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LIE SUPERAUTOMORPHISMS ON ASSOCIATIVE ALGEBRAS

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Abstract. The results on Lie homomorphisms of associative algebras are extended to certain associative superalgebras. It is shown that under appropriate conditions a Lie superautomorphism of $A = A_0 \oplus A_1$ is a sum of a superautomorphism or the negative of a superantiautomorphism and a central map. In particular we consider the situation when A is a central simple algebra and its $\mathbb{Z}_2$-grading is induced by an idempotent.

1. Introduction

Let $A = A_0 \oplus A_1$ be an associative superalgebra. Then A becomes a Lie superalgebra if we replace the associative product by the superbracket $[a, b]_s$. The relationship between the associative and Lie structure of A was studied by many authors; see for example [3, 4, 7, 10, 11, 13, 15, 16, 18]. However, to the best of our knowledge, the natural problem of finding the precise connection between the homomorphisms of these two structures did not receive due attention in the literature. The purpose of this paper is to begin filling this gap.

In the classical ungraded case, the results on Lie homomorphisms of associative rings and algebras have also been obtained with some "delay" if compared to other Lie structure results. These latter had already been obtained in the 1950s and 1960s by Herstein and some of his students; see [12]. Martindale solved various Lie map problems somewhat later (see e.g. [14]), however, under an assumption that the rings in question contain nontrivial idempotents. The first result avoiding idempotents was obtained in 1993 by the second author [8]. The methods from [8] and related papers were later extended in various directions, and eventually this resulted in the creation of the theory of functional identities; see [9]. Among various applications of functional identities, solutions of several Lie map problems are particularly notable. Functional identities will be used, indirectly but essentially, also in this paper.

We shall say that a bijective linear map $\varphi : A \to A$ is a *Lie superautomorphism* of A if $\varphi(A_i) = A_i$, $i = 0, 1$, and $\varphi([a, b]_s) = [\varphi(a), \varphi(b)]_s$ for all $a, b \in A_0 \cup A_1$. We will show that under certain favorable conditions φ can be expressed through superautomorphisms or superantiautomorphisms and central maps. The main result (Theorem 3.1) describes some abstract conditions, which are then applied to the

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case when A is a central simple algebra and the natural $\mathbb{Z}_2$-grading is induced by an idempotent (Corollary 3.2). The main idea of the proof is to introduce a usual Lie automorphism of the Grassmann envelope of A and then apply the theory from [9]. Here we were influenced by our recent work [1, 2], where we noticed that some results from [9] are applicable to the tensor products of commutative algebras with rather general algebras. In a certain sense, Grassman algebras are very close to commutative algebras and Grassman envelopes are very close to tensor products. This was a good motivation for the present work.

We remark that an analogous concept of a Jordan superhomomorphism was treated in [6], but using a more straightforward and elementary approach.

We do not try to push the results to their utmost generality in this short paper. Our main goal is to present the method which, as we hope, could be extended to more general contexts. We plan to continue the investigation of Lie superhomomorphisms in a more technical work in the near future.

After preparing a draft of this paper, we received a preprint of Wang [17] in which Lie superhomomorphisms have also been considered. While there is some overlap between his paper and ours, there are also essential differences. Wang does not reduce the problem to the usual Lie maps in associative algebras (as we do, using the Grassmann envelope), but studies functional identities directly in associative superalgebras. Also, he imposes the conditions on the odd part A_1, while our restrictions concern the even part A_0.

2. Preliminaries

By "an algebra" we shall always mean an algebra over a fixed field F with $\mathrm{char}(F) \neq 2$. Mostly we will consider associative algebras, but not exclusively. So the term "algebra" can mean a nonassociative algebra. For convenience we assume that all our *associative algebras have an identity element.*

2.1. **Superalgebras.** Recall that a *superalgebra* is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$, $A_iA_j \subseteq A_{i+j}$ where $i, j \in \mathbb{Z}_2$. Elements from A_i are said to be *homogeneous of degree* i, $i = 0, 1$. For $x \in A_i$ we shall write $|x| = i$.

