AN EXAMPLE OF AN ALMOST GREEDY BASIS IN $L^1(0, 1)$

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Abstract. We give an explicit construction of an almost greedy basis of $L^1(0, 1)$, complementing the results on existence of such a basis. The basis is described in terms of the Haar basis. We construct a quasi-greedy basis in a Banach space which is isomorphic to $L^1(0, 1)$, and then we calculate an isomorphic image of a quasi-greedy basis.

1. Introduction

Let $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ be a normalized basis in a Banach space $X$. For each $f \in X$ we have

$$f = \sum_{k=1}^{\infty} c_k(f, \Psi) \psi_k,$$

where $\lim_{k \to \infty} c_k(f, \Psi) = 0$. We define $\Lambda_0 = \emptyset$; then for each $m \geq 1$ we inductively define sets $\Lambda_m \subset \mathbb{N}$ to satisfy

$$\# \Lambda_m = m, \quad \Lambda_{m-1} \subset \Lambda_m \quad \text{and} \quad \min_{k \in \Lambda_m} |c_k(f, \Psi)| \geq \max_{k \not\in \Lambda_m} |c_k(f, \Psi)|.$$

Note that the sets $\Lambda_m$ are not uniquely determined. Denote the set of all such sequences $\{\Lambda_m\}$ by $D(f)$. For any $\Lambda \in D(f)$ we put

$$G_m(f) = G_m(f, \Psi) = G_m(f, \Psi, \Lambda) = \sum_{k \in \Lambda_m} c_k(f, \Psi) \psi_k.$$

This nonlinear method of approximation is known as a Thresholding Greedy Algorithm (TGA) or as a Greedy algorithm (see for example [6]). In fact $G_m(f)$ can be realized by the following procedure: take the expansion (1) and form a sum of $m$ terms with the biggest $|c_k(f, \Psi)|$. If the basis $\Psi$ is unconditional, then obviously

$$\|G_m(f, \Psi, \Lambda)\|_X \leq C \cdot \|f\|_X,$$

with a constant $C$ independent of $f$, $m$ and $\Lambda$.

Definition. A basis $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ is called quasi-greedy for $X$ if there exists a constant $C$ such that for any $f \in X$ and for any $\Lambda \in D(f)$ the inequality (2) holds for any $m \in \mathbb{N}$.

The following theorem was proved in [8].

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**Theorem A.** A basis \( \Psi = \{ \psi_k \}_{k=1}^{+\infty} \) is quasi-greedy for \( X \) if and only if for any \( f \in X \) there exists \( \Lambda \in D(f) \) such that

\[
\lim_{m \to +\infty} \| f - G_m(f, \Psi, \Lambda) \|_X = 0.
\]

**Definition 2.** A basis \( \Psi = \{ \psi_k \}_{k=1}^{+\infty} \) is called almost greedy for \( X \) if there exists a constant \( C \) such that for all \( f \in X \) and \( m \in \mathbb{N} \) we have

\[
\| f - G_m(f, \Psi) \|_X \leq C \inf_{A \subset \mathbb{N} : |A| = m} \left\{ \left\| f - \sum_{j \in A} a_j \psi_j \right\|_X : a_j \in \mathbb{R}, |A| = m \right\}.
\]

It follows from Definition 2 that every almost greedy basis is also a quasi-greedy basis. It was proved in [2] that \( \Psi \) is almost greedy if and only if \( \Psi \) is quasi-greedy and democratic.

**Definition 3.** A basis \( \Psi = \{ \psi_k \}_{k=1}^{+\infty} \) is called greedy for \( X \) if there exists a constant \( C \) such that for all \( f \in X \) and \( m \in \mathbb{N} \) we have

\[
\| f - G_m(f, \Psi) \|_X \leq C \inf_{A \subset \mathbb{N} : |A| = m} \left\{ \left\| f - \sum_{j \in A} a_j \psi_j \right\|_X : a_j \in \mathbb{R}, |A| = m \right\}.
\]

It was proved in [6] that a basis is greedy if and only if it is unconditional and democratic. So we have the following relations:

- Greedy \( \Rightarrow \) Unconditional \( \Rightarrow \) Quasi-greedy,
- Greedy \( \Rightarrow \) Almost greedy \( \Rightarrow \) Quasi-greedy,
- Greedy = Unconditional + Democratic,
- Almost greedy = Quasi-greedy + Democratic.

