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A CLASSIFICATION OF H-PRIMES OF QUANTUM PARTIAL FLAG VARIETIES

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Abstract. We classify the invariant prime ideals of a quantum partial flag variety under the action of the related maximal torus. As a result we construct a bijection between them and the torus orbits of symplectic leaves of the standard Poisson structure on the corresponding flag variety. It was previously shown by K. Goodearl and the author that the latter are precisely the Lusztig strata of the partial flag variety.

1. Introduction

Let G be a split, simply connected, semisimple algebraic group over a field $\mathbb{K}$ of characteristic 0, and let $\mathfrak{g}$ be its Lie algebra. Denote by B and B_- a pair of dual Borel subgroups and set $T = B \cap B_-$. Given a set of simple roots I, one defines the standard parabolic subgroup $P_I \supset B$ and the multicone

$$\mathrm{Spec}\Big(\bigoplus_{n_i \in \mathbb{Z}_{\geq 0}} H^0(G/P_I, \otimes_{i \notin I} \mathcal{L}_{\omega_i}^{n_i})\Big)$$

over the flag variety G/P_I. Here the tensor product involves the canonical line bundles $\mathcal{L}_{\omega_i}$ over G/P_I corresponding to the fundamental weights for the simple roots not in I. Its coordinate ring has a canonical deformation $R_q[G/P_I]$ defined by Lakshmibai–Reshetikhin [10] and Soibelman [18]. The group H of grouplike elements of $\mathcal{U}_q(\mathfrak{g})$ acts naturally on $R_q[G/P_I]$.

Currently, little is known about the spectrum of $R_q[G/P_I]$ beyond the case of the full flag variety G/B. For the quantized ring $R_q[G/B]$ Gorelik [7] classified all H-invariant prime ideals and described the inclusions between them and the strata of a related partition of $\mathrm{Spec} R_q[G/B]$. In the general case one can apply results of Goodearl and Letzter [5] to obtain a partition of $\mathrm{Spec} R_q[G/P_I]$ indexed by the H-prime ideals of $R_q[G/P_I]$ such that each stratum is homeomorphic to the spectrum of a Laurent polynomial ring. The H-primes of $R_q[G/P_I]$ are unknown except the case of Grassmannians which is due to Launois, Lenagan and Rigal [11].

In this paper we prove a classification of the H-invariant prime ideals of the rings $R_q[G/P_I]$ associated to all partial flag varieties (see Theorem 3.8):

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Theorem 1.1. *For an arbitrary partial flag variety G/P_I the H-invariant prime ideals of $R_q[G/P_I]$ (not containing the augmentation ideal* (3.1)*) are parametrized by*

$$S_{W,I} := \{(w,v) \in W^I \times W \mid v \leq w\}. \tag{1.1}$$

All such ideals are completely prime.

Here W^I denotes the set of minimal length representatives for the elements of W/W_I, where W is the Weyl group of G and W_I is the parabolic subgroup of W corresponding to P_I.

To put our results in a more geometric context, let us assume that the ground field is $\mathbb{K} = \mathbb{C}$. The action of the torus T on G/P_I preserves the *standard Poisson structure* π_I on G/P_I; cf. [6]. According to [6, Theorem 0.4] its T-orbits of symplectic leaves are

$$\mathcal{T}^I_{w,v} = q_I(B \cdot wB \cap B_- \cdot vB),\ (w,v) \in S_{W,I},$$

where $q_I \colon G/B \to G/P_I$ is the canonical projection, cf. [1] for the case of Grassmannians. The varieties $\mathcal{T}^I_{w,v}$ are precisely the strata of the Lusztig stratification [13] of G/P_I defined for the purposes of the study of total positivity on G/P_I.

The algebra $R_q[G/P_I]$ is a quantization of the projective Poisson variety $(G/P_I, \pi_I)$. In particular our results provide the first step of the orbit method program for this situation: we obtain a bijection between the H-primes of $R_q[G/P_I]$ and the T-orbits of leaves of $(G/P_I, \pi_I)$.

The Zariski closures of $\mathcal{T}^I_{w,v}$ were explicitly determined in [6, 17]:

$$\overline{\mathcal{T}^I_{w,v}} = \{\mathcal{T}^I_{w',v'} \mid \exists z \in W_I \text{ such that } w \geq w'z, v \leq v'z\}.$$

Denote by $\mathcal{I}^I_{w,v}$ the H-invariant prime ideal of $R_q[G/P_I]$ corresponding to $(w,v) \in S_{W,I}$ according to the parametrization of Theorem 1.1; see §3.5 for details.

Following the orbit method we make the following conjecture.

Conjecture 1.2. *Let $(w,v),(w',v') \in S_{W,I}$, defined in* (1.1)*. Then $\mathcal{I}^I_{w,v} \subseteq \mathcal{I}^I_{w',v'}$ if and only if there exists $z \in W_I$ such that*

$$w \geq w'z \ \text{ and } \ v \leq v'z.$$

This was established by Gorelik in the case $I = \emptyset$ of the full flag variety [7] in which case z always has to be the identity. In general, the conjecture is open even for Grassmannians.

Finally we prove results, which are analogous to Theorem 1.1, for the quantum deformations [10, 18] of the coordinate rings of the cones

$$\operatorname{Spec}\Big(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} H^0(G/P_I, \mathcal{L}_{n\lambda})\Big)$$

over G/P_I for certain dominant weights λ.

After the completion of this paper we learned that Stéphane Launois and Laurent Rigal independently found proofs of Theorems 3.8 and 3.12.

2. Generalities on quantum groups and quantum flag varieties

2.1. Let $\mathbb{K}$ be a field of characteristic 0 and let $q \in \mathbb{K}$ be transcendental over $\mathbb{Q}$. Let $\mathfrak{g}$ be a split semisimple Lie algebra over $\mathbb{K}$ of rank r with Cartan matrix (c_{ij}).

Denote by $\mathcal{U}_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$. It is a Hopf algebra over $\mathbb{K}$ with generators

$$X_i^{\pm}, K_i^{\pm 1},\ i = 1, \dots, r,$$

as in [2, §9.1]. Let $\{d_i\}_{i=1}^r$ be the standard choice of integers for which the matrix $(d_i c_{ij})$ is symmetric. Set $q_i = q^{d_i}$. Fix a nondegenerate invariant bilinear form $\langle .,. \rangle$ on $\mathfrak{g}$ such that the square length of a long root is equal to 2.

Let Q and Q^+ be the sets of all integral and dominant integral weights of $\mathfrak{g}$. The sets of simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$ will be denoted by $\{\alpha_i\}_{i=1}^r$, $\{\alpha_i^\vee\}_{i=1}^r$, and $\{\omega_i\}_{i=1}^r$, respectively. For $\lambda, \mu \in Q$ one sets $\lambda \geq \mu$ if $\mu = \lambda - \sum_{i=1}^r n_i \alpha_i^\vee$ for some $n_i \in \mathbb{Z}_{\geq 0}$, and $\lambda > \mu$ if $\lambda \geq \mu$ and $\lambda \neq \mu$.