An important example of a superalgebra is the *Grassman superalgebra*, G. As an algebra, G is just an associative algebra generated by elements $1, e_1, e_2, \ldots$ that satisfy $e_i^2 = e_ie_j + e_je_i = 0$ for all i, j; as a superalgebra, it is determined by the following rule: $1 \in G_0$, $e_{i_1}e_{i_2}\ldots e_{i_k} \in G_0$ if k is even and $e_{i_1}e_{i_2}\ldots e_{i_k} \in G_1$ if k is odd. Now let $A = A_0 \oplus A_1$ be an arbitrary superalgebra. The algebra $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$, which we view as a subalgebra of the tensor product $A \otimes G$, is called the *Grassman envelope* of A. If $G(A)$ is a Lie algebra, then we say that A is a *Lie superalgebra.* Similar definitions make sense for other varieties of algebras. In particular, if $G(A)$ is an associative algebra, then we say that A is an *associative superalgebra.* But actually it is easy to see that an associative superalgebra is nothing but a $\mathbb{Z}_2$-graded associative algebra. On the other hand, a Lie superalgebra is not a Lie algebra if its grading is nontrivial. Lie superalgebras can be equivalently defined through the super-anticommutativity of the product and the super-Jacobi identity. But we shall not need them in this paper.

Let $A = A_0 \oplus A_1$ be an associative superalgebra. The *superbracket* of two homogeneous elements $a, b \in A$ is defined as $[a, b]_s = ab - (-1)^{|a||b|}ba$. We extend $[\,.\,,\,.\,]_s$ by bilinearity to $A \times A$. Then A, endowed with the superbracket together with the original grading and the original vector space structure, becomes a Lie superalgebra. The *supercenter* of A is defined as the set of all $a \in A$ such that $[a, A]_s = 0$. Note that a Lie superautomorphism $\varphi : A \to A$ satisfies $\varphi([a_0, b]) = [\varphi(a_0), \varphi(b)]$ for all $a_0 \in A_0$, $b \in A$, and $\varphi(a_1 \circ b_1) = \varphi(a_1) \circ \varphi(b_1)$ for all $a_1, b_1 \in A_1$. Here, of course, $[u, v] = uv - vu$ and $u \circ v = uv + vu$.

Let A be an associative algebra and e an idempotent in A. If we set

$$A_0 = eAe + (1-e)A(1-e) \quad \text{and} \quad A_1 = eA(1-e) + (1-e)Ae, \tag{1}$$

then A becomes an associative superalgebra. This is the basic example of a superalgebra structure on an associative algebra, and often this is, in fact, the only example possible. Indeed, let $A = A_0 \oplus A_1$ be an arbitrary associative superalgebra. Then $\sigma(a_0 + a_1) = a_0 - a_1$ defines an automorphism of A such that $\sigma^2 = \mathrm{id}$. If σ is inner, then there exists an invertible $u \in A$ such that u^2 lies in the center of A and $A_0 = \{x \in A \,|\, [u, x] = 0\}$, $A_1 = \{x \in A \,|\, u \circ x = 0\}$. Assume further that u^2 can be written as a square of some central element, $u^2 = c^2$. Then we may replace u by $c^{-1}u$ and therefore assume without loss of generality that $u^2 = 1$. Hence $e = \frac{1}{2}(1 - u)$ is an idempotent, and one can easily show that A_0 and A_1 are given as in (1). Thus, for instance, if an associative algebra A is such that it has only inner automorphisms, its center is just F, and F is an algebraically closed field, then every superalgebra structure of A arises from an idempotent.

The prototype example of (1) is $M(p\,|\,q)$, the algebra of square matrices of order $p + q$ equipped with the following $\mathbb{Z}_2$-grading: $M(p\,|\,q)_0$ consists of matrices of the form $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$, $A \in M_p(F)$, $D \in M_q(F)$, and $M(p\,|\,q)_1$ consists of matrices of the form $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$, $B \in M_{p,q}(F)$, $C \in M_{q,p}(F)$.

2.2. **The strong degree.** The concept of the strong degree was introduced in [5] and is also exposed in [9, Chapter 2]. We will now give a very brief survey which is sufficient for our purposes.

Let A be an associative algebra. By $M(A)$ we denote the multiplication algebra of A, that is, the algebra of linear operators on A generated by all left and all right multiplications L_a and R_b, $a, b \in A$. Thus a typical element in $M(A)$ is an operator on A of the form $x \mapsto \sum_{i=1}^{n} a_i x b_i$, $a_i, b_i \in A$.