It was proved by V. N. Temlyakov (see [7]) that the Haar system is a greedy basis in \( L^p(0,1) \), \( 1 < p < \infty \).

It is known that there is no unconditional basis in \( L^1(0,1) \). But it was proved in [1] that there is an almost greedy basis in \( L^1(0,1) \). That proof was not constructive, and that basis was not constructed. In [3] it was shown that the Haar system is not a quasi-greedy basis in \( L^1(0,1) \). The same result for general Haar systems and for the Franklin system was proved in [4] and [5] correspondingly.

In this paper we give an example of an almost greedy basis in \( L^1(0,1) \). We denote

\[
M_i = 2^i - i - \sum_{p:2^p+1 < i} (2^{2p+1} - 1), \quad i = 0, 1, \ldots
\]

It is easy to check that

\[
M_0 = 1 \quad \text{and} \quad M_{i+1} - M_i = \begin{cases} 0, & \text{if } i = 2^p + 1 \text{ for some natural } p, \\ 2^i - 1, & \text{otherwise.} \end{cases}
\]

Let \( \{ h_n \} \) be a classical Haar system normalized in \( L^1(0,1) \) and let \( \psi_i = h_{2^i + 2} \) (i.e. the Haar function with support \( [2^{-i}, 2^{1-i}) \)).
Let \( i \geq 0 \) and \( i \neq 2^p + 1 \) for any \( p \geq 0 \). For any \( 0 \leq k \leq M_{i+1} - M_i = 2^i - 1 \) we define
\[
\tilde{f}_{(i,k)} = \begin{cases} 
  h_{i+1} - \frac{1}{2^i} \sum_{j=0}^{2^i-2} \psi_{M_i+j} + \psi_{M_i+k}, & \text{if } k < 2^i - 1, \\
  h_{i+1} - \frac{1}{2^i} \sum_{j=0}^{2^i-2} \psi_{M_i+j}, & \text{if } k = 2^i - 1.
\end{cases}
\]

**Theorem 1.** The system \( \{\tilde{f}_{(i,k)}\} \) is an almost greedy basis in \( L^1(0,1) \).

We construct a quasi-greedy basis in an auxiliary Banach space and show that it is an isomorphic image of \( \{\tilde{f}_{(i,k)}\} \).

2. Preliminaries and auxiliary results

We need results from [1]. Let \( X \) be a Banach space with a basis \( \{b_n\} \). By passing to an equivalent norm we may assume that \( \{b_n\} \) is normalized and bimonotone. Let \( S \) be a 1-symmetric and 1-unconditional sequence space with basis \( \{e_i\}, e_i = \{\delta_{ij}\} \).

Let \( \{e^*_i\} \) be the sequence of biorthogonal functionals in \( S^* \).

Define
\[
f(n) = \|e_1 + \ldots + e_n\|_S
\]
and
\[
g(n) = \frac{n}{f(n)} = \|e^*_1 + \ldots + e^*_n\|_{S^*}.
\]

We assume that \( \{e_i\} \) is not equivalent to the unit vector basis of \( c_0 \). Thus,
\[
f(n) \to \infty \text{ as } n \to \infty.
\]

For \( n \geq 1 \), let \( \sigma_n = [2^{n-1}; 2^n - 1] \),
\[
v_n = \frac{1}{f(2^{n-1})} \sum_{k \in \sigma_n} e_k \quad \text{and} \quad v^*_n = \frac{1}{g(2^{n-1})} \sum_{k \in \sigma_n} e^*_k.
\]

Let \( P \) be the norm-one projection on \( S \) defined by
\[
P\xi = \sum_{n=1}^{\infty} \langle \xi, v^*_n \rangle v_n,
\]
and let \( Q = I - P \). Define a norm on \( c_{00} \) by
\[
\|\xi\|_Y = \left\| \sum_{n=1}^{\infty} \langle \xi, v^*_n \rangle b_n \right\|_X + \|Q\xi\|_S
\]
and then complete it to obtain a sequence space \( Y \).

The following statements were proved in [1].