Recall that the weight spaces of a $\mathcal{U}_q(\mathfrak{g})$-module V are defined by

$$V_\lambda = \{v \in V \mid K_i v = q^{\langle \lambda, \alpha_i^\vee \rangle} v,\ \forall i = 1, \dots, r\},\ \lambda \in Q.$$

A $\mathcal{U}_q(\mathfrak{g})$-module is a weight module if it is the sum of its weight spaces. The irreducible finite dimensional weight $\mathcal{U}_q(\mathfrak{g})$-modules are parametrized by Q^+; see [2, §10.1] for details. For $\lambda \in Q^+$ let $V(\lambda)$ be the corresponding irreducible module and let v_λ be a highest weight vector. All duals of finite dimensional $\mathcal{U}_q(\mathfrak{g})$-modules will be considered as left modules using the antipode of $\mathcal{U}_q(\mathfrak{g})$.

Denote the Weyl and braid groups of $\mathfrak{g}$ by W and $\mathcal{B}_\mathfrak{g}$, respectively. Let $s_1, \dots, s_r$ be the simple reflections of W and let $T_1, \dots, T_r$ be the standard generators of $\mathcal{B}_\mathfrak{g}$. There is a natural action of $\mathcal{B}_\mathfrak{g}$ on $\mathcal{U}_q(\mathfrak{g})$ and the modules $V(\lambda)$; see [12, §5.2 and §37.1] for details. One has $T_w(x.v) = (T_w x).(T_w v)$ and $T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu}$ for all $w \in W$, $x \in \mathcal{U}_q(\mathfrak{g})$, $\lambda \in Q^+$, $v \in V(\lambda)$, $\lambda \in Q$.

2.2. Let G be the split, connected, simply connected algebraic group over $\mathbb{K}$ with Lie algebra $\mathfrak{g}$, and let B and B_- be a pair of opposite Borel subgroups. Let $T = B \cap B_-$.

The quantized coordinate ring $R_q[G]$ is the Hopf subalgebra of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by all matrix entries $c_{\xi,v}^\lambda$, $\lambda \in Q^+$, $v \in V(\lambda), \xi \in V(\lambda)^*$: $c_{\xi,v}^\lambda(x) = \langle \xi, xv \rangle$ for $x \in \mathcal{U}_q(\mathfrak{g})$. One has the left and right actions of $\mathcal{U}_q(\mathfrak{g})$ on $R_q[G]$,

$$x \rightharpoonup c = \sum c_{(2)}(x) c_{(1)},\ c \leftharpoonup x = \sum c_{(1)}(x) c_{(2)},\ x \in \mathcal{U}_q(\mathfrak{g}), c \in R_q[G], \tag{2.1}$$

where $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.

Denote by $\mathcal{U}_\pm$ the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_i^\pm\}_{i=1}^r$. Let H be the group generated by $\{K_i^{\pm 1}\}_{i=1}^r$. The subalgebra of $R_q[G]$ invariant under the left action of $\mathcal{U}_+$ will be denoted by R^+. It is spanned by all matrix entries $c_{\xi,v_\lambda}^\lambda$, where $\lambda \in Q^+$, $\xi \in V(\lambda)^*$ and v_λ is the fixed highest weight vector of $V(\lambda)$.

For $I \subset \{1, \dots, r\}$ denote by $P_I \supset B$ the corresponding standard parabolic subgroup. Let $I^c = \{1, \dots, r\} \backslash I$. Let $Q_I = \{\sum_i n_i \omega_i \mid i \in I^c, n_i \in \mathbb{Z}\}$, $Q_I^+ = \{\sum_i n_i \omega_i \mid i \in I^c, n_i \in \mathbb{Z}_{\geq 0}\}$, and $Q_I^{++} = \{\sum_i n_i \omega_i \mid i \in I^c, n_i \in \mathbb{Z}_{>0}\}$.

Denote by $\mathcal{U}_q(\mathfrak{l}_I)$ the Hopf subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_i^\pm, K_i\}_{i \in I}$. The quantized (multihomogeneous) coordinate ring $R_q[G/P_I]$ of the partial flag variety G/P_I is defined [10, 18] by

$$R_q[G/P_I] = \mathrm{Span}\{c_{\xi,v_\lambda}^\lambda \mid \lambda \in Q_I^+, \xi \in V(\lambda)\}.$$

It is the subalgebra of R_+ invariant under the left action (2.1) of the Hopf algebra $\mathcal{U}_q(\mathfrak{l}_I)$. Recall that each $\lambda \in Q_I^+$ gives rise to a line bundle $\mathcal{L}_\lambda$ on the flag variety

G/P_I. The ring $R_q[G/P_I]$ is a deformation of the coordinate ring of the multicone

$$\operatorname{Spec}\Big(\bigoplus_{\lambda\in Q_I^+} H^0(G/P_I,\mathcal{L}_\lambda)\Big)$$

over G/P_I.

A subset of $R_q[G]$ is invariant under the right action of H if and only if it is invariant under the rational action of the torus $(\mathbb{K}^*)^r$ given by

$$(a_1,\dots,a_r)\cdot c^\lambda_{\xi,v_\lambda} = \left(\prod_{i=1}^r a_i^{\langle\mu,\alpha_i^\vee\rangle}\right) c^\lambda_{\xi,v_\lambda}, \quad \text{for } \xi\in V(\lambda)_\mu.$$

In particular H-primes and $(\mathbb{K}^*)^r$-primes of $R_q[G/P_I]$ coincide.

2.3. Given a reduced expression

$$w = s_{i_1}\dots s_{i_k} \tag{2.2}$$

of an element $w\in W$, define the roots

$$\beta_1=\alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2},\dots,\beta_k = s_{i_1}\dots s_{i_{k-1}}\alpha_{i_k} \tag{2.3}$$

and the root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{s_{i_1}}X^\pm_{i_2},\dots,X^\pm_{\beta_k} = T_{s_{i_1}\dots s_{i_{k-1}}}X^\pm_{i_k}; \tag{2.4}$$

see [12, §39.3]. De Concini, Kac and Procesi defined [3] the subalgebras $\mathcal{U}^w_\pm$ of $\mathcal{U}_\pm$ generated by $X^\pm_{\beta_j}$, $j=1,\dots,k$ and proved:

Theorem 2.1 ([3, Proposition 2.2])**.** *The algebras $\mathcal{U}^w_\pm$ do not depend on the choice of a reduced decomposition of w and have the PBW basis*

$$(X^\pm_{\beta_k})^{n_k}\dots(X^\pm_{\beta_1})^{n_1},\ \ n_1,\dots,n_k\in\mathbb{Z}_{\geq 0}. \tag{2.5}$$

The fact that the vector space spanned by the monomials (2.5) does not depend on the choice of a reduced decomposition of w was independently obtained by Lusztig [12, Proposition 40.2.1].

Recall that the universal R-matrix associated to $w\in W$ is defined by

$$\mathcal{R}^w = \prod_{j=k,\dots,1}\exp_{q_{i_j}}\left((1-q_{i_j})^{-2}X^+_{\beta_j}\otimes X^-_{\beta_j}\right), \tag{2.6}$$

where

$$\exp_{q_i} = \sum_{n=0}^\infty q_i^{n(n+1)/2}\frac{n^k}{[n]_{q_i}!}.$$

In (2.6) the terms are multiplied in the order $j=k,\dots,1$. The R-matrix $\mathcal{R}^w$ belongs to a certain completion [12, §4.1.1] of $\mathcal{U}^w_+\otimes\mathcal{U}^w_-$ and does not depend on the choice of the reduced decomposition of w.

