BOCKSTEIN THEOREM FOR NILPOTENT GROUPS

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Abstract. We extend the definition of Bockstein basis $\sigma(G)$ to nilpotent groups $G$. A metrizable space $X$ is called a Bockstein space if $\dim_G(X) = \sup\{\dim_H(X) \mid H \in \sigma(G)\}$ for all Abelian groups $G$. The Bockstein First Theorem says that all compact spaces are Bockstein spaces.

Here are the main results of the paper:

Theorem 0.1. Let $X$ be a Bockstein space. If $G$ is nilpotent, then $\dim_G(X) \leq 1$ if and only if $\sup\{\dim_H(X) \mid H \in \sigma(G)\} \leq 1$.

Theorem 0.2. $X$ is a Bockstein space if and only if $\dim_{\mathbb{Z}(l)}(X) = \dim_{\hat{\mathbb{Z}(l)}}(X)$ for all subsets $l$ of prime numbers.

1. Introduction

We use the Kuratowski notation $X \tau M$ in the case when every map from a closed subset of $X$ to $M$ can be extended over all $X$.

Recall that the cohomological dimension of a space $X$ with respect to an Abelian group $G$ is less than or equal to $n$, denoted by $\dim_G X \leq n$, if $H^{n+1}(X, A; G) = 0$ for all closed $A \subset X$.

Of basic importance in cohomological dimension theory is the Bockstein basis $\sigma(G)$ of an Abelian group $G$ (see Definition 3.1) and the following result of Bockstein (see [12] or [1]):

Theorem 1.1 (Bockstein First Theorem). If $X$ is a compact space, then

$$\dim_G(X) = \sup\{\dim_H(X) \mid H \in \sigma(G)\}.$$ 

The aim of this paper is to generalize Theorem 1.1 to nilpotent groups. There are two issues to resolve first:

1. Define cohomological dimension with respect to non-Abelian groups.
2. Define the Bockstein basis of nilpotent groups.

The definition of $\dim_G(X)$ for $G$ non-Abelian was first introduced by A. Dranishnikov and D. Repovš [5] as follows: By [2] Theorem 1.1, $\dim_G(X) \leq n$ is equivalent (for Abelian groups $G$) to $X \tau K(G, n)$, where $K(G, n)$ is an Eilenberg-MacLane space.

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space. One ought to use the same equivalence in the definition of \( \dim_G(X) \) for non-Abelian groups. For nilpotent groups \( G \) this definition was used for the characterization of nilpotent CW complexes as absolute extensors of metric compacta (see [3]). There is no Eilenberg-MacLane space \( K(G, n) \), \( 1 < n < \infty \), for non-Abelian groups \( G \), so \( \dim_G X \in \{0, 1, \infty \} \). Since \( \dim_G X = 0 \) means \( \dim(X) = 0 \) (see Lemma 1.2), the only interesting question is if \( X \tau K(G, 1) \) holds.

Our definition of the Bockstein basis for nilpotent groups can be found in Definition 3.2.

The remainder of this section is devoted to elementary properties of cohomological dimension over non-Abelian groups.

**Lemma 1.2.** If \( X \) is a metrizable space and \( \dim_G(X) = 0 \) for some group \( G \neq 1 \), then \( \dim_H(X) = 0 \) for any group \( H \).

**Proof.** We will show that for a nontrivial group \( G \) we have \( \dim_G(X) = 0 \) if and only if \( \dim_{\mathbb{Z}_2}(X) = 0 \) and \( \dim_{\mathbb{Z}_2}(X) = 0 \) means \( \dim(X) = 0 \).

Suppose \( \dim_G(X) = 0 \) for some \( G \neq 1 \). Let \( i: \mathbb{Z}_2 \to G \) be an injection and let \( r: G \to \mathbb{Z}_2 \) be a map such that \( r \circ i = id_{\mathbb{Z}_2} \). Let \( A \) be a closed subset of \( X \) and \( f: A \to K(\mathbb{Z}_2, 0) = \mathbb{Z}_2 \) a map. Then there exists an extension \( f': X \to G \) of \( i \circ f: A \to G \). Then \( F = r \circ f': X \to \mathbb{Z}_2 \) is an extension of \( f \).

Suppose that \( \dim_{\mathbb{Z}_2}(X) = 0 \). Let \( A \subseteq X \) be a closed subset and let \( f: A \to G \) be a continuous map. For every \( g \in G \) we define \( A_g = f^{-1}(g) \) and \( B_g = \{ x \in X \mid d(x, A \setminus A_g) \leq d(x, A_g) \} \). Because \( A \setminus A_g \) is a closed subset of \( X \), \( B_g \) is a closed subset of \( X \). Because \( B_g \cap A_g = \emptyset \) we can define a continuous map \( f_g: A_g \cup B_g \to \mathbb{Z}_2 \) by \( f_g(A_g) = 1 \) and \( f_g(B_g) = 0 \). Let \( F_g: X \to \mathbb{Z}_2 \) be an extension of \( f_g \). For every \( g \in G \) we define \( X_g = F_g^{-1}(1) \). Sets \( X_g \) are open and closed in \( X \) and pairwise disjoint. So we can define a continuous map \( F: X \to G \) as

\[
F(x) = \begin{cases} 
  g & ; x \in X_g, \\
  e & ; x \notin \bigcup_{g \in G} X_g.
\end{cases}
\]

The map \( F \) is an extension of \( f \). \( \square \)

**Lemma 1.3.** Let \( X \) be a metrizable space. If \( 1 \to J \to G \to I \to 1 \) is an exact sequence of groups and \( \dim_J X \leq 1 \), then \( \dim_I X \leq 1 \) if and only if \( \dim_G(X) \leq 1 \).

