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ILL-POSEDNESS OF THE BASIC EQUATIONS OF FLUID DYNAMICS IN BESOV SPACES

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Abstract. We give a construction of a divergence-free vector field $u_0 \in H^s \cap B^{-1}_{\infty,\infty}$, for all $s < 1/2$, with arbitrarily small norm $\|u_0\|_{B^{-1}_{\infty,\infty}}$ such that any Leray-Hopf solution to the Navier-Stokes equation starting from u_0 is discontinuous at $t = 0$ in the metric of $B^{-1}_{\infty,\infty}$. For the Euler equation a similar result is proved in all Besov spaces $B^s_{r,\infty}$ where $s > 0$ if $r > 2$, and $s > n(2/r - 1)$ if $1 \le r \le 2$. This includes the space $B^{1/3}_{3,\infty}$, which is known to be critical for the energy conservation in ideal fluids.

1. Introduction

In recent years, numerous results have appeared in the literature on the well-posedness theory of the Euler and Navier-Stokes equations in Besov spaces (see, for example, [1, 4, 5, 10, 13] and the references therein). The best local existence and uniqueness result known for the Euler equation states that for any initial condition $u_0 \in B^{\frac{n}{r}+1}_{r,1}$ with $1 < r \le \infty$, where n is the dimension of the fluid domain, there exists a unique weak solution u in the space $C([0,T]; B^{\frac{n}{r}+1}_{r,1})$, for some $T = T(u_0) > 0$, such that $u(t) \to u_0$ in $B^{\frac{n}{r}+1}_{r,1}$. The case of $r = 2$, $n = 3$ is especially interesting for it constitutes the borderline space for applicability of the standard energy method in proving such a result (see [9]). Notice that $B^{5/2}_{2,1}$ is a proper subspace of the Sobolev space $H^{5/2} = B^{5/2}_{2,2}$, where local existence is an outstanding open problem. In this paper we present a construction of u_0 which demonstrates ill-posedness of the Euler equations in a range of Besov spaces with the opposite extreme summation index, namely in $B^s_{r,\infty}$. In particular, there exists a $u_0 \in B^{\frac{n}{r}+1}_{r,\infty}$ such that any energy bounded weak solution to the Euler equation that starts from u_0 does not converge back to u_0 is the metric of $B^{\frac{n}{r}+1}_{r,\infty}$ as time goes to zero. Another particular instance includes the space $B^{1/3}_{3,\infty}$, which defines a critical regularity of solutions to obey the energy conservation law in ideal fluids (see [6]).

In the second part of this paper we address the question of ill-posedness for the Navier-Stokes equations in the critical Besov space $X = B^{-1}_{\infty,\infty}$. We recall

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that the homogeneous space $\dot{X} = \dot{B}^{-1}_{\infty,\infty}$ is invariant with respect to the natural scaling of the equation in $\mathbb{R}^3$. Moreover it is the largest such space (see [4]). The non-homogeneous space considered in this paper is even larger although quasi-invariant only with respect to the small scale dilations. In a recent work of Bourgain and Pavlović [3] a mild solution to NSE was constructed with initial condition $\|u_0\|_{\dot{X}} < \delta$ such that at a time $t < \delta$ the solution satisfies $\|u(t)\|_{\dot{X}} > 1/\delta$. This shows that the evolution under NSE is not continuous from $\dot{X}$ into $C([0,T];\dot{X})$. In our Proposition 3.2, similar to the case of the Euler equation, we construct an initial condition U which belongs to all Besov spaces $B^{3/r-1}_{r,\infty}$ in the range $1 < r \leq \infty$ (in particular, U has finite energy) such that any Leray-Hopf weak solution starting from U does not return to U as $t \to 0$ in the metric of the inhomogeneous space X. This demonstrates an even more dramatic breakdown of NSE evolution in X as there is no continuous trajectory in X at all. More importantly, our construction gives a simple model for the forward energy cascade, which is typically observed in turbulent flows [8]. Incidentally, the result proved in [7] shows that any left-continuous Leray-Hopf solution in X is necessarily regular.

We consider periodic boundary conditions for two reasons. Firstly, we do not make use of infinitesimally small frequencies in our analysis, and secondly, our construction is more transparent when the frequency space is a lattice. However, with the technique developed in [6], the results can be carried over to the case of $\mathbb{R}^n$ as well.