Let $t \in A$ be a nonzero element, and let $n \geq 0$ be an integer. We say that the *strong degree* of t is greater than n, s-$\deg(t) > n$, if for every $0 \leq i \leq n$ there exists $\mathcal{E}_i \in M(A)$ such that $\mathcal{E}_i(t^j) = \delta_{ij}$ for each $j = 0, 1, \ldots, n$ (here δ_{ij} is the "Kronecker delta", and $t^0 = 1$). Clearly, in this case, $1, t, \ldots, t^n$ are linearly independent. If s-$\deg(t) > n - 1$ but s-$\deg(t) \not> n$, then we say that the strong degree of t is n (s-$\deg(t) = n$). If s-$\deg(t) > n$ for every positive integer n, then s-$\deg(t) = \infty$. Finally, the *strong degree* of A is s-$\deg(A) = \sup\{s\text{-}\deg(t) \mid t \in A\}$. Trivially, s-$\deg(A) \geq 1$ for every algebra A.

Let us record three simple lemmas. The first two can be very easily checked, and we omit the proofs.

Lemma 2.1. *If A' is a subalgebra of A such that A' contains the identity element of A, then s-$\deg(A) \geq s$-$\deg(A')$.*

Lemma 2.2. *If A_1 and A_2 are algebras, then*

$$s\text{-}\deg(A_1 \oplus A_2) = \min\{s\text{-}\deg(A_1),\ s\text{-}\deg(A_2)\}.$$

Lemma 2.3. *If A_1 and A_2 are algebras, then*

$$s\text{-}\deg(A_1 \otimes A_2) \geq \max\{s\text{-}\deg(A_1),\ s\text{-}\deg(A_2)\}.$$

Proof. If the strong degree of $t_1 \in A_1$ is $> n$, then the strong degree of $t_1 \otimes 1 \in A_1 \otimes A_2$ is also $> n$. Using this, one easily completes the proof. $\square$

By a *central simple algebra* we mean a simple algebra such that its center coincides with F. The next lemma follows from [9, Lemma 2.3 and Corollary C.3].

Lemma 2.4. *If A is a central simple algebra, then $s\text{-deg}(A) = \sqrt{\dim A}$.*

Let us point out that the case when A is infinite dimensional is not excluded here; in this case we have $s\text{-deg}(A) = \infty$.

For other examples of algebras whose strong degree can be computed, we refer the reader to [5] and [9].

2.3. **Lie (super)automorphisms.** Let B be an associative algebra. Recall that a bijective linear map $\Phi : B \to B$ is said to be a Lie automorphism if $\Phi([a, b]) = [\Phi(a), \Phi(b)]$ for all $a, b \in B$. Clearly, the restriction of a Lie superautomorphism of $A = A_0 \oplus A_1$ on A_0 is a Lie automorphism of A_0. The next result is an immediate corollary to [9, Theorem 2.19 and Theorem 6.1]. Its proof is a typical application of the general theory of functional identities.

Theorem 2.5. *Let B be an associative algebra such that $s\text{-deg}(B) \geq 3$. Assume that the center Z of B does not contain idempotents different from 0 and 1. If Φ is a Lie automorphism of B, then $\Phi = \Theta + \Omega$, where Θ is either a homomorphism of B or the negative of an antihomomorphism of B, and Ω is a map from B into Z which vanishes on commutators.*

Note that this is the optimal description of a Lie automorphism through associative maps. Namely, a map of the form $\Theta + \Omega$, where Θ and Ω are as in the theorem, preserves the Lie bracket $[\,.\,,\,.\,]$. It is easy to guess what are the counterparts of these maps in the superalgebra setting. Let $A = A_0 \oplus A_1$ be an associative superalgebra. A linear map $\theta : A \to A$ is called a *superhomomorphism* if it is a homomorphism of the algebra A (i.e., it satisfies $\theta(ab) = \theta(a)\theta(b)$) and if it preserves the $\mathbb{Z}_2$-grading (i.e., $\theta(A_i) \subseteq A_i$, $i = 0, 1$). Of course, superhomomorphisms also preserve the superbracket $[\,.\,,\,.\,]_s$. Next, a linear $\mathbb{Z}_2$-grading-preserving map $\theta : A \to A$ is called a *superantihomomorphism* if $\theta(ab) = (-1)^{|a||b|}\theta(b)\theta(a)$ for all homogeneous elements $a, b \in A$. Note that the negative of a superantihomomorphism preserves the superbracket. Finally, if θ is either a superhomomorphism or a superantihomomorphism and τ is a map from A into its center such that $\tau([A_0, A_0]) = \tau(A_1 \circ A_1) = \tau(A_1) = 0$, then $\theta + \tau$ also preserves the superbracket (here by the center we mean the usual center, not supercenter). Moreover, if the range of τ lies in A_0, then $\theta + \tau$ also preserves the $\mathbb{Z}_2$-grading.