**Proof.** In view of Lemma 1.2 the only interesting case is that of \( \dim_J X = 1 \). Use the fibration \( K(J, 1) \to K(G, 1) \to K(I, 1) \) and the fact that \( X \tau K(J, 1) \) to conclude \( X \tau K(I, 1) \) if and only if \( X \tau K(G, 1) \).

If \( G \) is a group, then \( \text{Ab}(G) \) is its abelianization.

**Lemma 1.4.** If \( X \) is a metrizable space, then \( \dim_{\text{Ab}(G)}(X) \leq \dim_G(X) \) for any group \( G \).

**Proof.** In view of Lemma 1.2 the only interesting case is that of \( G \) non-Abelian and \( \dim_G X = 1 \). Since \( X \tau K(G, 1) \) one gets \( X \tau K(H_1(K(G, 1)), 1) \) by Theorem 3.4 of [5]. As \( H_1(K(G, 1)) = \text{Ab}(G) \), we are done. \( \square \)

### 2. Nilpotent Groups

If \( A, B \subset G \) are subgroups, then the commutator subgroup \( [A, B] \) is a group generated by all commutators \( [a, b] \), \( a \in A \) and \( b \in B \). The lower central series \( \{ \Gamma_n(G) \} \) for a group \( G \) is defined as follows: \( \Gamma_1(G) = G \), and \( \Gamma_{n+1}(G) = [\Gamma_n(G), G] \).
If a group $G$ is nilpotent, then there exists an integer $c$ such that $\Gamma_c(G) \neq \{1\}$ but $\Gamma_{c+1}(G) = \{1\}$. The number $c$ (denoted by $h(G)$) is called the nilpotency class of the nilpotent group $G$ or its Hirsch length. Abelian groups are nilpotent of Hirsch length 1. By \cite[Theorem 3.1]{14}, for every $n$ there exists an epimorphism

$$\otimes^n \text{Ab}(G) \to \Gamma_n(G)/\Gamma_{n+1}(G).$$

In particular, there is an epimorphism $\otimes^c \text{Ab}(G) \to \Gamma_c(G)$. It follows from the definition that $\Gamma_c(G)$ is in the center of $G$. Therefore $1 \to \Gamma_c(G) \to G \to G/\Gamma_c(G) \to 1$ is a central extension. A short calculation shows that the epimorphism $G \to G/\Gamma_c(G)$ induces the trivial homomorphism $\Gamma_c(G) \to \Gamma_c(G/\Gamma_c(G)) = 1$. Therefore $G/\Gamma_c(G)$ is nilpotent of nilpotency class strictly less than $c$. This motivates the following definition.

**Definition 2.1.** A central extension $K \to G \to I$ of groups where $G$ is nilpotent (or equivalently $I$ is nilpotent), for which there exists an epimorphism $\otimes^n \text{Ab}(G) \to K$ for some $n$, is called a nilpotent central extension.

Thus, for every (non-Abelian) nilpotent group $G$, there exists a nilpotent central extension $K \to G \to I$ such that the Hirsch length of $I$ is less than the Hirsch length of $G$.

**Lemma 2.2.** Let $1 \to K \to G \xrightarrow{\pi} I \to 1$ be a central extension of nilpotent groups.

(a) If $K$ and $I$ are $p$-divisible, then $G$ is $p$-divisible.

(b) If the extension is a nilpotent central extension and $G$ is $p$-divisible, then $K$ and $I$ are $p$-divisible.

**Proof.** Suppose $K$ and $I$ are $p$-divisible. Let $g \in G$. Then $\pi(g) = i^p$ for some $i \in I$. Let $\bar{g} \in G$ be such that $\pi(\bar{g}) = i$. Then $\pi(\bar{g}^p g^{-1}) = 1$, so $\bar{g}^p g^{-1} = k^p$ for some $k \in K$. Because $k \in K \subset C(G)$, $g = (\bar{g} k^{-1})^p$, so $G$ is $p$-divisible.

If $G$ is $p$-divisible, then any epimorphic image of $G$ is $p$-divisible. Thus both $I$ and $\text{Ab}(G)$ are $p$-divisible. As there is an epimorphism $\otimes^n \text{Ab}(G) \to K$, $K$ is also $p$-divisible. \hfill $\square$

**Lemma 2.3.** Suppose $\mathcal{P}_i$, $i = 1, 2$, are two classes of nilpotent groups such that for any nilpotent central extension $1 \to K \to G \to I \to 1$ where $h(I) < h(G)$ the following conditions hold:

(a) $K$ and $I$ belong to $\mathcal{P}_1$ if $G \in \mathcal{P}_1$,

(b) $G \in \mathcal{P}_2$ if $K, I \in \mathcal{P}_2$.

If $A \in \mathcal{P}_1 \implies A \in \mathcal{P}_2$ for all Abelian groups $A$, then $\mathcal{P}_1 \subset \mathcal{P}_2$.