Let us now introduce the notation and spaces used in this paper. We will fix the notation for scales $\lambda_q = 2^q$ in some inverse length units. Let us fix a nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1/2$, and $\chi(\xi) = 0$ for $|\xi| \geq 1$. We define $\varphi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$, and $\varphi_q(\xi) = \varphi(\kappa_q^{-1}\xi)$ for $q \geq 0$, and $\varphi_{-1} = \chi$. For a tempered distribution vector field u on the torus $\mathbb{T}^n$ we consider the Littlewood-Paley projections

$$u_q(x) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k)\varphi_q(k)e^{ik\cdot x}, \quad q \geq -1. \tag{1}$$

So, we have $u = \sum_{q=-1}^{\infty} u_q$ in the sense of distributions. We also use the following notation: $u_{\leq q} = \sum_{p=-1}^{q} u_p$ and $\tilde{u}_q = u_{q-1} + u_q + u_{q+1}$.

Let us recall the definition of Besov spaces. A tempered distribution u belongs to $B^s_{r,l}(\mathbb{T}^n)$ for $s \in \mathbb{R}$, $1 \leq l, r \leq \infty$ iff

$$\|u\|_{B^s_{r,l}} = \left[\sum_{q\geq -1} (\lambda_q^s \|u_q\|_r)^l\right]^{1/l} < \infty.$$

We use $\|u\|_p$ to denote the norm in the Lebesgue space $L^p(\mathbb{T}^n)$.

2. Inviscid case

The Euler equations for the evolution of an ideal fluid are given by

$$u_t + (u\cdot\nabla)u = -\nabla p, \tag{2}$$

$$\operatorname{div} u = 0. \tag{3}$$

As noted in the introduction we assume throughout that the fluid domain is the torus $\mathbb{T}^n$, $n \geq 2$. By a weak solution to (2) we understand an L^2-valued weakly continuous field u satisfying (2)–(3) in the distributional sense. Let us recall that

all such solutions have absolutely continuous-in-time Fourier coefficients (see for example [11]).

We denote by $\vec{e}_1, \dots, \vec{e}_n$ the vectors of the standard unit basis. Let us fix an $s > 0$ and define

$$u_0(x_1, \dots, x_n) = \vec{e}_1 \cos(x_2) + \vec{e}_2 \sum_{q=0}^{\infty} \frac{1}{\lambda_q^s} \cos(\lambda_q x_1).$$

Notice that u_0 is divergence free and $u_0 \in B^s_{r,\infty}(\mathbb{T}^n)$ for any $r \in [1, \infty]$. Furthermore, u_0 is in fact two dimensional.

Proposition 2.1. *Suppose u is a weak solution to the Euler equation* (2) *with initial condition $u(0) = u_0$. Then there is $\delta = \delta(n, r, s) > 0$ independent of u such that*

$$\limsup_{t \to 0^+} \|u(t) - u_0\|_{B^s_{r,\infty}} \geq \delta, \tag{4}$$

where $s > 0$ if $r > 2$, and $s > n(2/r - 1)$ in the case that $1 \leq r \leq 2$.

The rest of the section is devoted to the proof of Proposition 2.1.

Let us denote $X = B^s_{r,\infty}$ for notational convenience. We can assume without loss of generality that for some $t_0 > 0$, $u \in L^\infty([0, t_0]; X)$. Indeed, otherwise (4) follows immediately. Further proof is based on the fact that u_0 produces a strong forward energy transfer, which forces u to actually escape from $B^s_{r,\infty}$ unless (4) is satisfied. Let us consider frequencies

$$\xi_q = (\lambda_q, 1, 0, \dots, 0), \text{ for } q = 0, 1, \dots.$$

Let $p(\xi)$ be the symbol of the Leray-Hopf projection, i.e.

$$p(\xi) = I - \frac{\xi \otimes \xi}{|\xi|^2}.$$

By a direct computation we obtain

$$f_q = p(\xi_q)(u_0 \cdot \nabla u_0)^\wedge(\xi_q) = i\lambda_q^{1-s} \vec{e}_2 + O(1/\lambda_q^s). \tag{5}$$