3. Main results

Let us first reveal the main idea on which this paper is based. Let $A = A_0 \oplus A_1$ be an associative superalgebra, and let $\varphi : A \to A$ be a Lie superautomorphism. We "extend" φ to $\Phi : G(A) \to G(A)$ in an obvious way, i.e., as the restriction of $\varphi \otimes \mathrm{id}$ to $G(A)$. Thus

$$\Phi(a_i \otimes g_i) = \phi(a_i) \otimes g_i, \quad a_i \in A_i,\, g_i \in G_i,\, i = 0, 1.$$

One easily checks that Φ is a Lie automorphism of $G(A)$. For example, if $a_1, b_1 \in A_1$ and $g_1, g_1' \in G_1$, then $g_1g_1' + g_1'g_1 = 0$ and hence

$$\Phi([a_1 \otimes g_1, b_1 \otimes g_1']) = \Phi(a_1 \circ b_1 \otimes g_1g_1') = \varphi(a_1 \circ b_1) \otimes g_1g_1'$$
$$= \big(\varphi(a_1) \circ \varphi(b_1)\big) \otimes g_1g_1' = [\varphi(a_1) \otimes g_1, \varphi(b_1) \otimes g_1'] = [\Phi(a_1 \otimes g_1), \Phi(b_1 \otimes g_1')].$$

Similarly one considers the action of Φ on other commutators.

Now assume that the algebra $B = G(A)$ satisfies the conditions of Theorem 2.5. Then $\Phi = \Theta + \Omega$, where Θ and Ω are as in this theorem. We now have to use this information to describe φ. This is the idea of our proof.

Let us simplify our task by assuming slightly more than required in Theorem 2.5. The assumptions that we impose are:

(a) s-$\deg(A_0) \geq 3$,
(b) the supercenter of A is equal to F.

Note that **(a)** yields that s-$\deg(A_0 \otimes G_0) \geq 3$ (by Lemma 2.3) and therefore s-$\deg(G(A)) \geq 3$ (by Lemma 2.1). Further, it is easy to see that **(b)** implies that the center Z of $G(A)$ is equal to $1 \otimes G_0$. Since G_0 does not contain nontrivial idempotents, the same holds for Z. Therefore **(a)** and **(b)** indeed imply all assumptions of Theorem 2.5. We thus have $\Phi = \Theta + \Omega$. Note also that we can write $\Omega(r) = 1 \otimes \omega(r)$, $r \in G(A)$, where $\omega : G(A) \to G_0$. Finally we remark that **(b)** implies that elements from F are the only elements that lie in both A_0 and the center of the algebra A.

We now have to treat two cases, the first, where Θ is a homomorphism and the second, where Θ is the negative of an antihomomorphism. Let us consider in detail the second case, which is apparently the less favorable.

We begin by considering $\varphi(a_0b_0) \otimes 1$ with $a_0, b_0 \in A_0$. We have