For all $\lambda\in Q^+$ and $w\in W$ fix $\xi_{w,\lambda}\in(V(\lambda)^*)_{-w\lambda}$ such that $\langle\xi_{w,\lambda},T_wv_\lambda\rangle=1$. Let

$$c^\lambda_w = c^\lambda_{\xi_{w,\lambda},v_\lambda}.$$

Then $c^\lambda_w c^\mu_w = c^{\lambda+\mu}_w = c^\mu_w c^\lambda_w$, $\forall\lambda,\mu\in Q^+$; see e.g. [19, §2.5].

Denote

$$c^I_w = \{c^\lambda_w \mid \lambda\in Q^I_+\}.$$

According to [9, Lemma 9.1.10] the set $c^{\{1,\dots,r\}}_w$ is Ore in R^+. Similarly one proves that c^I_w is an Ore subset of $R_q[G/P_I]$.

3. H-invariant prime ideals of $R_q[G/P_I]$

3.1. Denote by $H-\mathrm{Spec}_+ R_q[G/P_I]$ the set of H-invariant prime ideals of $R_q[G/P_I]$ under the right action of H which do not contain the ideal

$$\mathcal{J}_+^I = \mathrm{Span}\{c^{\lambda}_{\xi,v_\lambda} \mid \lambda \in Q_I^{++}, \xi \in V(\lambda)^*\} \tag{3.1}$$

of $R_q[G/P_I]$.

To classify the ideals $\mathcal{I}$ in $H-\mathrm{Spec}_+ R_q[G/P_I]$, we first partition the set according to the maximal quantum Schubert ideal contained in $\mathcal{I}$, using techniques of Joseph [9], similar to Hodges–Levasseur [8] and Gorelik [7]. We then relate those strata to H-invariant prime ideals of the algebras $\mathcal{U}_-^w$ along the lines of our previous work [19], similarly to De Concini–Procesi [4], and finally use results of Mériaux–Cauchon [15] and the author [19].

3.2. Recall from the introduction that W_I denotes the parabolic subgroup of the Weyl group W generated by s_i, $i \in I$, and W^I denotes the set of minimal length representatives of the cosets in W/W_I.

We will need the following known lemma. We include its proof for completeness.

Lemma 3.1. *Assume that $\lambda_j \in Q_I^+$ and μ_j are weights of $V(\lambda_j)$ for $j = 1, 2$. Then $\langle \mu_1, \mu_2\rangle \le \langle \lambda_1, \lambda_2\rangle$. If $\lambda_2 \in Q_I^{++}$, then equality implies $\mu_1 = w\lambda_1$ for some $w \in W^I$. If in addition $\lambda_1 \in Q_I^{++}$, then $\mu_2 = w\lambda_2$ for the same w.*

Proof. There exists $w \in W$ such that $w^{-1}\mu_1 \in Q^+$. Then $w^{-1}\mu_2 = \lambda_2 - \sum_{i=1}^r n_i\alpha_i^\vee$, for some $n_i \in \mathbb{Z}_{\ge 0}$, and

$$\begin{aligned}\langle \mu_1, \mu_2\rangle = \langle w^{-1}\mu_1, w^{-1}\mu_2\rangle &= \langle w^{-1}\mu_1, \lambda_2\rangle \\ &\quad - \sum_{i=1}^r n_i\langle w^{-1}\mu_1, \alpha_i^\vee\rangle \le \langle w^{-1}\mu_1, \lambda_2\rangle \le \langle \lambda_1, \lambda_2\rangle.\end{aligned} \tag{3.2}$$

Assume that $\lambda_2 \in Q_I^{++}$ and that equality holds. Then $\lambda_1 - w^{-1}\mu_1 = \sum_{i=1}^r m_i\alpha_i^\vee$ and $\sum_{i=1}^r m_i\langle \lambda_2, \alpha_i^\vee\rangle = 0$. Thus $m_i = 0$ for all $i \in I^c$. Since $X_i^- v_{\lambda_1} = 0$ for $i \in I$ and $w^{-1}\mu_1 = \lambda_1 - \sum_{i\in I} m_i\alpha_i^\vee$ is a weight of $V(\lambda_1)$, $m_i = 0$ for all i and $w^{-1}\mu_1 = \lambda_1$.

Now assume in addition that $\lambda_1 \in Q_I^{++}$ and that equality holds in (3.2). Then $n_i = 0$ for all $i \in I^c$ and $w^{-1}\mu_2 = \lambda_2 - \sum_{i\in I} n_i\alpha_i^\vee$. Since the latter is a weight of $V(\lambda_2)$, in the same way we obtain $w^{-1}\mu_2 = \lambda_2$. □

For $\lambda \in Q_I^+$ and $\mu \in Q$ denote by $\mathcal{J}_\lambda(\mu)$ the ideal of $R_q[G/P_I]$ generated by $c^{\lambda}_{\xi,v_\lambda}$ for $\xi \in V(\lambda)^*_{-\mu'}$, $\mu' < \mu$. Analogously to [9, Proposition 9.1.5 (i)] we have

$$c^{\lambda_1}_{\xi_1,v_{\lambda_1}} c^{\lambda_2}_{\xi_2,v_{\lambda_2}} - q^{\langle\lambda_1,\lambda_2\rangle - \langle\mu_1,\mu_2\rangle} c^{\lambda_2}_{\xi_2,v_{\lambda_2}} c^{\lambda_1}_{\xi_1,v_{\lambda_1}} \in \mathcal{J}_{\lambda_2}(\mu_2) \tag{3.3}$$

if $\xi_j \in V(\lambda_j)^*_{-\mu_j}$ for $j = 1, 2$.

3.3. Following Joseph [9, §9.3.8] for an ideal $\mathcal{I}$ of $R_q[G/P_I]$ and $\lambda \in Q_I^+$ define

$$C_{\mathcal{I}}^+(\lambda) = \{\mu \in Q \mid \exists\, \xi \in V(\lambda)^*_{-\mu} \text{ such that } c^{\lambda}_{\xi,v_\lambda} \notin \mathcal{I}\}.$$

If $C_{\mathcal{I}}^+(\lambda)$ is empty, let $D_{\mathcal{I}}^+(\lambda) = \emptyset$. Otherwise denote by $D_{\mathcal{I}}^+(\lambda)$ the set of minimal elements of $C_{\mathcal{I}}^+(\lambda)$.