We will prove the following estimate for the nonlinear term:

$$\|(u \cdot \nabla v)_q\|_1 \lesssim \lambda_q^{1-s} \|u\|_X \|v\|_X, \tag{6}$$

for all $u, v \in X$ and $q \geq -1$. First, let us assume that $r \leq 2$, and let r^* be the conjugate of r, i.e. $\frac{1}{r} + \frac{1}{r^*} = 1$. Using the identity $\operatorname{div}(u \otimes v) = u \cdot \nabla v$ and the Bernstein inequality we obtain

$$\|\operatorname{div}(u \otimes v)_q\|_1 \lesssim \lambda_q \|(u \otimes v)_q\|_1 \leq \lambda_q \sum_{\substack{p', p'' \geq q \\ |p' - p''| \leq 2}} \|u_{p'}\|_r \|v_{p''}\|_{r^*} \tag{7}$$

$$+ \lambda_q \|u_q\|_r \sum_{p \leq q} \|v_p\|_{r^*} + \lambda_q \|v_q\|_r \sum_{p \leq q} \|u_p\|_{r^*}. \tag{8}$$

Using that

$$\|w_p\|_{r^*} \lesssim \lambda_p^{n(2/r-1)} \|w_p\|_r,$$

we have for the first sum

$$\lambda_q \sum_{\substack{p',p''\geq q\\ |p'-p''|\leq 2}} \|u_{p'}\|_r \|v_{p''}\|_{r^*} \lesssim \lambda_q \sum_{\substack{p',p''\geq q\\ |p'-p''|\leq 2}} \|u_{p'}\|_r \lambda_{p'}^s \|v_{p''}\|_r \lambda_{p''}^s \lambda_{p''}^{n(2/r-1)-2s}$$
$$\lesssim \lambda_q^{1+n(2/r-1)-2s} \|u\|_X \|v\|_X.$$

For the second sum we obtain

$$\lambda_q \|u_q\|_r \sum_{p\leq q} \|v_p\|_{r^*} \lesssim \lambda_q^{1-s} \lambda_q^s \|u_q\|_r \sum_{p\leq q} \|v_p\|_r \lambda_p^s \lambda_p^{n(2/r-1)-s} \lesssim \lambda_q^{1-s} \|u\|_X \|v\|_X.$$

A similar estimate holds for the third term. We thus obtain (6).

In the case $r > 2$, we use the basic embedding $L^r \hookrightarrow L^{r^*}$ instead of Bernstein's inequalities in (7)–(8). The rest of the argument is similar.

We have

$$\hat{u}(\xi_q, t) = \hat{u}(\xi_q, 0) + \int_0^t p(\xi_q)(u \cdot \nabla u)^\wedge(\xi_q, s) ds, \tag{9}$$

for all $t > 0$. By our construction, $\hat{u}(\xi_q, 0) = 0$. On the other hand, we can estimate using (6) that

$$\begin{aligned} |p(\xi_q)(u \cdot \nabla u)^\wedge(\xi_q, s) - f_q| &\leq |(u \cdot \nabla u)^\wedge(\xi_q, s) - (u_0 \cdot \nabla u_0)^\wedge(\xi_q)| \\ &= |(u \cdot \nabla u)_q^\wedge(\xi_q, s) - (u_0 \cdot \nabla u_0)_q^\wedge(\xi_q)| \\ &\leq \|(u \cdot \nabla u)_q(s) - (u_0 \cdot \nabla u_0)_q\|_1 \\ &\lesssim \lambda_q^{1-s} (\|u(s)\|_X + \|u_0\|_X) \|u(s) - u_0\|_X. \end{aligned}$$

Thus, from (9) we obtain

$$\lambda_q^s |\hat{u}(\xi_q, t)| \geq t\lambda_q - tO(1) - C\lambda_q \int_0^t (\|u(s)\|_X + \|u_0\|_X) \|u(s) - u_0\|_X ds.$$

We can see that if the limit in (4) does not exceed $\delta = 1/(10C)$, then the integral becomes less than $t/2$. This implies that $u(t) \notin X$.