**Proof.** We prove the implication $G \in \mathcal{P}_1 \implies G \in \mathcal{P}_2$ by induction on the Hirsch length of $G$. By assumption the implication holds for an Abelian group $G$. Therefore suppose $1 \to K \to G \to I \to 1$ is a nilpotent central extension of groups and $I$ is of lower Hirsch length than the Hirsch length of $G$. By (a) we have $K, I \in \mathcal{P}_1$. By the inductive hypothesis, $K, I \in \mathcal{P}_2$. The condition (b) yields $G \in \mathcal{P}_2$. \hfill $\square$

**Corollary 2.4.** Let $A$ be an Abelian group and $n \in \{1, 2\}$. Consider the following statements:

1. $\tilde{H}_n(G; A) = 0$,
2. $\tilde{H}_1(G; A) = 0$ for all $i \leq n$.

If (1) is equivalent to (2) for all Abelian groups $G$, then the two statements are equivalent for all nilpotent groups $G$. 

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Proof. Let $P_1 = P_1^r$ (respectively, $P_2 = P_2^r$) be the class of all nilpotent groups $G$ such that $H_i(G; A) = 0$ for all $i \leq n$ (respectively, $H_i(G; A) = 0$ for all $i$) and $h(G) \leq r$. Our goal is to prove, by induction on $r$, that $P_1^r = P_2^r$. It is clearly so for $r = 1$. Assume $P_1^m = P_2^m$ for all $m < r$.

Suppose $1 \rightarrow K \rightarrow G \rightarrow I \rightarrow 1$ is a nilpotent central extension of groups such that $h(I) < h(G) = r$. If $G \in P_j^r$, then $H_i(K; A) = 0$, which implies $H_i(G; A) \rightarrow H_i(I; A)$ is an epimorphism for $i \leq 2$, so $H_i(I; A) = 0$ for $1 \leq i \leq n$. By the inductive assumption, $H_i(I; A) = 0$ for all $i \geq 1$. If $H_i(K; A)$ is the first nontrivial reduced homology group of $K$, then the Leray-Serre spectral sequence implies $H_{i+1}(I; A) \neq 0$, a contradiction. Thus both $K$ and $I$ belong to $P_j^r$. Conversely, if $K, I \in P_j^{r-1}$, then (by the inductive assumption) they have trivial homology with coefficients in $A$ resulting in $G$ having trivial homology with coefficients in $A$ and $G \in P_j^r$. Applying Lemma 3.3 one gets $P_1^r = P_2^r$. □

3. BOCKSTEIN BASIS

If $G$ is a group, then $\text{Tor}(G)$ is the subgroup generated by torsion elements of $G$, $\text{Tor}_p(G)$ is the subgroup generated by all elements of $G$ whose order is a power of $p$, $F_p(G) = G/\text{Tor}_p(G)$, and $F(G) = G/\text{Tor}(G)$.

The Bockstein groups are: rationals $\mathbb{Q}$, cyclic groups $\mathbb{Z}/p$ of $p$ elements, $p$-adic circles $\mathbb{Z}/p^{\infty}$, and $p$-localizations of integers $\mathbb{Z}_{(p)} = \{ \frac{n}{m} \in \mathbb{Q} \mid n$ is not divisible by $p\}$, where $p$ is a prime number. Here is a classical definition of the maximal Bockstein basis of an Abelian group $G$:

**Definition 3.1.** $\sigma(G)$ is a subset of all Bockstein groups satisfying the following conditions:

1. $\mathbb{Q} \in \sigma(G)$ if and only if $F(G) \neq 1$.
2. $\mathbb{Z}/p^{\infty} \in \sigma(G)$ if and only if $\text{Tor}_p(G) \neq 1$ or $F(G)$ is not divisible by $p$.
3. $\mathbb{Z}/p \in \sigma(G)$ if and only if $\text{Tor}_p(G)$ is not divisible by $p$ or $F(G)$ is not divisible by $p$.
4. $\mathbb{Z}_{(p)} \in \sigma(G)$ if and only if $F(G)$ is not divisible by $p$.

Notice that our Definition 3.1 differs from that in [1] in the sense that ours is maximal (if $\mathbb{Z}/p \in \sigma(G)$, then $\mathbb{Z}/p^{\infty} \in \sigma(G)$ and if $\mathbb{Z}_{(p)} \in \sigma(G)$, then $\mathbb{Z}/p \in \sigma(G)$) and the one in [1] is minimal (if $\mathbb{Z}/p \in \sigma(G)$ or $\mathbb{Z}_{(p)} \in \sigma(G)$, then $\mathbb{Z}/p^{\infty} \notin \sigma(G)$). From the point of view of the First Bockstein Theorem, both definitions are equivalent.

Here is a definition for nilpotent groups which is more convenient in this paper as it allows using localization of short exact sequences of nilpotent groups. Recall that a (multiplicative) group is $\tilde{p}$-local iff the map $x \mapsto x^p$ is a bijection. We call a nilpotent group $p$-local if it is $\tilde{q}$-local for all primes $q \neq p$.