Theorem 3.2. *For each prime ideal $\mathcal{I}$ of $R_q[G/P_I]$ which does not contain $\mathcal{J}_+^I$ there exists $w \in W^I$ such that $D_{\mathcal{I}}^+(\lambda) = \{w\lambda\}$ for all $\lambda \in Q_I^+$.*

For $w \in W^I$ denote by X_w^I the set of those H-invariant prime ideals $\mathcal{I}$ of $R_q[G/P_I]$ which satisfy $D_{\mathcal{I}}^+(\lambda) = \{w\lambda\}$ for all $\lambda \in Q_I^+$. Note that $X_w^I \subset H - \mathrm{Spec}_+ R_q[G/P_I]$ since $D_{\mathcal{I}}^+(\lambda) = \{w\lambda\}$ implies that $\mathcal{I}$ does not contain $\mathcal{J}_+^I$. Thus we have the set theoretic decomposition:

$$H - \mathrm{Spec}_+(R_q[G/P_I]) = \bigsqcup_{w \in W^I} X_w^I. \tag{3.4}$$

Proof of Theorem 3.2. We follow the idea of the proof of [9, Proposition 9.3.8]. Assume that $\mathcal{I}$ is a prime ideal of $R_q[G/P_I]$ which does not contain $\mathcal{J}_+^I$.

Assume that $D_{\mathcal{I}}^+(\lambda_j) \neq \emptyset$ and $\mu_j \in D_{\mathcal{I}}^+(\lambda_j)$ for $j = 1, 2$. It follows from the definition of the ideal $\mathcal{J}_{\lambda_j}(\mu_j)$ (see §3.2) that $\mathcal{I} \supset \mathcal{J}_{\lambda_j}(\mu_j)$ for $j = 1, 2$. Fix $\xi_j \in V(\lambda_j)^*_{-\mu_j}$ such that $c^{\lambda_j}_{\xi, v_{\lambda_j}} \notin \mathcal{I}$. Then the images $\bar{c}_j$ of $c^{\lambda_j}_{\xi, v_{\lambda_j}}$ in $R_q[G/P_I]/\mathcal{I}$ are normal by (3.3) and thus are not zero divisors since $\mathcal{I}$ is prime. Applying (3.3) one more time leads to

$$\bar{c}_1\bar{c}_2 = q^{\langle\lambda_1,\lambda_2\rangle - \langle\mu_1,\mu_2\rangle}\bar{c}_2\bar{c}_1 \ \text{ and } \ \bar{c}_2\bar{c}_1 = q^{\langle\lambda_1,\lambda_2\rangle - \langle\mu_1,\mu_2\rangle}\bar{c}_1\bar{c}_2.$$

Therefore $\langle\lambda_1, \lambda_2\rangle = \langle\mu_1, \mu_2\rangle$.

Since $\mathcal{I} \not\supset \mathcal{J}_+^I$ there exists $\lambda \in Q_I^{++}$ such that $C_{\mathcal{I}}^+(\lambda) \neq \emptyset$. Then the above argument and Lemma 3.1 imply that $D_{\mathcal{I}}^+(\lambda) = \{w\lambda\}$ for some $w \in W^I$. Let $\xi \in V(\lambda)^*_{-w\lambda}$, $\xi \neq 0$. Since $\dim V(\lambda)^*_{-w\lambda} = 1$, $c^{\lambda}_{\xi, v_\lambda} \notin \mathcal{I}$. Moreover the image of $c^{\lambda}_{\xi, v_\lambda}$ in $R_q[G/P_I]/\mathcal{I}$ is normal because of (3.3) and all of its powers do not vanish. As a consequence $C_{\mathcal{I}}^+(n\lambda) \neq \emptyset$ for all $n \in \mathbb{Z}_{>0}$, and the above argument and Lemma 3.1 imply $D_{\mathcal{I}}^+(n\lambda) = \{nw\lambda\}$ for the same w. Furthermore, if $C_{\mathcal{I}}^+(\lambda')$ is nonempty for another $\lambda' \in Q_I^{++}$, then the above argument combined with the last assertion of Lemma 3.1 yields $D_{\mathcal{I}}^+(\lambda') = \{w\lambda'\}$ for the same $w \in W^I$.

We claim that for all $\mu \in Q_I^+$, $w\mu \in C_{\mathcal{I}}^+(\mu)$. There exists $n \in \mathbb{Z}_{>0}$ such that $\mu \leq n\lambda$. If $c^{\mu}_{\xi, v_\mu} \in \mathcal{I}$ for some $\xi \in V(\mu)^*_{-w\mu}$, $\xi \neq 0$, then this would force $c_w^{n\lambda - \mu} c^{\mu}_{\xi, v_\mu} \in \mathcal{I}$ for $n \in \mathbb{Z}_{>0}$ with the above property. Recall the well-known fact that $\dim V(\eta)^*_{-w\eta} = 1$ for all $\eta \in Q_I^+$. It follows that c^{μ}_{ξ, v_μ} is a nonzero scalar multiple of c_w^{μ}. Hence, by §2.3, $c_w^{n\lambda - \mu} c^{\mu}_{\xi, v_\mu}$ is a nonzero scalar multiple of $c_w^{n\lambda}$. In particular, $c_w^{n\lambda} \in \mathcal{I}$ and $w(n\lambda) \notin C_{\mathcal{I}}^+(n\lambda)$, since $\dim V(n\lambda)^*_{-w(n\lambda)} = 1$. This is a contradiction.

Next, we verify that $w\mu \in D_{\mathcal{I}}^+(\mu)$ for all $\mu \in Q_I^+$. If not, then there exists $\gamma = \sum_{i=1}^r n_i \alpha_i^\vee$, $n_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=1}^r n_i > 0$ such that $w\mu - \gamma \in D_{\mathcal{I}}^+(\mu)$. It follows that $\mathcal{J}_\lambda(w\mu - \gamma) \subset \mathcal{I}$. Let $\xi \in V(\mu)^*_{-w\mu+\gamma}$ be such that $c^{\mu}_{\xi, v_\mu} \notin \mathcal{I}$. Then the image of c^{μ}_{ξ, v_μ} in $R_q[G/P_I]/\mathcal{I}$ is normal by (3.3) and thus $c_w^{\lambda} c^{\mu}_{\xi, v_\mu} \notin \mathcal{I}$. This implies $w(\lambda + \mu) - \gamma \in C_{\mathcal{I}}^+(\lambda + \mu)$. At the same time $\lambda + \mu \in Q_I^{++}$ forces $D_{\mathcal{I}}^+(\lambda + \mu) = \{w(\lambda + \mu)\}$, which is a contradiction.

To obtain $D_{\mathcal{I}}^+(\mu) = \{w\mu\}$ for all $\mu \in Q_I^+$, one needs to show that $D_{\mathcal{I}}^+(\mu) \supsetneq \{w\mu\}$ is impossible. If $\eta \in D_{\mathcal{I}}^+(\mu) \backslash \{w\mu\}$, then the argument at the beginning of the proof and Lemma 3.1 imply $\eta = y\mu$ for some $y \in W^I$, $y \neq w$. Using normality again one gets $w\lambda + y\mu \in C_{\mathcal{I}}^+(\lambda + \mu)$, so $w\lambda + y\mu \geq w(\lambda + \mu)$ since $\lambda + \mu \in Q_I^{++}$. Thus $w\mu \leq y\mu$, which is a contradiction to $y\mu \in D_{\mathcal{I}}^+(\mu)$. This completes the proof of the lemma. $\square$

For $w \in W^I$ define the quantum Schubert ideals

$$\mathcal{Q}(w)_I^+ = \mathrm{Span}\{c^\lambda_{\xi,v_\lambda} \mid \lambda \in Q_I^+, \xi \in V(\lambda)^*,\ \xi \perp \mathcal{U}_+ T_w v_\lambda\} \tag{3.5}$$

of $R_q[G/P_I]$, where "$\perp$" means orthogonal with respect to the pairing between $V(\lambda)^*$ and $V(\lambda)$, cf. [10, 18, 9, 7]. The ideal $\mathcal{Q}(w)_I^+$ is completely prime since it is the intersection of the completely prime ideal $\mathcal{Q}(w)^+_{\{1,\dots,r\}}$ of R^+ (see [9, Proposition 10.1.8]) with $R_q[G/P_I]$.