**Statement 1** (Proposition 6.1). Suppose that \( \{b_n\} \) is a bimonotone basis for \( X \). Then \( \{e_n\} \) is a basis for \( Y \) such that
\[
\frac{1}{8} \sup_n \eta_n f(n) \leq \|\xi\|_Y \leq 6 \sum_{n=1}^{\infty} \frac{f(n)}{n} \eta_n
\]
for all real scalars \( \xi = (\xi_n) \) in \( c_{00} \), where \( \{\eta_n\} \) is the nonincreasing rearrangement of \( \{\|\xi_i\|\} \).

**Statement 2.** Suppose that \( X \) is a Banach space with a basis that contains a complemented subspace isomorphic to \( S \). Then \( Y \sim X \).
Statement 3 (Theorem 7.1). Suppose that \( \{b_n\} \) is a basis for \( X \) and \( S = \ell_1 \). Then \( \{e_n\} \) is a quasi-greedy basis for \( Y \).

Let us apply Statements 1-3 when \( X = L^1(0,1) \) and \( S = \ell_1, \{b_i\} = \{h_i\} \) (the Haar system).

Lemma 1. The system \( \{e_i\} \) is an almost greedy basis in \( Y \) and

\[
Q(e_{2^i+k}) = \left(0, \ldots, 0, \underbrace{-2^{-i}, \ldots, -2^{-i}}_{2^i-1}, 1 - 2^{-i}, \underbrace{2^{-i}, \ldots, 2^{-i}}_{2^i-k-1}, 0, 0 \ldots}\right)
\]

for all \( 0 \leq k < 2^i \).

Proof. We have \( f(n) = n \) and \( g(n) = 1 \). Let \( \xi = \sum_{k \in A} e_k \) with \( |A| = m \). Then

\[
\eta_n = \begin{cases} 
1, & n \leq m, \\
0, & n > m.
\end{cases}
\]

Hence, according to Statement 1 we conclude that

\[
\frac{m}{8} \leq \| \sum_{k \in A} e_k \|_Y \leq 6m.
\]

Hence \( \| \sum_{k \in A} e_k \|_Y \asymp |A| \). So \( \{e_n\} \) is democratic in \( Y \). Combined with Statement 3 we conclude that \( \{e_n\} \) is an almost greedy basis. Equality (4) immediately follows from the definition of \( Q \). Lemma 1 is proved.

According to the definition of \( Y \) there exists an isomorphic operator \( R : Y \mapsto L^1(0,1) \oplus Q(\ell_1) \) such that

\[
R(e_{2^i+k}) = (h_{i+1}, Q(e_{2^i+k}))
\]

for every \( k = 0, \ldots, 2^i-1; \ i = 0, 1, \ldots \).

3. ISOMORPHIC OPERATORS

Let \( \{M_i\} \) be the sequence defined by (3) and let \( \Lambda_n = [2^{n-1}, 2^n - 2] \). Denote

\[
\Lambda = \{n_1, n_2, n_3, \ldots\} = \bigcup_{p=0}^{\infty} \Lambda_{2^p+2} \quad \text{and} \quad \{m_1, m_2, \ldots\} = \left(\bigcup_{n=1}^{\infty} \Lambda_n\right) \setminus \Lambda,
\]

where sequences \( \{n_i\} \) and \( \{m_i\} \) are increasing. For every \( \{a_i\} \in Q(\ell_1) \) we define the operator \( S : Q(\ell_1) \mapsto \ell_1 \oplus \ell_1 \) by

\[
S(\{a_i\}) = \left(\{a_{m_1}, a_{m_2}, \ldots\}, \{a_{n_1}, a_{n_2}, \ldots\}\right).
\]

Since \( \{a_i\} \in Q(\ell_1) \) if and only if \( \{a_i\} \in \ell_1 \) and \( a_{2^i-1} = -\sum_{i \leq n} a_i \) for all natural \( n \), we conclude that \( S \) is an isomorphic operator. The next lemma follows from the definitions of \( Q \) and \( S \).
Lemma 2. 1) Let \( i \neq 2^p + 1 \) for any \( p \geq 0 \). Then

\[
S(Q(e_{2^{-i}+k})) = \begin{cases} 
\left(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}, 1 - 2^{-i}, -2^{-i}, \ldots, -2^{-i}, 0 \ldots \right), & \text{if } k < 2^i - 1, \\
\left(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}, 0 \ldots \right), & \text{if } k = 2^i - 1.
\end{cases}
\]