3. Ill-posedness of NSE

Now we turn to the analogous question for the viscous model. The Navier-Stokes equations are given by

$$u_t + (u \cdot \nabla)u = \nu \Delta u - \nabla p, \tag{10}$$

$$\operatorname{div} u = 0. \tag{11}$$

Our fluid domain here is the three dimensional torus $\mathbb{T}^3$. We refer to [12] for the classical well-posedness theory for this equation. Let us recall that for every divergence-free field $U \in L^2(\mathbb{T}^3)$ there exists a weak solution $u \in C_w([0,T); L^2) \cap L^2([0,T); H^1)$ to (10)-(11) satisfying the energy equality

$$\|u(t)\|_2^2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2 ds \leq \|U\|_2^2, \tag{12}$$

for all $t > 0$, and such that $u(t) \to U$ strongly in L^2 as $t \to 0$. In what follows we do not actually use inequality (12), which allows us to formulate a more general statement below in Proposition 3.2.

Let us fix a small $\epsilon > 0$. Let us choose a sequence $q_1 < q_2 < \cdots$ with elements sufficiently far apart so that $\lambda_{q_i}^2/\lambda_{q_{i+1}} < \epsilon$. Let us fix a small $c > 0$ and consider the following blocks of integers:

$$\begin{aligned}
&A_j = [(1-c)\lambda_{q_j}, (1+c)\lambda_{q_j}] \times [-c\lambda_{q_j}, c\lambda_{q_j}]^2 \cap \mathbb{Z}^3,\\
&B_j = [-c\lambda_{q_j-1}, c\lambda_{q_j-1}]^2 \times [(1-c)\lambda_{q_j-1}, (1+c)\lambda_{q_j-1}] \cap \mathbb{Z}^3,\\
&C_j = A_j + B_j,\\
&A_j^* = -A_j,\ B_j^* = -B_j,\ C_j^* = -C_j.
\end{aligned}$$

Thus, A_j, C_j and their conjugates lie in the λ_{q_j}-th shell, while B_j, B_j^* lie in the contiguous λ_{q_j-1}-th shell. Let us denote

$$\vec{e}_1(\xi) = p(\xi)\vec{e}_1,\ \vec{e}_2(\xi) = p(\xi)\vec{e}_2,$$

for $\xi \in \mathbb{Z}^3 \backslash \{0\}$. We define

$$U = \sum_{j\geq 1} (U_{q_j} + U_{q_j-1}), \tag{13}$$

where

$$\widehat{U_{q_j}}(\xi) = \frac{1}{\lambda_{q_j}^2}\left(\vec{e}_2(\xi)\chi_{A_j\cup A_j^*} + i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j} - i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j^*}\right)$$

and

$$\widehat{U_{q_j-1}}(\xi) = \frac{1}{\lambda_{q_j}^2}\vec{e}_1(\xi)\chi_{B_j\cup B_j^*}.$$

Since U has no modes in the (q_j+1)-st shell, we have $\tilde{U}_{q_j} = U_{q_j-1} + U_{q_j}$.

Lemma 3.1. *One has $U \in B_{r,\infty}^{\frac{3}{r}-1}$, for all $1 < r \leq \infty$. Henceforth, $U \in H^s$ for any $s < \frac{1}{2}$.*

Proof. We give the estimate only for one block. Using boundedness of the Leray-Hopf projection, we have for $1 < r < \infty$,

$$\|\lambda_{q_j}^{-2}(\vec{e}_2(\cdot)\chi_{A_j})^\vee\|_r \lesssim \lambda_{q_j}^{-2}\|(\chi_{A_j})^\vee\|_r \leq \lambda_{q_j}^{-2}\|D_{c\lambda_{q_j}}\|_r^3,$$

where D_N denotes the Dini kernel. By a well-known estimate, we have $\|D_N\|_r \leq N^{1-\frac{1}{r}}$, which implies the lemma.

