3(C,4,3)/p^{3/2} = C^3(s)a(p),$$

where $s \in \mathbb{F}_p$ is defined up to sign by $s^2 = -3$ and where, up to sign, $a(p)$ is the p-th coefficient of a weight 2 eigenform on $\Gamma_0(108)$ with quadratic nebentypus of conductor 12 and eigenfield $\mathbb{Q}(\sqrt{-2}, \sqrt{6})$. For $n = 4$,

$$T_4(C,8)/(pG(C^9)) = -C^9(-6)a(p),$$

where $a(p)$ is the p-th coefficient of a weight 2 eigenform with integer coefficients on $\Gamma_0(768)$ with trivial nebentypus, namely MFE#51550.

Note added in proof. Some of the conjectures in the Appendix have been proved in a paper of Booher, Etropolski, and Hittson to appear in Int. J. Number Theory. Our conjecture for $T_4(\phi, 0)$ has been proved in a submitted paper by Dummit, Goldberg, and Perry.

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