$$\begin{aligned}
&\varphi(a_0b_0) \otimes 1 = \Phi(a_0b_0 \otimes 1)\\
&= \Theta((a_0 \otimes 1)(b_0 \otimes 1)) + 1 \otimes \omega(a_0b_0 \otimes 1)\\
&= -\,\Theta(b_0 \otimes 1)\Theta(a_0 \otimes 1) + 1 \otimes \omega(a_0b_0 \otimes 1)\\
&= -\,\big(\varphi(b_0) \otimes 1 - 1 \otimes \omega(b_0 \otimes 1)\big)\big(\varphi(a_0) \otimes 1 - 1 \otimes \omega(a_0 \otimes 1)\big) + 1 \otimes \omega(a_0b_0 \otimes 1)\\
&= -\,\varphi(b_0)\varphi(a_0) \otimes 1 + \varphi(b_0) \otimes \omega(a_0 \otimes 1) + \varphi(a_0) \otimes \omega(b_0 \otimes 1)\\
&\quad + 1 \otimes \big(\omega(a_0b_0 \otimes 1) - \omega(b_0 \otimes 1)\omega(a_0 \otimes 1)\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\big(\varphi(a_0b_0) + \varphi(b_0)\varphi(a_0)\big) \otimes 1 &= \varphi(b_0) \otimes \omega(a_0 \otimes 1)\\
&\quad + \varphi(a_0) \otimes \omega(b_0 \otimes 1) + 1 \otimes \big(\omega(a_0b_0 \otimes 1) - \omega(b_0 \otimes 1)\omega(a_0 \otimes 1)\big).
\end{aligned} \tag{2}$$

For every $a_0 \in A_0$ we write $\omega(a_0 \otimes 1) = \tau(a_0) + \epsilon(a_0)$, where $\tau(a_0) \in F$ and $\epsilon(a_0)$ lies in the linear span of the products of e_i's. Suppose there exists $a_0 \in A_0$ such that $\epsilon(a_0) \neq 0$. Then it readily follows from (2) that every $\varphi(b_0)$ lies in the linear span of $\varphi(a_0)$ and 1. But this is impossible since **(a)** in particular implies that A_0 contains elements that are not algebraic of degree ≤ 2. Consequently $\epsilon(a_0) = 0$ for every $a_0 \in A_0$, and so $\omega(a_0 \otimes 1) = \tau(a_0) \in F$. Therefore (2) reduces to

$$\varphi(a_0b_0) + \varphi(b_0)\varphi(a_0) = \tau(a_0)\varphi(b_0) + \tau(b_0)\varphi(a_0) + \tau(a_0b_0) - \tau(b_0)\tau(a_0). \tag{3}$$

We now define $\theta : A_0 \to A_0$ by

$$\theta(a_0) = \varphi(a_0) - \tau(a_0),$$

so that $\Theta(a_0 \otimes 1) = \theta(a_0) \otimes 1$. Note that (3) can now be written as

$$\theta(a_0 b_0) = -\theta(b_0)\theta(a_0). \tag{4}$$

In a similar fashion we consider $\varphi(a_0 b_0) \otimes e_1 e_2$ with $a_0, b_0 \in A_0$:

$$\begin{aligned}
&\varphi(a_0 b_0) \otimes e_1 e_2 = \Phi(a_0 b_0 \otimes e_1 e_2) \\
&= \Theta((a_0 \otimes 1)(b_0 \otimes e_1 e_2)) + 1 \otimes \omega(a_0 b_0 \otimes e_1 e_2) \\
&= -\Theta(b_0 \otimes e_1 e_2)\Theta(a_0 \otimes 1) + 1 \otimes \omega(a_0 b_0 \otimes e_1 e_2) \\
&= -\big(\varphi(b_0) \otimes e_1 e_2 - 1 \otimes \omega(b_0 \otimes e_1 e_2)\big)\big(\theta(a_0) \otimes 1\big) + 1 \otimes \omega(a_0 b_0 \otimes e_1 e_2) \\
&= -\varphi(b_0)\theta(a_0) \otimes e_1 e_2 + \theta(a_0) \otimes \omega(b_0 \otimes e_1 e_2) + 1 \otimes \omega(a_0 b_0 \otimes e_1 e_2).
\end{aligned}$$

Thus

$$(\varphi(a_0 b_0) + \varphi(b_0)\theta(a_0)) \otimes e_1 e_2 = \theta(a_0) \otimes \omega(b_0 \otimes e_1 e_2) + 1 \otimes \omega(a_0 b_0 \otimes e_1 e_2).$$

Using (4) we see that $\varphi(a_0 b_0) + \varphi(b_0)\theta(a_0) = \tau(b_0)\theta(a_0) + \tau(a_0 b_0)$, and so we get

$$\theta(a_0) \otimes \big(\omega(b_0 \otimes e_1 e_2) - \tau(b_0) e_1 e_2\big) + 1 \otimes \big(\omega(a_0 b_0 \otimes e_1 e_2) - \tau(a_0 b_0) e_1 e_2\big) = 0.$$