If $r = \infty$, we simply use the triangle inequality to obtain

$$\|U_{q_j}\|_\infty \lesssim \lambda_{q_j}.$$

The conclusion $U \in H^s$ is a consequence of the embedding $B_{2,\infty}^{\frac{1}{2}} \hookrightarrow H^s$ for any $s < \frac{1}{2}$. □

Let us now examine the trilinear term. We will use the following notation for convenience:

$$u \otimes v : \nabla w = \int_{\mathbb{T}^3} v_i \partial_i w_j u_j dx. \tag{14}$$

Using the antisymmetry we obtain

$$\begin{aligned} U\otimes U:\nabla U_{q_j} &= \sum_{k\geq j+1}\tilde U_{q_k}\otimes\tilde U_{q_k}:\nabla U_{q_j}+\tilde U_{q_j}\otimes\tilde U_{q_j}:\nabla U_{q_j}\\ &+U_{\leq q_{j-1}}\otimes\tilde U_{q_j}:\nabla U_{q_j}+\tilde U_{q_j}\otimes U_{\leq q_{j-1}}:\nabla U_{q_j}\\ &=\sum_{k\geq j+1}\tilde U_{q_k}\otimes\tilde U_{q_k}:\nabla U_{q_j}+U_{q_j-1}\otimes U_{q_j}:\nabla U_{q_j}\\ &-U_{q_j}\otimes U_{q_j}:\nabla U_{\leq q_{j-1}}=A+B-C. \end{aligned}$$

Using Bernstein's inequalities we estimate

$$|A|\lesssim\lambda_{q_j}\|U_{q_j}\|_\infty\sum_{k\geq j+1}\|\tilde U_{q_k}\|_2^2\lesssim\frac{\lambda_{q_j}^2}{\lambda_{q_{j+1}}}\leq\epsilon, \tag{15}$$

$$|C|\lesssim\|U_{q_j}\|_2^2\sum_{k\leq j-1}\lambda_{q_k}\|\tilde U_{q_k}\|_\infty\lesssim\frac{\lambda_{q_{j-1}}^2}{\lambda_{q_j}}\leq\epsilon. \tag{16}$$

On the other hand, a straightforward computation shows that $B\sim\lambda_{q_j}$. Combining this with estimates (15), (16) we obtain

$$U\otimes U:\nabla U_{q_j}\sim\lambda_{q_j}. \tag{17}$$

Proposition 3.2. *Let $u\in C_w([0,T);L^2)\cap L^2([0,T);H^1)$ be a weak solution solution to the NSE with initial condition $u(0)=U$. Then there is $\delta=\delta(u)>0$ such that*

$$\limsup_{t\to0+}\|u(t)-U\|_{B^{-1}_{\infty,\infty}}\geq\delta. \tag{18}$$

If in addition u is a Leray-Hopf solution satisfying the energy inequality (12)*, then δ can be chosen to be independent of u.*

Proof. Using u_{q_j} as a test function we can write

$$\partial_t(\tilde u_{q_j}\cdot u_{q_j})=-\nu\nabla\tilde u_{q_j}\cdot\nabla u_{q_j}+u\otimes u:\nabla u_{q_j}.$$

Denote $E(t)=\int_0^t\|\nabla u\|_2^2ds$. Using (17) and integrating in time we obtain

$$\|\tilde u_{q_j}(t)\|_2^2\geq\|U_{q_j}\|_2^2-\nu E(t)+c_1\lambda_{q_j}t \tag{19}$$

$$-c_2\int_0^t\left|u\otimes u:\nabla u_{q_j}-U\otimes U:\nabla U_{q_j}\right|ds, \tag{20}$$

for some positive constants c_1 and c_2. We now show that if the conclusion of the proposition fails, then for some small $t>0$ the integral term in (20) is less than $c_1\lambda_{q_j}t/2$ uniformly for all large j. This forces the inequalities $\|\tilde u_{q_j}(t)\|_2^2\gtrsim\lambda_{q_j}t$ to hold for all large j. Hence, u has infinite energy, which is a contradiction.