**Definition 3.2.** Let $G$ be a nilpotent group. Then the Bockstein basis $\sigma(G)$ is defined as follows:

1. $\mathbb{Q} \notin \sigma(G)$ if and only if $G = \text{Tor}(G)$.
2. $\mathbb{Z}/p^{\infty} \notin \sigma(G)$ if and only if $G$ is $p$-local.
3. $\mathbb{Z}/p \notin \sigma(G)$ if and only if $G$ is divisible by $p$.
4. $\mathbb{Z}_{(p)} \notin \sigma(G)$ if and only if $F(G)$ is $\tilde{p}$-local.

**Remark 3.3.** Note that according to the above definition we have $\mathbb{Z}_{(p)} \in \sigma(G) \Rightarrow \mathbb{Z}/p \in \sigma(G) \Rightarrow \mathbb{Z}/p^{\infty} \in \sigma(G)$. 

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Corollary 3.4. For a nilpotent group $G$ the following statements are equivalent:

1. $Q \notin \sigma(G)$,
2. $H_s(G; Q) = 0$, and
3. $H_1(G; Q) = 0$.

Proof. For Abelian groups, (2) and (3) are equivalent [1] Theorem V.6.4(ii)]. With Corollary 2.4 we extend this equivalence to all nilpotent groups.

Let $\mathcal{P}_1$ be the class of torsion nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G; Q) = 0$ (i.e., Ab(G) is torsion). Use Lemma 2.3 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$ and thus (1) and (3) are equivalent.

Corollary 3.5. For a nilpotent group $G$, $\mathbb{Z}/p \notin \sigma(G)$ if and only if $H_1(G; \mathbb{Z}/p) = 0$.

Proof. Let $\mathcal{P}_1$ be the class of $p$-divisible nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G; \mathbb{Z}/p) = 0$. Use Lemmas 2.3 and 2.2 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$.

Corollary 3.6. For a nilpotent group $G$ the following statements are equivalent:

1. $\mathbb{Z}/p^{\infty} \notin \sigma(G)$,
2. $H_s(G; \mathbb{Z}/p^{\infty}) = 0$, and
3. $H_1(G; \mathbb{Z}/p^{\infty}) = H_2(G; \mathbb{Z}/p^{\infty}) = 0$.

Proof. If $\mathbb{Z}/p^{\infty} \notin \sigma(G)$, then $G$ is $p$-local, so all its integral homology groups are $p$-local and $H_i(G; \mathbb{Z}/p^{\infty}) = 0$ for all $i \geq 1$, which proves the implication (1) $\implies$ (2).

Notice (2) $\iff$ (3) by Corollary 2.4. Indeed, if $A$ is Abelian and $H_1(A; \mathbb{Z}/p^{\infty}) = H_2(A; \mathbb{Z}/p^{\infty}) = 0$, then $A$ must be $p$-local.

Let $\mathcal{P}_1$ be the class of $p$-local nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G; \mathbb{Z}/p^{\infty}) = H_2(G; \mathbb{Z}/p^{\infty}) = 0$. Use Lemma 2.3 to conclude that $\mathcal{P}_1 = \mathcal{P}_2$.

Corollary 3.7. For a nilpotent group $G$ the following statements are equivalent:

1. $\mathbb{Z}/p \notin \sigma(G)$,
2. $H_s(F(G); \mathbb{Z}/p^{\infty}) = 0$, and
3. $H_1(F(G); \mathbb{Z}/p^{\infty}) = H_2(F(G); \mathbb{Z}/p^{\infty}) = 0$.

It is obvious that $\mathbb{Z}/p^{\infty} \notin \sigma(G)$ if and only if $G \to G_{(p)}$ is an isomorphism. The following lemma characterizes $\mathbb{Z}/p \notin \sigma(G)$ via localizations.

Lemma 3.8. For a nilpotent group $G$ the following statements are equivalent:

1. $\mathbb{Z}/p \notin \sigma(G)$,
2. $G \to G_{(p)}$ is an epimorphism.

Proof. If $G \to G_{(p)}$ is an epimorphism, then its kernel has trivial $p$-localization (by exactness of the localization functor) and must be a torsion group. Therefore $F(G) = F(G_{(p)})$ is $p$-local. If $F(G)$ is $p$-local, then apply exactness of the localization functor to the short exact sequence $1 \to \text{Tor}(G) \to G \to F(G) \to 1$ and derive that $G \to G_{(p)}$ is an epimorphism.

Definition 3.9. The 

Definition 3.9. The torsion-divisible Bockstein basis $\sigma_{TD}(G)$ of $G$ consists of all $\mathbb{Z}/p^{\infty}$ belonging to $\sigma(G)$. We set $\sigma_{NTD}(G) = \sigma(G) \setminus \sigma_{TD}(G)$.

Lemma 3.10. If $G \to I$ is an epimorphism of nilpotent groups. Then $\sigma_{NTD}(I) \subset \sigma_{NTD}(G)$.
Proof. Suppose $Q \not\in \sigma(G)$. Then $G$ is a torsion group. So $I$ is a torsion group; hence $Q \not\in \sigma(I)$.

Let $Z/p \not\in \sigma(G)$. Then $G$ is $p$-divisible and then also $I$ is $p$-divisible.

Let $Z_{(p)} \not\in \sigma(G)$. Then $G \to G_{(p)}$ is an epimorphism. Because $\hat{p}$-localization is an exact functor, the map $G_{(p)} \to I_{(p)}$ is an epimorphism and hence $I \to I_{(p)}$ is an epimorphism.