Proposition 3.3. *For all $w \in W^I$, if $\mathcal{I}$ is a prime ideal of $R_q[G/P_I]$ with $D^+_{\mathcal{I}}(\lambda) = \{w\lambda\}$ for all $\lambda \in Q_I^+$, then*

$$\mathcal{Q}(w)_I^+ \subseteq \mathcal{I}.$$

Proof. We use the idea of the proof of [9, Corollary 10.1.13]. Let $\mathcal{I} \in X_w^I$. We need to prove that

(*) *If $\lambda \in Q_I^+$, $\xi \in V(\lambda)_{-\mu}$, $\xi \perp \mathcal{U}_+ v_\lambda$, $\mu \in Q$, then $c^\lambda_{\xi,v_\lambda} \in \mathcal{I}$.*

First we show (*) for $\lambda \in Q_I^{++}$. Fix $\lambda \in Q_I^{++}$. Assume that (*) is not correct, and choose $c^\lambda_{\xi,v_\lambda}$ with the property $c^\lambda_{\xi,v_\lambda} \notin \mathcal{I}$, $c^\lambda_{\xi,v_\lambda} \in \mathcal{Q}(w)_I^+$ such that $\mu \in Q$ is minimal. Using $\mathcal{J}_\lambda(w\lambda) \subseteq \mathcal{I}$ and applying (3.3), we obtain

$$c^\lambda_{\xi,v_\lambda} c^\lambda_w - q^{\langle\lambda,\lambda\rangle - \langle\mu,w\lambda\rangle} c^\lambda_w c^\lambda_{\xi,v_\lambda} \in \mathcal{I}. \tag{3.6}$$

From [9, Lemma 10.1.11 (i)] one has

$$c^\lambda_{\xi_{1,\lambda},v_\lambda} c^\lambda_{\xi,v_\lambda} = q^{\langle\lambda,\lambda\rangle - \langle\mu,\lambda\rangle} c^\lambda_{\xi,v_\lambda} c^\lambda_{\xi_{1,\lambda},v_\lambda} \tag{3.7}$$

(see §2.4 for the definition of $\xi_{y,\lambda}$ for $y \in W$). Denote by $\mathcal{U}_+^+$ the subalgebra of $\mathcal{U}_+$ generated by $\{X_i^+\}_{i=1}^r$. The minimality property of μ and the fact that $\mathcal{Q}(w)_I^+$ is invariant under the right action of $\mathcal{U}_+$ (see (2.1)) imply that $c^\lambda_{a\xi,v_\lambda} \in \mathcal{I}$ for all $a \in \mathcal{U}_+^+$. Choose $a \in \mathcal{U}_+^+$ such that $\xi_{w,\lambda} = a\xi_{1,\lambda}$. Acting by $S(a)$ on (3.7) and using the right action (2.1) of $\mathcal{U}_+$ one gets

$$c^\lambda_w c^\lambda_{\xi,v_\lambda} - q^{\langle\lambda,\lambda\rangle - \langle\mu,w\lambda\rangle} c^\lambda_{\xi,v_\lambda} c^\lambda_w \in \mathcal{I}. \tag{3.8}$$

Comparing (3.6) and (3.8) and using the fact that the image of $c^\lambda_w \notin \mathcal{I}$ in $R_q[G/P_I]/\mathcal{I}$ is normal imply that $\langle\lambda,\lambda\rangle = \langle\mu,w\lambda\rangle$. Lemma 3.1 implies that $\mu = w\lambda$, which is a contradiction to the fact that $c^\lambda_w \notin \mathcal{Q}(w)_I^+$.

Finally we prove (*) for $\lambda \in Q_I^+$. Let $\lambda' \in Q_I^{++}$. If $c^\lambda_{\xi,v_\lambda} \in \mathcal{Q}(w)_I^+$, then $c^{\lambda'}_w c^\lambda_{\xi,v_\lambda} = c^{\lambda'+\lambda}_{\xi',v_{\lambda'+\lambda}}$ for some $\xi' \in V(\lambda'+\lambda)^*$, $\xi' \perp \mathcal{U}_+ v_{\lambda'+\lambda}$. Since $\lambda'+\lambda \in Q_I^{++}$, $c^{\lambda'+\lambda}_{\xi',v_{\lambda'+\lambda}} \in \mathcal{I}$. Because the image of $c^\lambda_w \notin \mathcal{I}$ in $R_q[G/P_I]/\mathcal{I}$ is normal, $c^\lambda_{\xi,v_\lambda} \in \mathcal{I}$. □

3.4. The next Lemma describes further invariance properties of $H-\mathrm{Spec}_+ R_q[G/P_I]$.

Lemma 3.4. *Every $\mathcal{I} \in H-\mathrm{Spec}_+ R_q[G/P_I]$ is also invariant under the left action of H.*

Proof. Let $\mathcal{I} \in X_w^I$, $w \in W^I$. For all $\lambda \in Q_I^+$ the images $\bar{c}^\lambda_w$ of c^λ_w in $R_q[G/P_I]/\mathcal{I}$ are normal and do not vanish; thus they are not zero divisors. Let $\mu \in Q$, $\lambda_j \in Q_I^+$ and $\xi_j \in V(\lambda_j)_\mu$, $j = 1,\dots,l$, be such that $\sum_{j=1}^l c^{\lambda_j}_{\xi_j,v_{\lambda_j}} \in \mathcal{I}$ and $\lambda_1,\dots,\lambda_l$ are distinct. Equation (3.3) implies

$$0 = \Big(\sum_{j=1}^l c^{\lambda_j}_{\xi_j,v_{\lambda_j}} + \mathcal{I}\Big)\bar{c}^\lambda_w = \bar{c}^\lambda_w\Big(\sum_{j=1}^l q^{\langle\lambda_j,\lambda\rangle - \langle\mu,w\lambda\rangle} c^{\lambda_j}_{\xi_j,v_{\lambda_j}} + \mathcal{I}\Big).$$

Since $\bar{c}_w^\lambda$ are not zero divisors,

$$\sum_{j=1}^{l} q^{\langle \lambda_j, \lambda\rangle - \langle \mu, w\lambda\rangle} c^{\lambda_j}_{\xi, v_{\lambda_j}} \in \mathcal{I}.$$

Since $\lambda \in Q_I^+$ is arbitrary, this implies that $c^{\lambda_j}_{\xi_j, v_{\lambda_j}} \in \mathcal{I}$ for all j. So $\mathcal{I}$ is invariant under the left action of H. $\square$