2) Let \( i = 2^p + 1 \) for some \( p \geq 0 \) and \( 0 \leq k < 2^i \). Then

\[
S(Q(e_{2^{-i}+k})) = \begin{cases} 
\left(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}, 1 - 2^{-i}, -2^{-i}, \ldots, -2^{-i}, 0 \ldots \right), & \text{if } k < 2^i - 1, \\
\left(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}, 0, 0 \ldots \right), & \text{if } k = 2^i - 1.
\end{cases}
\]

Functions \( \psi_i \) have disjoint supports \( (\text{supp}(\psi_i) = (2^{-i}, 2^{1-i})) \). Therefore we have

\[
\| \sum_{i=1}^{\infty} a_i \psi_i \|_{L^1(0,1)} = \| \{a_i\} \|_{\ell_1}.
\]

Denote

\[
Z = \{ f \in L^1(0,1) : \int_0^1 f(t) \psi_n(t) dt = 0 \text{ for all } n \in \mathbb{N} \}.
\]

Let us define the operator \( T : L^1(0,1) \to (Z, \ell_1) \) by the formula

\[
T(f) = T\left(\sum_{n=1}^{\infty} a_n h_n\right) = \left(\sum_{n=1, n \neq 2^{p+2}}^{\infty} a_n h_n, \{a_{2^{p+2}}\}\right).
\]

Note that

\[
T(h_n) = \begin{cases} 
(0, \{\delta_{p+1,1}\}), & \text{if } n = 2^p + 2, \\
(h_n, \{0\}), & \text{otherwise}.
\end{cases}
\]

The next lemma is obvious.

Lemma 3. The operator \( T \) represents an isomorphism between \( L^1(0,1) \) and \( Z \oplus \ell_1 \).

4. An almost greedy basis in \( L^1(0,1) \)

In the proof of Statement \( \boxed{} \) the following isomorphic chain was used:

\[
Y \sim X \oplus Q(S) \sim Z \oplus S \oplus Q(S) \sim Z \oplus S \oplus P(S) \oplus Q(S) \sim Z \oplus S \sim Z \oplus S \sim X.
\]
In the proof of Theorem 1 we will use the following:
\[ Y \sim L^1(0, 1) \oplus Q(\ell_1) \]
\[ \sim Z \oplus \ell_1 \oplus Q(\ell_1) \quad \text{(by operator } T) \]
\[ \sim Z \oplus \ell_1 \oplus \ell_1 \oplus \ell_1 \quad \text{(by operator } S) \]
\[ \sim L^1(0, 1) \oplus \ell_1 \oplus \ell_1. \quad \text{(by operator } T). \]

**Proof of Theorem 1.** For \( y \in Y \) denote
\[
\begin{align*}
R(y) &= (y_1, q_1), \text{ where } y_1 \in L^1(0, 1), \ q_1 \in Q(\ell_1); \\
T(y_1) &= (z_1, a_1), \text{ where } z_1 \in Z, \ a_1 \in \ell_1; \\
S(q_1) &= (a_2, a_3), \text{ where } a_2, a_3 \in \ell_1; \\
T^{-1}(z_1, a_2) &= y_2, \text{ where } y_2 \in L^1(0, 1);
\end{align*}
\]
and set
\[ D(y) = \{y_2, a_1, a_3\}. \]

It is clear that \( D \) is an isomorphic operator from \( Y \) to \( L^1(0, 1) \oplus \ell_1 \oplus \ell_1 \). Now, let us calculate \( D(e_n) \). Let \( n = 2^i + k \) with \( 0 \leq k < 2^i \). According to (5) we have
\[ R(e_{2^i+k}) = (h_{i+1}, Q(e_{2^i+k})), \]
which means that \( y_1 = h_{i+1} \) and \( q_1 = Q(e_{2^i+k}) \). We consider four cases.