Choosing a_0 so that $\theta(a_0) \notin F$ (its existence is a trivial consequence of **(a)**) it follows that for every $b_0 \in A_0$ we have

$$\omega(b_0 \otimes e_1 e_2) = \tau(b_0) e_1 e_2. \tag{5}$$

Next we consider $\varphi(a_0 b_1) \otimes e_1$ with $a_0 \in A_0$, $b_1 \in A_1$. We have

$$\begin{aligned}
&\varphi(a_0 b_1) \otimes e_1 = \Phi(a_0 b_1 \otimes e_1) \\
&= \Theta((a_0 \otimes 1)(b_1 \otimes e_1)) + 1 \otimes \omega(a_0 b_1 \otimes e_1) \\
&= -\Theta(b_1 \otimes e_1)\Theta(a_0 \otimes 1) + 1 \otimes \omega(a_0 b_1 \otimes e_1) \\
&= -\big(\varphi(b_1) \otimes e_1 - 1 \otimes \omega(b_1 \otimes e_1)\big)\big(\theta(a_0) \otimes 1\big) + 1 \otimes \omega(a_0 b_1 \otimes e_1) \\
&= -\varphi(b_1)\theta(a_0) \otimes e_1 + \theta(a_0) \otimes \omega(b_1 \otimes e_1) + 1 \otimes \omega(a_0 b_1 \otimes e_1),
\end{aligned}$$

and hence

$$\big(\varphi(a_0 b_1) + \varphi(b_1)\theta(a_0)\big) \otimes e_1 = \theta(a_0) \otimes \omega(b_1 \otimes e_1) + 1 \otimes \omega(a_0 b_1 \otimes e_1).$$

Since $\omega(b_1 \otimes e_1), \omega(a_0 b_1 \otimes e_1) \in G_0$ it follows that $\varphi(a_0 b_1) + \varphi(b_1)\theta(a_0) = 0$. Consequently, choosing a_0 so that $\theta(a_0) \notin F$ we obtain $\omega(b_1 \otimes e_1) = 0$ for every $b_1 \in A_1$.

We now extend θ to A by setting

$$\theta(a_1) = \varphi(a_1)$$

for every $a_1 \in A_1$. Note that we have

$$\theta(a_0 b_1) = -\theta(b_1)\theta(a_0) \tag{6}$$

and $\Theta(b_1 \otimes e_1) = \theta(b_1) \otimes e_1$. Of course, similarly we have $\Theta(b_1 \otimes e_i) = \theta(b_1) \otimes e_i$ for every i.

Considering $\theta(b_1 a_0) \otimes e_1 = \varphi(b_1 a_0) \otimes e_1$ in a similar (although now more straightforward) way as we considered $\varphi(a_0 b_1) \otimes e_1$, one obtains

$$\theta(b_1 a_0) = -\theta(a_0)\theta(b_1) \tag{7}$$

for all $a_0 \in A_0$, $b_1 \in A_1$.

Finally we consider $\varphi(a_1b_1)\otimes e_1e_2$ with $a_1, b_1 \in A_1$. Using (5) we obtain

$$\begin{aligned}\varphi(a_1b_1)\otimes e_1e_2 &= \Phi(a_1b_1\otimes e_1e_2)\\ &= \Theta((a_1\otimes e_1)(b_1\otimes e_2)) + 1\otimes\omega(a_1b_1\otimes e_1e_2)\\ &= -\Theta(b_1\otimes e_2)\Theta(a_1\otimes e_1) + 1\otimes\tau(a_1b_1)e_1e_2\\ &= -(\theta(b_1)\otimes e_2)(\theta(a_1)\otimes e_1) + 1\otimes\tau(a_1b_1)e_1e_2\\ &= (\theta(b_1)\theta(a_1)+\tau(a_1b_1))\otimes e_1e_2.\end{aligned}$$

Therefore $\varphi(a_1b_1) = \theta(b_1)\theta(a_1) + \tau(a_1b_1)$, which yields

$$\theta(a_1b_1) = \theta(b_1)\theta(a_1). \tag{8}$$

From (4), (6), (7) and (8) we now see that θ is the negative of a superantihomomorphism.