Denote by $\widetilde{c}_w^\lambda$ the image of c_w^λ in $R_q[G/P_I]/\mathcal{Q}(w)_I^+$. Set $\widetilde{c}_w^I = \{\widetilde{c}_w^\lambda \mid \lambda \in Q_I^+\}$ and

$$R_{I,w} := \big(R_q[G/P_I]/\mathcal{Q}(w)_I^+\big)[(\widetilde{c}_w^I)^{-1}].$$

For $\mu \in Q_I$ denote $c_w^\mu = c_w^{-\lambda_1} c_w^{\lambda_2} \in R_{I,w}$ whenever $\mu = \lambda_1 - \lambda_2$, $\lambda_1, \lambda_2 \in Q_I^+$. This is independent of the choice of λ_1, λ_2; cf. §2.3. Then:

$$R_{I,w} = \mathrm{Span}\{\widetilde{c}_w^{-\lambda'}(c^\lambda_{\xi, v_\lambda} + \mathcal{Q}(w)_I^+) \mid \lambda, \lambda' \in Q_I^+, \xi \in V(\lambda)^*\}.$$

Denote by $R_{I,w}^H$ the invariant subalgebra of $R_{I,w}$ with respect to the induced left action of H. We have

$$R_{I,w}^H = \{\widetilde{c}_w^{-\lambda}(c^\lambda_{\xi, v_\lambda} + \mathcal{Q}(w)_I^+) \mid \lambda \in Q_I^+, \xi \in V(\lambda)^*\}. \tag{3.9}$$

There is no need to take the Span of the right hand side of (3.9) because:

(**) *For all $\lambda, \lambda' \in Q_I^+$, $\xi \in V(\lambda)^*$, there exists $\xi' \in V(\lambda + \lambda')^*$ such that $c_w^{-\lambda} c^\lambda_{\xi, V_\lambda} = c_w^{-\lambda-\lambda'} c^\lambda_{\xi', v_{\lambda+\lambda'}}$.*

Denote by $H - \mathrm{Spec} R_{I,w}$ and $H - \mathrm{Spec} R_{I,w}^H$ the sets of H-invariant prime ideals of $R_{I,w}$ and $R_{I,w}^H$ with respect to the induced right action of H.

If $\mathcal{I} \in X_w^I$, then $c_w^I \cap \mathcal{I} = \emptyset$ and $\mathcal{I} \supset \mathcal{Q}(w)_I^+$ by Theorem 3.2 and Proposition 3.3. Therefore [14, 2.1.16(vii)] the map

$$\mathcal{I} \mapsto (\mathcal{I}/\mathcal{Q}(w)_I^+) R_{I,w} \subset R_{I,w} \tag{3.10}$$

defines an order preserving bijection between X_w^I and $H - \mathrm{Spec} R_{I,w}$. We have $R_{I,w} = R_{I,w}^H\{\widetilde{c}_w^\mu \mid \mu \in Q_I\}$. The weight lattice Q acts on $R_{I,w}^H$ by ring automorphisms by

$$\begin{aligned}\mu \cdot \widetilde{c}_w^{-\lambda}(c^\lambda_{\xi, v_\lambda} + \mathcal{Q}(w)_I^+) &= \widetilde{c}_w^\mu \widetilde{c}_w^{-\lambda}(c^\lambda_{\xi, v_\lambda} + \mathcal{Q}(w)_I^+)\widetilde{c}_w^{-\mu} \\ &= q^{\langle \mu' - w\lambda, w\mu\rangle} \widetilde{c}_w^{-\lambda}(c^\lambda_{\xi, v_\lambda} + \mathcal{Q}(w)_I^+), \quad \text{for } \xi \in V(\lambda)^*_{\mu'}.\end{aligned}$$

It is clear that

$$R_{I,w} \cong R_{I,w}^H * Q_I \tag{3.11}$$

(see (3.3)), where $*$ stands for skew-group ring.

Let $\mathcal{J} \in H - \mathrm{Spec} R_{I,w}$. Lemma 3.4 implies that each ideal in X_w^I is invariant under both the left and right actions of H. From the bijection (3.10) we obtain that the same is true for the ideal $\mathcal{J}$ and thus

$$\mathcal{J} = \mathcal{J}^H\{\widetilde{c}_w^\mu \mid \mu \in Q_I\} = \{\widetilde{c}_w^\mu \mid \mu \in Q_I\}\mathcal{J}^H, \tag{3.12}$$

where $\mathcal{J}^H := \mathcal{J} \cap R_{I,w}^H$. From (3.11) one obtains that $R_{I,w}/\mathcal{J} \cong (R_{I,w}^H/\mathcal{J}^H) * Q_I$. Since $R_{I,w}/\mathcal{J}$ is prime, $R_{I,w}^H/\mathcal{J}^H$ is Q_I-prime[1] (see for instance the remark after [16, Theorem II]). This implies that $R_{I,w}^H/\mathcal{J}^H$ is H-prime, because a subset of $R_{I,w}^H/\mathcal{J}^H$

[1]We recall that a ring R acted upon a group M by ring automorphisms is called M-prime if there are no nontrivial M-invariant ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ of R such that $\mathcal{J}_1\mathcal{J}_2 = 0$. An M-invariant ideal $\mathcal{I}$ of R is called M-prime if $R/\mathcal{I}$ is M-prime.

which is closed under the right action of H induced from (2.1) is necessarily closed under the action of Q_I. Therefore $\mathcal{J}^H$ is an H-prime ideal of $R^H_{I,w}$. In Theorem 3.6 below we prove that $R^H_{I,w}$ and $\mathcal{U}^w_-$ are isomorphic H-algebras. The latter is an iterated skew polynomial ring, and Proposition 4.2 of Goodearl and Letzter [5] applies to give that all H-primes of $\mathcal{U}^w_-$ are completely prime. In particular, $\mathcal{J}^H$ is an H-invariant prime ideal of $R^H_{I,w}$.

In the opposite direction, if $\mathcal{J}_0 \in H-\mathrm{Spec}R^H_{I,w}$, then $\mathcal{J} := \mathcal{J}_0\{\widetilde{c}^\mu_w \mid \mu \in Q_I\}$ is a two sided ideal of $R_{I,w}$ (invariant under both actions of H) and $R_{I,w}/\mathcal{J} \cong (R^H_{I,w}/\mathcal{J}_0) * Q_I$. Since $R_{I,w}/\mathcal{J}_0$ is prime and Q_I is torsion free, Theorem II of Passman [16] implies that $R_{I,w}/\mathcal{J}$ is prime and thus $\mathcal{J} \in H-\mathrm{Spec}R_{I,w}$. Therefore (3.12) defines an order preserving bijection between $H-\mathrm{Spec}R_{I,w}$ and $H-\mathrm{Spec}R^H_{I,w}$. We obtain:

Proposition 3.5. *The map*

$$\mathcal{I} \in X^I_w \mapsto (\mathcal{I}/\mathcal{Q}(w)^+_I)R_{I,w} \cap R^H_{I,w} \in H-\mathrm{Spec}R^H_{I,w}$$

defines an order preserving bijection from X^I_w to $H-\mathrm{Spec}R^H_{I,w}$. All ideals in X^I_w are completely prime.