Lemma 3.11. Let $1 \to K \to G \to I \to 1$ be a central extension of nilpotent groups. Then $\sigma(G) \subset \sigma(K) \cup \sigma(I)$.

Proof. Let $Q \not\in \sigma(K) \cup \sigma(I)$. Then $K$ and $I$ are torsion groups. Hence $G$ is torsion, so $Q \not\in \sigma(G)$.

Let $Z/p \not\in \sigma(K) \cup \sigma(I)$. Then $K$ and $I$ are $p$-divisible. By Lemma 2.2, also $G$ is $p$-divisible.

Let $Z/p^\infty \not\in \sigma(K) \cup \sigma(I)$. Then $K \to K_{(p)}$ and $I \to I_{(p)}$ are isomorphisms. Using the Five Lemma and the fact that $\hat{p}$-localization is an exact functor, we conclude that also $G \to G_{(p)}$ is an isomorphism.

Let $Z_{(p)} \not\in \sigma(K) \cup \sigma(I)$. Then $K \to K_{(p)}$ and $I \to I_{(p)}$ are epimorphisms. By the Three Lemma [10], also $G \to G_{(p)}$ is an epimorphism.

Lemma 3.12. Let $1 \to K \to G \to I \to 1$ be a nilpotent central extension of groups. If $Z_{(p)} \not\in \sigma(\text{Ab}(G))$ for some prime $p$, then $Z_{(p)} \not\in \sigma(K)$ and $Z_{(p)} \not\in \sigma(\text{Ab}(I))$.

Proof. Assume $Z_{(p)} \not\in \sigma(\text{Ab}(G))$. That means the map $F(\text{Ab}(G)) \to F(\text{Ab}(G))_{(p)}$ is an isomorphism. Because $\text{Ab}(G) \to \text{Ab}(I)$ is an epimorphism and $F$ is a right exact functor, the map $F(\text{Ab}(G)) \to F(\text{Ab}(I))$ is an epimorphism. Hence the map $F(\text{Ab}(I)) \to F(\text{Ab}(I))_{(p)}$ is an epimorphism. Its kernel is a $p$-torsion group, so the kernel is trivial and the map $F(\text{Ab}(I)) \to F(\text{Ab}(I))_{(p)}$ is an isomorphism. That means $Z_{(p)} \not\in \sigma(\text{Ab}(I))$.

There exists an epimorphism $\otimes^n \text{Ab}(G) \to K$ for some integer $n$. Because $F(\text{Ab}(\otimes^n G)) \to F(\text{Ab}(\otimes^n G))_{(p)}$ is an isomorphism in the same way as in the previous paragraph, we can prove that $F(K) \to F(K)_{(p)}$ is an isomorphism. Hence $Z_{(p)} \not\in \sigma(K)$.

Lemma 3.13. Let $G$ be a nilpotent group. Then $\sigma_{\text{NTD}}(G) = \sigma_{\text{NTD}}(\text{Ab}(G))$ and $\sigma_{\text{NTD}}(\text{Ab}(G)) \subset \sigma(G)$.

Proof. The inclusion $\sigma_{\text{NTD}}(\text{Ab}(G)) \subset \sigma_{\text{NTD}}(G)$ follows from Lemma 3.10.

Let us prove $\sigma_{\text{NTD}}(G) \subset \sigma_{\text{NTD}}(\text{Ab}(G))$. Suppose $Q \not\in \sigma_{\text{NTD}}(\text{Ab}(G))$. Because $G$ is a torsion group if and only if $\text{Ab}(G)$ is a torsion group, $Q \not\in \sigma_{\text{NTD}}(G)$.

Suppose $Z/p \not\in \sigma_{\text{NTD}}(\text{Ab}(G))$. Because $G$ is $p$-divisible if and only if $\text{Ab}(G)$ is $p$-divisible, $Z/p \not\in \sigma_{\text{NTD}}(G)$.

Consider the class $\mathcal{P}$ of all nilpotent groups such that $Z_{(p)} \not\in \sigma_{\text{NTD}}(\text{Ab}(G))$ implies $Z_{(p)} \not\in \sigma_{\text{NTD}}(G)$. $\mathcal{P}$ clearly contains all Abelian groups. To show $\mathcal{P}$ equals the class $\mathcal{N}$ of all nilpotent groups it suffices to show (see Lemma 2.3) that for any nilpotent central extension $A \to G \to G'$ such that the Hirsch length of $G'$ is less than $h(G)$, $A, G' \in \mathcal{P}$ implies $G \in \mathcal{P}$. Assume $Z_{(p)} \not\in \sigma_{\text{NTD}}(\text{Ab}(G))$. By Lemma 3.12 we conclude $Z_{(p)} \not\in \sigma_{\text{NTD}}(G')$ as $G' \in \mathcal{P}$ and $Z_{(p)} \not\in \sigma_{\text{NTD}}(A)$. By Lemma 3.11 $Z_{(p)} \not\in \sigma_{\text{NTD}}(G)$.