So, let us suppose that for every $\delta>0$ there exists $t_0=t_0(\delta)>0$ such that $\|u(t)-U\|_{B^{-1}_{\infty,\infty}}<\delta$ for all $0<t\leq t_0$. Denoting $w=u-U$ we write

$$\begin{aligned} u\otimes u:\nabla u_{q_j}-U\otimes U:\nabla U_{q_j}&=w\otimes U:\nabla U_{q_j}+u\otimes w:\nabla U_{q_j}\\ &+u\otimes u:\nabla w_{q_j}=A+B+C. \end{aligned}$$

We will now decompose each triplet into three terms according to the type of interaction (cf. Bony [2]) and estimate each of them separately:

$$A = \sum_{\substack{p',p''\geq q_j\\ |p'-p''|\leq 2}} w_{p'}\otimes U_{p''} : \nabla U_{q_j} + w_{\leq q_j}\otimes \tilde{U}_{q_j} : \nabla U_{q_j} + \tilde{w}_{q_j}\otimes U_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = A_1 + A_2 + A_3.$$

Using Lemma 3.1 along with the Hölder and Bernstein inequalities we obtain

$$\begin{aligned}
|A_1| &\leq \|\nabla U_{q_j}\|_4 \sum \|w_{p'}\|_\infty \|U_{p''}\|_{4/3} \lesssim \lambda_{q_j}^{5/4} \sum \|w_{p'}\|_\infty \lambda_{p''}^{-5/4} \lesssim \delta\lambda_{q_j},\\
|A_2| &= |U_{q_j}\otimes \tilde{U}_{q_j} : \nabla w_{\leq q_j}| \leq \|\tilde{U}_{q_j}\|_2^2 \|\nabla w_{\leq q_j}\|_\infty\\
&\lesssim \lambda_{q_j}^{-1} \sum_{p\leq q_j} \lambda_p^2 \lambda_p^{-1} \|w_p\|_\infty < \delta\lambda_{q_j}.
\end{aligned}$$

As to A_3, we choose an $r>1$ close enough to 1 and r^*, $\frac{1}{r}+\frac{1}{r^*}=1$, and estimate using Lemma 3.1,

$$\begin{aligned}
|A_3| &\leq \lambda_{q_j} \|U_{\leq q_j}\|_{r^*} \|U_{q_j}\|_r \|\tilde{w}_{q_j}\|_\infty \lesssim \lambda_{q_j} \lambda_{q_j}^{1-\frac{3}{r^*}} \lambda_{q_j}^{1-\frac{3}{r}} \|\tilde{w}_{q_j}\|_\infty\\
&\leq \|\tilde{w}_{q_j}\|_\infty \leq \delta\lambda_{q_j}.
\end{aligned}$$

So, we have proved the following estimate:

$$|A| \lesssim \delta\lambda_{q_j}. \tag{21}$$

As to B we decompose analogously,

$$B = \sum_{\substack{p',p''\geq q_j\\ |p'-p''|\leq 2}} u_{p'}\otimes w_{p''} : \nabla U_{q_j} + u_{\leq q_j}\otimes \tilde{w}_{q_j} : \nabla U_{q_j} + \tilde{u}_{q_j}\otimes w_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = B_1 + B_2 + B_3.$$

Again, using Lemma 3.1 and the Bernstein and Hölder inequalities we obtain

$$\begin{aligned}
|B_1| &\lesssim \lambda_{q_j} \|U_{q_j}\|_2 \sum \|u_{p'}\|_2 \|w_{p''}\|_\infty \leq \delta\lambda_{q_j}^{1/2} \|\nabla u\|_2,\\
|B_2| &= \left|U_{q_j}\otimes \tilde{w}_{q_j} : \nabla u_{\leq q_j}\right| \leq \|U_{q_j}\|_2 \|\tilde{w}_{q_j}\|_\infty \|\nabla u_{\leq q_j}\|_2,\\
&\leq \lambda_{q_j}^{-1/2} \|\tilde{w}_{q_j}\|_\infty \|\nabla u\|_2 \leq \delta\lambda_{q_j}^{1/2} \|\nabla u\|_2,\\
|B_3| &\leq \|\tilde{u}_{q_j}\|_2 \|w_{\leq q_j}\|_\infty \|\nabla U_{q_j}\|_2 \lesssim \lambda_{q_j}^{1/2} \|\tilde{u}_{q_j}\|_2 \sum_{p\leq q_j} \lambda_p^{-1} \|w_p\|_\infty \lambda_p\\
&\lesssim \delta\lambda_{q_j}^{1/2} \|\nabla u\|_2.
\end{aligned}$$