Extending τ to A by simply setting $\tau(A_1) = 0$ we thus have

$$\varphi(a) = \theta(a) + \tau(a)$$

for every $a \in A$. Let us finally make use of the condition that Ω vanishes on commutators (see Theorem 2.5). Considering commutators in $[A_0\otimes 1, A_0\otimes 1]$ we obtain $\tau([A_0, A_0]) = 0$. Similarly, considering commutators in $[A_1\otimes e_1, A_1\otimes e_2]$ and also applying (5) we get $\tau(A_1\circ A_1) = 0$.

As θ is the negative of a superantihomomorphism, we have $\theta(1)\theta(a) = \theta(a)\theta(1) = -\theta(a)$ for every $a \in A$. In particular, $\theta(1)$ thus commutes with all elements from $\varphi(A) = A$, which implies that $\theta(1) \in F$ (see the paragraph following **(b)**). But then $\theta(1) = -1$.

Suppose that $a = a_0 + a_1$ is such that $\theta(a) = 0$. Then $\varphi(a_0) + \tau(a_0) = 0$ and $\varphi(a_1) = 0$. The second identity yields $a_1 = 0$. The first identity implies that $\varphi([a_0, A]) = [\varphi(a_0), \varphi(A)] = -[\tau(a_0), A] = 0$. Thus $[a_0, A] = 0$ and so $a = a_0 \in F$. However, since $\theta(1) = -1$, this is possible only if $a = 0$. This proves that θ is injective. From $\theta(a-\tau(a)) = \theta(a) - \tau(a)\theta(1) = \theta(a) + \tau(a) = \varphi(a)$ we see that θ is also surjective. Thus θ is the negative of a *superantiautomorphism*.

Recall that we have derived all these conclusions under the assumption that Θ is the negative of an antihomomorphism. If Θ were a homomorphism, then following the same procedure we would arrive at analogous conclusions, such as θ is then a *superautomorphism*.

To summarize, we have the desired conclusion $\varphi = \theta + \tau$ under the assumption that the conditions **(a)** and **(b)** are fulfilled. More precisely, the following is true.

Theorem 3.1. *Let $A = A_0 \oplus A_1$ be an associative superalgebra such that its supercenter is F and s-deg$(A_0) \geq 3$. Then every Lie superautomorphism φ of A takes the form $\varphi = \theta + \tau$, where θ is either a superautomorphism of A or the negative of a superantiautomorphism of A, and τ is a map from A into F satisfying $\tau([A_0, A_0]) = \tau(A_1\circ A_1) = \tau(A_1) = 0$.*

In our final result we apply Theorem 3.1 to a more concrete situation.

Corollary 3.2. *Let A be a central simple associative algebra. Let e be an idempotent in A and consider A as an associative superalgebra with respect to* (1). *If* $\dim eAe > 4$ *and* $\dim(1-e)A(1-e) > 4$, *then every Lie superautomorphism φ of A is of the form $\varphi = \theta + \tau$ where θ is either a superautomorphism of A or the negative of a superantiautomorphism of A, and τ is a map from A into F satisfying* $\tau([A_0, A_0]) = \tau(A_1\circ A_1) = \tau(A_1) = 0$.

Proof. Both algebras eAe and $(1-e)A(1-e)$ are also central simple. The simplicity can be easily checked. Let us show that they are central. This is undoubtedly known, but we still give a short proof for completeness. We want to show that the center of eAe is equal to Fe. Let eae be a nonzero element in the center of eAe. Since A is simple, there exists $x_i, y_i \in A$ such that $\sum_i x_i eaey_i = 1$. For every $x \in A$ we then have $xe = \sum_i x_i eaey_i xe$. Since eae commutes with $ey_i xe$ this implies $xe = \sum_i x_i ey_i xeae$. Thus $xe = bxeae$ for every $x \in A$ where $b = \sum_i x_i ey_i$. Accordingly, $ybxeae = yxe = byxeae$ for all $x, y \in A$. That is, $[b, A]Aeae = 0$. Since A is simple it follows that $[b, A] = 0$, and hence, A being central, we have $b = \lambda \in F$. Returning to $xe = bxeae$ it now follows that $eae = \lambda^{-1}e$, as claimed.