**Case 1.** \( i = 2^p + 1 \) for some \( p \geq 0 \) and \( k < 2^i - 1 \). According to (7)
\[
\begin{align*}
(z_1, a_1) &= T(h_{i+1}) = (0, \{\delta_{p+1,j}\}), \\
(a_2, a_3) &= S(Q(e_{2^i+k})) \\
&= \left(\{0\}, \{(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}, 1 - 2^{-i}, -2^{-i}, \ldots, -2^{-i}0 \ldots)\}\right), \\
y_2 &= T^{-1}(z_1, a_2) = T^{-1}(0, \{0\}) = 0.
\end{align*}
\]
Hence
\[ D(e_{2^i+k}) = \left(0, \{\delta_{p+1,j}\}, \{(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}, 1 - 2^{-i}, -2^{-i}, \ldots, -2^{-i}0 \ldots)\}\right). \]

**Case 2.** \( i = 2^p + 1, \ k = 2^i - 1 \).
\[
\begin{align*}
(z_1, a_1) &= T(h_{i+1}) = (0, \{\delta_{p+1,j}\}), \\
(a_2, a_3) &= S(Q(e_{2^i+k})) = \left(\{0\}, \{(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}0 \ldots)\}\right), \\
y_2 &= T^{-1}(z_1, a_2) = T^{-1}(0, \{0\}) = 0.
\end{align*}
\]
Hence \[ D(e_{2^i+k}) = \left(0, \{\delta_{p+1,j}\}, \{(0, \ldots, 0, -2^{-i}, \ldots, -2^{-i}0 \ldots)\}\right). \]
Case 3. \(i \neq 2^p + 1, \ k \neq 2^i - 1.\)

\[
\begin{align*}
(z_1, a_1) &= T(h_{i+1}) = (h_{i+1}, \{0\}), \\
(a_2, a_3) &= S(Q(e_{2^i+k})) \\
&= \left(\left(0, \ldots, 0, \underbrace{-2^{-i}, \ldots, -2^{-i}}_{M_i-1}, 1 - 2^{-i}, -2^{-i}, \ldots, -2^{-i}, 0 \ldots}, \{0\}\right), \\
y_2 &= T^{-1}(z_1, a_2) \\
&= T^{-1}\left(h_{i+1}, \left(0, \ldots, 0, \underbrace{-2^{-i}, \ldots, -2^{-i}}_{M_i-1}, 1 - 2^{-i}, -2^{-i}, \ldots, -2^{-i}, 0 \ldots}\right) \\
&= h_{i+1} - \frac{1}{2^i} \sum_{j=M_i}^{M_{i+1}} \psi_j + \psi_{M_{i+k}} = h_{i+1} - \frac{1}{2^i} \sum_{j=M_i}^{M_{i+1}-1} \psi_j + \psi_{M_{i+k}}.
\end{align*}
\]

Therefore \(D(e_{2^i+k}) = \left(f_{(i,k)}, \{0\}, \{0\}\right).\)

Case 4. \(i \neq 2^p + 1, \ k = 2^i - 1.\)

\[
\begin{align*}
(z_1, a_1) &= T(h_{i+1}) = (h_{i+1}, \{0\}), \\
(a_2, a_3) &= S(Q(e_{2^i+k})) = \left(\left(0, \ldots, 0, \underbrace{-2^{-i}, \ldots, -2^{-i}}_{M_i-1}, 0 \ldots}, \{0\}\right), \\
y_2 &= T^{-1}(z_1, a_2) = T^{-1}\left(h_{i+1}, \left(0, \ldots, 0, \underbrace{-2^{-i}, \ldots, -2^{-i}}_{M_i-1}, 0 \ldots\right)\right) \\
&= h_{i+1} - \frac{1}{2^i} \sum_{j=M_i}^{M_{i+1}-1} \psi_j.
\end{align*}
\]

Therefore \(D(e_{2^i+k}) = \left(f_{(i,k)}, \{0\}, \{0\}\right).\)

Finally we have

\[
D(e_{2^i+k}) = \begin{cases} 
\{0, \ldots, 0, \ldots\}, & \text{if } i = 2^p + 1, \\
\left(f_{(i,k)}, \{0\}, \{0\}\right), & \text{if } i \neq 2^p + 1.
\end{cases}
\]

Since \(\{D(e_{2^i+k})\}\) is an almost-greedy basis in \(L^1(0, 1) \oplus \ell_1 \oplus \ell_1\), then the system \(\{f_{(i,k)}\}\) is an almost greedy basis in \(L^1(0, 1)\). Theorem 1 is proved. \(\square\)

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References


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