Let us now prove that $\sigma(\text{Ab}(G)) \subset \sigma(G)$. By Lemma 3.10 $\sigma_{\text{NTD}}(\text{Ab}(G)) \subset \sigma_{\text{NTD}}(G)$. Suppose $Z/p^\infty \not\in \sigma(G)$. Then $G$ is uniquely $p$-divisible, so $H_*(G; Z)$ is
uniquely $p$-divisible. In particular $\text{Ab}(G) = H_1(G; \mathbb{Z})$ is uniquely $p$-divisible; hence $
abla/\mathbb{Z} \not\in \sigma(\text{Ab}(G))$. \hfill \qed

**Theorem 3.14.** If $1 \rightarrow K \rightarrow G \rightarrow I \rightarrow 1$ is a nilpotent central extension, then $\sigma(G) = \sigma(K) \cup \sigma(I)$.

*Proof.* By Lemma 3.11 $\sigma(G) \subset \sigma(K) \cup \sigma(I)$.

Let us prove that $\sigma(K) \cup \sigma(I) \subset \sigma(G)$. Suppose $Q \not\in \sigma(G)$. Then $G$ is a torsion group. Therefore $K$ and $I$ are also torsion groups, so $Q \not\in \sigma(K) \cup \sigma(I)$.

Suppose $Z/p \not\in \sigma(G)$. Then $G$ is $p$-divisible. By Lemma 2.2, $K$ and $I$ are $p$-divisible, so $Z/p \not\in \sigma(K) \cup \sigma(I)$.

Suppose $Z/p^\infty \not\in \sigma(G)$. Then $G \rightarrow G(p)$ is an isomorphism. Because $p$-localization is an exact functor, the map $K \rightarrow K(p)$ is a monomorphism and the map $\text{Ab}(G) \rightarrow (\text{Ab}(G))_{(p)}$ is an epimorphism. Because there exists an epimorphism $\otimes^n\text{Ab} \rightarrow K$, the map $K \rightarrow K(p)$ is also an epimorphism; hence it is an isomorphism. By the Five Lemma, also the map $I \rightarrow I(p)$ is an isomorphism, so $Z/p^\infty \not\in \sigma(K) \cup \sigma(I)$.

If $Z(p) \not\in \sigma(G)$, then $Z(p) \not\in \sigma(K)$ and $Z(p) \not\in \sigma(I)$ by Lemmas 3.12 and 3.13 \hfill \qed

4. BOCKSTEIN SPACES

**Definition 4.1.** A metrizable space $X$ is called a Bockstein space if $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ for all Abelian groups $G$.

**Remark 4.2.** In the above definition observe that $\dim_G X$ is an element of $\mathbb{N} \cup \{0, \infty\}$ and not only in $\{0, 1, \infty\}$ as in the case of non-Abelian groups $G$.

Dranishnikov-Repovš-Shchepin [4] showed the existence of a separable metric space $X$ of dimension 2 such that $\dim_{\mathbb{Z}(p)} X = 1$ for all primes $p$. Thus, $X$ is not a Bockstein space as $\dim_{\mathbb{Z}} X = 2 > 1 = \sup\{\dim_H X \mid H \in \sigma(\mathbb{Z})\}$.

**Problem 4.3.** Is every metric ANR a Bockstein space?

**Proposition 4.4.** Suppose $X = \bigcup_{n=1}^{\infty} X_n$ is metrizable and each $X_n$ is closed in $X$. If all $X_n$ are Bockstein spaces, then so is $X$.

*Proof.* Suppose $G$ is an Abelian group and $H \in \sigma(G)$. If $\dim_G(X) \leq m$, then $\dim_G(X_n) \leq m$ for all $n$ and $\dim_H(X_n) \leq m$ for all $n$ resulting in $\dim_H(X) \leq m$.

If $\dim_H(X) < m$ for all $H \in \sigma(G)$, then $\dim_H(X_n) < m$ for all $n$ and $\dim_G(X_n) < m$ for all $n$ resulting in $\dim_G(X) < m$. \hfill \qed

For a subset $l \subset \mathbb{P}$ of prime integers let $\mathbb{Z}_l = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is not divisible by any } p \in l\}$ and let $\mathbb{Z}_l$ be the group of $l$-adic integers.

**Theorem 4.5.** A metrizable space $X$ is a Bockstein space if and only if $\dim_{\mathbb{Z}(l)} X = \dim_{\mathbb{Z}(l)} X$ for all subsets $l \subset \mathbb{P}$ of the set of prime numbers.