We thus obtain

$$|B| \lesssim \delta\lambda_{q_j}^{1/2} \|\nabla u\|_2. \tag{22}$$

Continuing in a similar fashion we write

$$C = \sum_{\substack{p',p''\geq q_j\\ |p'-p''|\leq 2}} u_{p'}\otimes u_{p''} : \nabla w_{q_j} + u_{\leq q_j}\otimes \tilde{u}_{q_j} : \nabla w_{q_j} + \tilde{u}_{q_j}\otimes u_{\leq q_j} : \nabla w_{q_j} - \text{repeated} = C_1 + C_2 + C_3,$$

$$|C_1| \leq \|\nabla w_{q_j}\|_\infty \sum_{p \geq q_j - 2} \|\tilde{u}_p\|_2^2 \lesssim \lambda_{q_j} \|w_{q_j}\|_\infty \lambda_{q_j}^{-2} \|\nabla u\|_2^2 \leq \delta \|\nabla u\|_2^2,$$
$$|C_2| \leq \|\nabla u\|_2 \|\tilde{u}_{q_j}\|_2 \|w_{q_j}\|_\infty \lesssim \lambda_{q_j}^{-1} \|\nabla u\|_2^2 \|w_{q_j}\|_\infty \leq \delta \|\nabla u\|_2^2.$$

Now using a uniform bound on the energy $\|u(t)\|_2^2 \lesssim 1$ for almost all $t \geq 0$, we estimate

$$|C_3| \lesssim \lambda_{q_j} \|w_{q_j}\|_\infty \|\tilde{u}_{q_j}\|_2 \leq \delta \lambda_{q_j} \|\nabla \tilde{u}_{q_j}\|_2.$$

Thus,

$$|C| \lesssim \delta \|\nabla u\|_2^2 + \delta \lambda_{q_j} \|\nabla \tilde{u}_{q_j}\|_2. \tag{23}$$

Now combining estimates (21), (22), (23) along with the boundedness of $E(t_0)$ we obtain

$$\int_0^{t_0} \left| u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j} \right| ds \lesssim \delta \lambda_{q_j} t_0 + \delta \lambda_{q_j}^{1/2} t_0^{1/2} + \delta + \delta \lambda_{q_j} \int_0^{t_0} \|\nabla \tilde{u}_{q_j}(s)\|_2 ds. \tag{24}$$

Using that

$$\int_0^{t_0} \|\nabla \tilde{u}_{q_j}(s)\|_2 ds \to 0$$

as $j \to \infty$ we can choose δ small enough and j_0 large enough so that the right-hand side of (24) is less than $\frac{c_1}{2c_2} \lambda_{q_j} t_0$, for all $j \geq j_0$. Going back to (19) this implies that

$$\|\tilde{u}_{q_j}(t_0)\|_2^2 \geq \|U_{q_j}\|_2^2 - \nu E(t_0) + c_1 \lambda_{q_j} t_0 / 2,$$

for all $j > j_0$, which shows that $u(t_0)$ has infinite energy, a contradiction.

The last statement of the proposition follows from the fact that we have the bounds on $\|u(t)\|_2 \leq \|U\|_2$ and $E(t_0) \leq (2\nu)^{-1} \|U\|_2^2$, which remove dependence of the constants on u. □

Remark 3.3. Using our vector field U one can obtain the approximate equality (up to a constant independent of q)

$$\lambda_q^5 U \otimes U : \nabla U_q \sim \|U\|_{B_{2,1}^{5/2}} \lambda_q^{5/2} \|U_q\|_2.$$

This shows that the conventional energy argument alone is not sufficient to prove local well-posedness for the 3D Euler equations in $H^{5/2}$. It is tempting to conjecture therefore that there is an example of an initial condition u_0 in any Besov space $B_{2,l}^{5/2}$, for $l > 1$, for which the analogue of Proposition 2.1 is true. However, this remains an open question.

Remark 3.4. To make a closer parallel with the result of Bourgain and Pavlović, [3], we can consider as our initial condition the field ϵU for $\epsilon > 0$ as small as we like. The conclusion of Proposition 3.2 will remain the same, although δ in (18) will naturally become dependent on ϵ. This also shows that a possible small initial data result in $B_{\infty,\infty}^{-1}$ will not include the condition of continuity in time, as for example is the case with $H^{1/2}$ or L^3.

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