One shows the last statement as follows. It was already indicated that all ideals in $H-\mathrm{Spec}R^H_{I,w}$ are completely prime. If $\mathcal{J} \in H-\mathrm{Spec}R_{I,w}$, then $\mathcal{J}^H := \mathcal{J} \cap R^H_{I,w} \in H-\mathrm{Spec}R^H_{I,w}$ and $R_{I,w}/\mathcal{J} \cong (R^H_{I,w}/\mathcal{J}^H) * Q_I$. Thus $\mathcal{J}$ is completely prime. Finally, if $\mathcal{I} \in X^I_w$, then $(\mathcal{I}/\mathcal{Q}(w)^+_I)R_{I,w} \in H-\mathrm{Spec}R_{I,w}$ has to be completely prime. Therefore $\mathcal{I}/\mathcal{Q}(w)^+_I$ and $\mathcal{I}$ are completely prime too.

3.5. Similarly to (3.9) (see (**)) one has

$$\Big(\big(R_q[G/P_I]\big)[(c^I_w)^{-1}]\Big)^H = \{c^{-\lambda}_w c^\lambda_{\xi,v_\lambda} \mid \lambda \in Q^+_I, \xi \in V(\lambda)^*\},$$

where the invariant subalgebra is computed with respect to the left action of H. Define

$$\mathcal{Q}(w)^+_{I,w} = \{c^{-\lambda}_w c^\lambda_{\xi,v_\lambda} \mid \lambda \in Q^+_I, \xi \in V(\lambda)^*, \xi \perp \mathcal{U}_+ T_w v_\lambda\} \subset \Big(\big(R_q[G/P_I]\big)[(c^I_w)^{-1}]\Big)^H;$$

cf. [7, §6.1.2] and (3.5). Clearly $\mathcal{Q}(w)^+_{I,w}$ is an ideal of $\Big(\big(R_q[G/P_I]\big)[(c^I_w)^{-1}]\Big)^H$ (see (**)), and one has the algebra isomorphism

$$\Big(\big(R_q[G/P_I]\big)[(c^I_w)^{-1}]\Big)^H/\mathcal{Q}(w)^+_{I,w} \cong R^H_{I,w}, \quad c^{-\lambda}_w c^\lambda_{\xi,v_\lambda} + \mathcal{Q}(w)^+_{I,w} \mapsto \widetilde{c}^{-\lambda}_w\big(c^\lambda_{\xi,v_\lambda} + \mathcal{Q}(w)^+_I\big), \ \ \lambda \in Q^+_I, \xi \in V(\lambda)^*. \tag{3.13}$$

Analogously to the proof of [19, Theorem 3.7] (cf. also [4, Theorem 3.2]), one shows that the $\mathbb{K}$-linear map

$$\phi_w\colon \Big(\big(R_q[G/P_I]\big)[(c^I_w)^{-1}]\Big)^H \to \mathcal{U}^w_-, \quad \phi_w(c^{-\lambda}_w c^\lambda_{\xi,v_\lambda}) = (c^\lambda_{\xi,T_w v_\lambda} \otimes \mathrm{id})(\mathcal{R}^w), \ \lambda \in Q^+_I, \xi \in V(\lambda)^*, \tag{3.14}$$

is well defined and is an H-equivariant algebra homomorphism. On the first algebra one uses the right action of H induced from (2.1). On the second algebra one uses the restriction of the action

$$K \cdot x = KxK^{-1}, \quad K \in H, x \in \mathcal{U}_q(\mathfrak{g}), \tag{3.15}$$

of H on $\mathcal{U}_q(\mathfrak{g})$ to $\mathcal{U}_-^w$; cf. [19, (3.18)].

Theorem 3.6. *The map* $\phi_w \colon \big(\big(R_q[G/P_I]\big)[(c_w^I)^{-1}]\big)^H \to \mathcal{U}_-^w$ *is a surjective* H*-equivariant algebra homomorphism with kernel* $\mathcal{Q}(w)_{I,w}^+$*. It induces an* H*-equivariant algebra isomorphism between* $R_{I,w}^H$ *and* $\mathcal{U}_-^w$*.*

Proof. Recall that each element of $\big(\big(R_q[G/P_I]\big)[(c_w^I)^{-1}]\big)^H$ is of the form $c_w^{-\lambda} c_{\xi,v_\lambda}^{\lambda}$ for some $\lambda \in Q_I^+$, $\xi \in V(\lambda)^*$. It belongs to the kernel of ϕ_w if and only if $\langle \xi, xT_w v_\lambda \rangle = 0$ for all $x \in \mathcal{U}_+^w$ (i.e. for all $x \in \mathcal{U}_+$). This is equivalent to $c_w^{-\lambda} c_{\xi,v_\lambda}^{\lambda} \in \mathcal{Q}(w)_{I,w}^+$.

The proof of the surjectivity of ϕ_w is similar to the one of [19, Proposition 3.6]. Assuming that ϕ_w is not surjective would imply that there exists $X \in \mathcal{U}_+^w$, $X \neq 0$, such that $\langle \xi, XT_w v_\lambda \rangle = 0$ for all $\lambda \in Q_I^+$, $\xi \in V(\lambda)^*$. Then $X_1 = T_w^{-1}(X) \in \mathcal{U}^-$, $X_1 \neq 0$, would satisfy $\langle \xi, X_1 v_\lambda \rangle = 0$ for all $\lambda \in Q_I^+$, $\xi \in V(\lambda)^*$. The latter is impossible, since as $\mathcal{U}^-$-modules one has

$$V(\lambda) \cong \mathcal{U}^- v_\lambda \Big/ \Big(\sum_{i \notin I} \mathcal{U}^- (X_i^-)^{\langle \lambda, \alpha_i^\vee \rangle + 1} v_\lambda \Big); \tag{3.16}$$

see e.g. [9, Theorem 4.3.6 (i)].

The second assertion now follows using (3.13). □

Recall [19, Theorem 3.8] proved using Gorelik's results [7]:

Theorem 3.7. *For the* H*-action* (3.15)*, the set* $H - \mathrm{Spec}\mathcal{U}_-^w$ *of* H*-invariant prime ideals of* $\mathcal{U}_-^w$ *ordered under inclusion is isomorphic to* $W^{\leq w}$ *as a poset.*

Equation (3.4), Proposition 3.5 and Theorem 3.7 imply the main result of this paper:

Theorem 3.8. *For any quantum partial flag variety* $R_q[G/P_I]$ *the* H*-invariant prime ideals of* $R_q[G/P_I]$ *(recall* (3.1)*) not containing* $\mathcal{J}_+^I$ *are parametrized by*

$$\{(w, v) \in W^I \times W \mid v \leq w\}. \tag{3.17}$$

All such ideals are completely prime.

Denote by $\mathcal{I}_{w,v}^I$ the H-invariant prime ideal of $R_q[G/P_I]$ in X_w^I which corresponds to the ideal $I_w(v)$ of [19, Theorem 3.8] under the order preserving bijection from Proposition 3.5 and the isomorphism from Theorem 3.6. Tracing back those bijections and using the poset part of the statement of Theorem 3.7, one obtains that for all $v, v' \leq w$:

$$\mathcal{I}_{w,v}^I \subseteq \mathcal{I}_{w,v'}^I \text{ if and only if } v \leq v'. \tag{3.18}$$

This proves the special case of Conjecture 1.2 when $w = w'$, but the general statement is harder.