Lemma 2.4 implies that the strong degree of both eAe and $(1-e)A(1-e)$ is ≥ 3. But then the strong degree of A_0 is also ≥ 3 by Lemma 2.2.

Using the fact that the center of eAe is Fe, one can easily show that the supercenter of A is just F. All conditions of Theorem 3.1 are thus fulfilled and the result follows. □

For example, Corollary 3.2 is applicable to the algebra $M(p\,|\,q)$ as long as $p > 2$ and $q > 2$. It is easy to see that in this situation the identities satisfied by τ imply that τ is necessarily a scalar multiple of the *supertrace*, i.e. the map given by $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \mapsto \mathrm{tr}(A) - \mathrm{tr}(D)$, where tr denotes the trace.

Corollary 3.2 shows both the power and the limitations of our approach based on the strong degree and functional identities. While it covers a rather large class of associative superalgebras (which are possibly infinite dimensional), it fails in some specific situations related to low dimensional algebras.

References

[1] Yu. Bahturin, M. Brešar, Lie gradings on associative algebras, *J. Algebra* **321** (2009), 264-283. MR2469360

[2] Yu. Bahturin, M. Brešar, I. Shestakov, Jordan gradings on associative algebras, *Algebr. Represent. Theory*, to appear.

[3] Yu. Bahturin, D. Fischman, S. Montgomery, On the generalized Lie structure of associative algebras, *Israel J. Math.* **96** (1996), 27-48. MR1432725 (98d:16048)

[4] Yu. Bahturin, O. H. Kegel, Universal sums of abelian subalgebras, *Comm. Algebra* **23** (1995), 2975-2990. MR1332159 (96g:16033)

[5] K. I. Beidar, M. Brešar, M. A. Chebotar, Functional identities revised: The fractional and the strong degree, *Comm. Algebra* **30** (2002), 935-969. MR1883042 (2002k:16048)

[6] K. I. Beidar, M. Brešar, M. A. Chebotar, Jordan superhomomorphisms, *Comm. Algebra* **31** (2003), 633-644. MR1968917 (2004a:16060)

[7] G. Benkart, X. Xu, K. Zhao, Classical Lie superalgebras over simple associative algebras, *Proc. London Math. Soc. (3)* **92** (2006), 581-600. MR2223537 (2007c:17006)

[8] M. Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993), 525-546. MR1069746 (93d:16044)

[9] M. Brešar, M. A. Chebotar, W. S. Martindale III, *Functional Identities*, Birkhäuser Verlag, 2007. MR2332350 (2008j:16068)

[10] C. Gómez-Ambrosi, J. Laliena, I. P. Shestakov, On the Lie structure of the skew elements of a prime superalgebra with superinvolution, *Comm. Algebra* **28** (2000), 3277-3291. MR1765316 (2001g:17007)

[11] C. Gómez-Ambrosi, I. P. Shestakov, On the Lie structure of the skew elements of a simple superalgebra with superinvolution, *J. Algebra* **208** (1998), 43-71. MR1643975 (99j:17032)

[12] I. N. Herstein, *Topics in Ring Theory*, University of Chicago Press, 1969. MR0271135 (42:6018)

[13] J. Laliena, S. Sacristán, Lie structure in semiprime superalgebras with superinvolution, *J. Algebra* **315** (2007), 751-760. MR2351892 (2008j:16123)

[14] W. S. Martindale III, Lie isomorphisms of prime rings, *Trans. Amer. Math. Soc.* **142** (1969), 437-455. MR0251077 (40:4308)
[15] F. Montaner, On the Lie structure of associative superalgebras, *Comm. Algebra* **26** (1998), 2337-2349. MR1626634 (99g:16055)
[16] S. Montgomery, Constructing simple Lie superalgebras from associative graded algebras, *J. Algebra* **195** (1997), 558-579. MR1469640 (99g:17043)
[17] Y. Wang, Functional identities and Lie superhomomorphisms on prime superalgebras, *Comm. Algebra*, to appear.
[18] K. Zhao, Simple Lie color algebras from graded associative algebras, *J. Algebra* **269** (2003), 439-455. MR2015286 (2004m:17044)

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