*Proof.* Since $\sigma(\mathbb{Z}(l)) = \sigma(\mathbb{Z}(l))$ for all $l \subset \mathbb{P}$, $\dim_{\mathbb{Z}(l)} X = \dim_{\mathbb{Z}(l)} X$ holds for any Bockstein space $X$.
Assume \( \dim_{\mathbb{Z}} X = \dim_{\mathbb{Z}} l \) for all subsets \( l \subset \mathbb{P} \) of the set of prime numbers. Suppose \( G \) is a torsion-free Abelian group \( G \). If \( \mathbb{Z}_{(p)} \in \sigma(G) \), then Theorem B(d) of \( [7] \) says that \( \dim_{\mathbb{Z}_{(p)}} X \leq \dim_{\mathbb{G}} X \). Therefore \( \dim_{\mathbb{G}} X \geq \sup \{ \dim_{\mathbb{H}} X \mid H \in \sigma(G) \} \). Suppose \( \sup \{ \dim_{\mathbb{H}} X \mid H \in \sigma(G) \} = n \) and consider \( l = \{ p \mid p \cdot G \neq G \} \). Theorem B(f) of \( [7] \) says that \( \dim_{\mathbb{G}} X \leq \dim_{\mathbb{Z}_{(p)}} X \). Since \( \sigma(G) = \sigma(\mathbb{Z}_{(p)}) \), \( \dim_{\mathbb{G}} X \leq \dim_{\mathbb{Z}_{(p)}} X = \sup \{ \dim_{\mathbb{H}} X \mid H \in \sigma(\mathbb{Z}_{(p)}) \} = \sup \{ \dim_{\mathbb{H}} X \mid H \in \sigma(G) \} = n \). That proves \( \dim_{\mathbb{G}} X = \sup \{ \dim_{\mathbb{H}} X \mid H \in \sigma(G) \} \) for all torsion-free Abelian groups. The same equality holds for all torsion Abelian groups by Theorem B(a) of \( [7] \). In the case of arbitrary Abelian groups, as \( \sigma(G) = \sigma(F(G)) \cup \sigma(Tor(G)) \) and \( \dim_{\mathbb{G}} = \max(\dim_{F(G)} X, \dim_{\text{Tor}(G)} X) \) (see Theorem B(b) of \( [7] \)) one gets \( \dim_{\mathbb{G}} X = \sup \{ \dim_{\mathbb{H}} X \mid H \in \sigma(G) \} \) as well. \( \square \)

Remark 4.6. Notice that it is not sufficient to assume \( \dim_{\mathbb{Z}_{(p)}} X = \dim_{\mathbb{Z}_{(p)}} Y \) for all primes \( p \) in Theorem 4.7. Indeed, the space \( X \) in \( [6] \) has that property as \( 1 = \dim_{\mathbb{Z}_{(p)}} X \geq \dim_{\mathbb{Z}_{(p)}} Y \geq 1 \) for all primes \( p \).

Theorem 4.7. Let \( X \) be a Bockstein space. If \( G \) is nilpotent, then \( \dim_{\mathbb{G}}(X) \leq 1 \) if and only if \( \sup \{ \dim_{\mathbb{H}}(X) \mid H \in \sigma(G) \} \leq 1 \).

Proof. Let \( \mathcal{P}_1 \) be the class of all nilpotent groups and let \( \mathcal{P}_2 \) be the class of nilpotent groups \( G \) such that \( \dim_{\mathbb{G}}(X) \leq 1 \) if and only if \( \sup \{ \dim_{\mathbb{H}}(X) \mid H \in \sigma(G) \} \leq 1 \). Since \( \mathcal{P}_2 \) contains all Abelian groups, in view of Lemma 2.8 it suffices to show that for any nilpotent central extension \( K \to G \to I \) the conditions \( K, I \in \mathcal{P}_2 \) imply \( G \in \mathcal{P}_2 \). It is so if \( G \) is Abelian, so assume \( G \) is not Abelian. Moreover, as \( \sigma(\text{Ab}(G)) \subset \sigma(G) \) by Lemma 3.12 and \( \dim_{\mathbb{G}} X \leq 1 \) implies \( \dim_{\text{Ab}(G)} X \leq 1 \) (see Lemma 1.4), either \( \dim_{\mathbb{G}}(X) \leq 1 \) or \( \sup \{ \dim_{\mathbb{H}}(X) \mid H \in \sigma(G) \} \leq 1 \) implies \( \dim_{\text{Ab}(G)} X \leq 1 \), so we may as well assume \( \dim_{\text{Ab}(G)} X \leq 1 \).

In view of Lemma 4.2 and the fact that \( \text{Ab}(G) = 1 \) implies \( G = 1 \), the equivalence of conditions \( \dim_{\mathbb{G}}(X) \leq 1 \) and \( \sup \{ \dim_{\mathbb{H}}(X) \mid H \in \sigma(G) \} \leq 1 \) may fail only if \( \dim_{\text{Ab}(G)} X = 1 \), so assume \( \dim_{\text{Ab}(G)} X = 1 \).

Suppose \( \dim_{\mathbb{H}}(X) = n > 1 \) for some \( H \in \sigma(K) \). If \( H \neq \mathbb{Z}/p^\infty \), then \( H \in \sigma_{\text{NTD}}(G) = \sigma_{\text{NTD}}(\text{Ab}(G)) \) (Theorem 3.11 and Lemma 3.13), a contradiction. So \( H = \mathbb{Z}/p^\infty \) for some prime \( p \) and \( \mathbb{Z}/p^\infty \notin \sigma(G) \). Hence \( \text{Ab}(G) \) is \( p \)-divisible and this is equivalent to \( G \) being \( p \)-divisible [2, Lemma 5.1]. Because \( \mathbb{Z}/p^\infty \notin \sigma(\text{Ab}(G)) \), by Lemma 3.6 \( H_1(\text{Ab}(G); \mathbb{Z}/p^\infty) = 0 \), so also \( H_1(G; \mathbb{Z}/p^\infty) = 0 \). By Lemma 3.6 \( H_2(G; \mathbb{Z}/p^\infty) \neq 0 \) as \( \mathbb{Z}/p^\infty \in \sigma(G) \). This implies that \( F(H_2(G; \mathbb{Z})) \) is not \( p \)-divisible, so \( \mathbb{Z}_{(p)} \notin \sigma(H_2(G; \mathbb{Z})) \). If \( G \) is not a torsion group, then \( \text{Ab}(G) \) is not a torsion group; hence by definition \( Q \in \sigma(\text{Ab}(G)) \). Therefore \( \dim_{\mathbb{Q}}(X) \leq 1 \). Using that fact and the Bockstein Inequalities (B15, B16 [2]), we get \( \dim_{\mathbb{Z}/p^\infty}(X) = \dim_{\mathbb{Z}_{(p)}}(X) - 1 \). Because \( \mathbb{Z}_{(p)} \in \sigma(H_2(G; \mathbb{Z})) \), the dimension \( \dim_{\mathbb{Z}_{(p)}}(X) \leq \dim_{H_2(G; \mathbb{Z})}(X) \leq 2 \) as \( X \) is a Bockstein space and then \( \dim_{\mathbb{Z}/p^\infty}(X) = \dim_{\mathbb{Z}_{(p)}}(X) - 1 \leq 1 \), a contradiction.

Thus \( G \) is a torsion group and is a product of \( q \)-groups \( G = \prod_{q \in \mathbb{P}} G_q \). Hence \( \text{Ab}(G) = \prod_{q \in \mathbb{P}} \text{Ab}(G_q) \). Because \( G \) is not \( p \)-local, \( G_p \neq 1 \), but \( \text{Ab}(G) \) is uniquely \( p \)-divisible, so \( \text{Ab}(G_p) = 1 \). Therefore \( G_p \) is a perfect nilpotent group, but such a group is trivial, a contradiction.

Thus \( \dim_{\mathbb{H}} X \leq 1 \) for all \( H \in \sigma(K) \) and \( \dim_X X = \sup \{ \dim_{\mathbb{H}}(X) \mid H \in \sigma(K) \} \leq 1 \) as \( K \) is Abelian and \( X \) is a Bockstein space.
Suppose $\text{sup}\{\text{dim}_H(X)|H \in \sigma(G)\} \leq 1$. By Theorem 3.13 $\sigma(G) = \sigma(K) \cup \sigma(I)$. Therefore $\text{sup}\{\text{dim}_H(X)|H \in \sigma(I)\} \leq 1$ and $\text{dim}_I X \leq 1$ as $I \in \mathcal{P}_2$. Consequently, $\text{dim}_G X \leq 1$ by Lemma 1.3.

Suppose $\text{dim}_I X \leq 1$. By Lemma 1.3 one gets $\text{dim}_I X \leq 1$ and $\text{sup}\{\text{dim}_H(X)|H \in \sigma(I)\} \leq 1$ as $I \in \mathcal{P}_2$. By Theorem 3.14 $\sigma(G) = \sigma(K) \cup \sigma(I)$; hence

$$\text{sup}\{\text{dim}_H(X)|H \in \sigma(G)\} = \text{sup}\{\text{dim}_H(X)|H \in \sigma(K) \cup \sigma(I)\} \leq 1.$$

\[\square\]

**Corollary 4.8.** Let $L$ be a connected nilpotent CW complex. If $X$ is a Bockstein space and $\text{dim}_{\pi_n(L)} X \leq n$ for all $n \geq 1$, then $\text{dim}_{\pi_n(L)} X \leq n$ for all $n \geq 1$.

**Proof.** It is shown in [2] that $\text{dim}_{\pi_n(L)} X \leq n$ for all $n \geq 2$, so it suffices to prove $\text{dim}_{\pi_1(L)} X \leq 1$. If that inequality is false, then there is $H \in \sigma(\pi_1(L))$ such that $\text{dim}_H(X) > 1$ (see Theorem 4.7). In view of Lemma 3.13 as $\text{dim}_{\pi_1(L)} X \leq 1$, $H = \mathbb{Z}/p^\infty$ for some prime $p$. Also, $H_1(L)$ is not a torsion group, so $\text{dim}_H(X) \leq 1$. Using Bockstein Inequalities one gets $\text{dim}_{\mathbb{Z}/p}(X) \geq 3$. Therefore the $i$-th homology groups of both $L$ and $\tilde{L}$ with coefficients in $\mathbb{Z}/p^\infty$ vanish for $i \leq 2$. From the fibration $\tilde{L} \to L \to K(\pi_1(L), 1)$ one gets that the $i$-th homology groups of $K(\pi_1(L), 1)$ with coefficients in $\mathbb{Z}/p^\infty$ vanish for $i \leq 2$. However, in view of Corollary 3.6 that means $\mathbb{Z}/p^\infty \notin \sigma(\pi_1(L))$, a contradiction. \[\square\]

**Corollary 4.9.** Let $L$ be a connected nilpotent CW complex and let $X$ be a Bockstein space such that $\text{dim}_{\pi_n(L)} X \leq n$ for all $n \geq 1$. If $X$ is finite dimensional or $X \in \text{ANR}$, then $X \tau L$.

**Proof.** By Corollary 4.8 one gets $\text{dim}_{\pi_n(L)} X \leq n$ for all $n \geq 1$ and by Theorem G in [7] one has $X \tau L$. \[\square\]

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