Remark 3.9. One can define the algebras $\mathcal{U}_q(\mathfrak{g})$, $R_q[G/P_I]$, $\mathcal{U}_-^w$ over any field $\mathbb{K}$ (not necessarily of characteristic 0), for $q \in \mathbb{K}$ which is not a root of unity. In this more general setting Mériaux and Cauchon [15] proved that the H-invariant prime ideals of $\mathcal{U}_-^w$ are parametrized by $W^{\leq w}$ (though the inclusions between them are unknown). All results of this section trivially carry out to this more general setting. As a result one obtains that there is a bijection between $H - \mathrm{Spec}_+ R_q[G/P_I]$ and the set (3.17) for the case when $\mathcal{U}_q(\mathfrak{g})$ is defined over an arbitrary field $\mathbb{K}$ and $q \in \mathbb{K}$ is not a root of unity.

3.6. Throughout this subsection fix $\lambda \in Q_I^{++}$. Consider the subalgebra [10, 18]

$$R_q[G/P_I]^\lambda = \mathrm{Span}\{c^{n\lambda}_{\xi,v_{n\lambda}} \mid n \in \mathbb{Z}_{\geq 0}, \xi \in V(n\lambda)\}$$

of $R_q[G/P_I]$. It is a deformation of the coordinate ring of the cone

$$\mathrm{Spec}\Big(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} H^0(G/P_I, \mathcal{L}_{n\lambda})\Big)$$

over G/P_I associated to $\lambda \in Q_I^{++}$; see §2.2 for the definition of the line bundles $\mathcal{L}_{n\lambda}$.

Define the ideal

$$\mathcal{J}_+^\lambda = \mathrm{Span}\{c^{n\lambda}_{\xi,v_{n\lambda}} \mid n \in \mathbb{Z}_{>0}, \xi \in V(n\lambda)^*\}$$

of $R_q[G/P_I]^\lambda$, cf. (3.1). Denote by $H - \mathrm{Spec}_+ R_q[G/P_I]^\lambda$ the set of H-invariant prime ideals of $R_q[G/P_I]^\lambda$ under the right action (2.1) of H which do not contain the ideal $\mathcal{J}_+^\lambda$.

For an ideal $\mathcal{I}$ of $R_q[G/P_I]^\lambda$ and $n \in \mathbb{Z}_{>0}$ define

$$C_{\mathcal{I}}^+(n) = \{\mu \in Q \mid \exists\, \xi \in V(n\lambda)^*_{-\mu} \text{ such that } c^{n\lambda}_{\xi,v_{n\lambda}} \notin \mathcal{I}\}.$$

If $C_{\mathcal{I}}^+(n) = \emptyset$, let $D_{\mathcal{I}}^+(n) = \emptyset$. Otherwise denote by $D_{\mathcal{I}}^+(n)$ the set of minimal elements of $C_{\mathcal{I}}^+(n)$. Denote the quantum Schubert ideal [10, 18, 9, 7]

$$\mathcal{Q}(w)_\lambda^+ = \mathrm{Span}\{c^{n\lambda}_{\xi,v_{n\lambda}} \mid n \in \mathbb{Z}_{>0}, \xi \in V(\lambda)^*, \xi \perp \mathcal{U}_+ T_w v_\lambda\}$$

of $R_q[G/P_I]^\lambda$; cf. (3.5). Analogously to Theorem 3.2 and Proposition 3.3 one shows:

Proposition 3.10. (1) *For each prime ideal $\mathcal{I}$ of $R_q[G/P_I]^\lambda$ which does not contain $\mathcal{J}_+^\lambda$ there exists $w \in W^I$ such that $D_{\mathcal{I}}^+(n) = \{nw\lambda\}$ for all $n \in \mathbb{Z}_{>0}$.*

(2) *For a given $w \in W^I$, all prime ideals of $R_q[G/P_I]^\lambda$ satisfying the condition in* (1) *contain the ideal $\mathcal{Q}(w)_\lambda^+$.*

Given $w \in W^I$, let X_w^λ be the set of H-invariant prime ideals $\mathcal{I}$ of $R_q[G/P_I]^\lambda$ such that $D_{\mathcal{I}}^+(n) = \{nw\lambda\}$, $\forall n \in \mathbb{Z}_{>0}$. Then $X_w^\lambda \subset H - \mathrm{Spec}_+ R_q[G/P_I]^\lambda$ and

$$H - \mathrm{Spec}_+ R_q[G/P_I]^\lambda = \bigsqcup_{w\in W^I} X_w^\lambda.$$

Similarly to [9, Lemma 9.1.10] one shows that $\{(c_w^\lambda)^n\}_{n\in\mathbb{Z}_{\geq 0}}$ is an Ore subset of $R_q[G/P_I]^\lambda$. Let $\overline{c}_w^\lambda$ be the image of c_w^λ in $R_q[G/P_I]^\lambda/\mathcal{Q}(w)_\lambda^+$. Set

$$R_{\lambda,w} := \big(R_q[G/P_I]^\lambda/\mathcal{Q}(w)_\lambda^+\big)[(\overline{c}_w^\lambda)^{-1}].$$

Consider the induced left action of H on $R_{\lambda,w}$ from (2.1), and denote by $R^H_{\lambda,w}$ the corresponding invariant subalgebra. Similarly to Proposition 3.5 one establishes that:

There is an order preserving bijection between X^λ_w and $H-\mathrm{Spec}R^H_{\lambda,w}$ given by

$$\mathcal{I}\in X^\lambda_w\mapsto(\mathcal{I}/\mathcal{Q}(w)^+_\lambda)R_{\lambda,w}\cap R^H_{\lambda,w}\in H-\mathrm{Spec}R^H_{\lambda,w}, \tag{3.19}$$

where $H-\mathrm{Spec}$ refers to the set of H-invariant prime ideals with respect to the induced right action from (2.1)*. All ideals in X^λ_w are completely prime.*

One has (see (**))

$$\Big(\big(R_q[G/P_I]^\lambda\big)[(c^\lambda_w)^{-1}]\Big)^H=\{c^{-n\lambda}_w c^{n\lambda}_{\xi,v_{n\lambda}}\mid n\in\mathbb{Z}_{\geq 0},\xi\in V(n\lambda)^*\},$$

where $(.)^H$ denotes the invariant subalgebra with respect to the induced left H-action from (2.1).

Similarly to Theorem 3.6 one proves:

Proposition 3.11. *The $\mathbb{K}$-linear map*

$$\psi_w\colon\Big(\big(R_q[G/P_I]^\lambda\big)[(c^\lambda_w)^{-1}]\Big)^H\to\mathcal{U}^w_-,$$
$$\psi_w(c^{-n\lambda}_w c^{n\lambda}_{\xi,v_{n\lambda}})=(c^{n\lambda}_{\xi,T_wv_{n\lambda}}\otimes\mathrm{id})(\mathcal{R}^w),\ n\in\mathbb{Z}_{\geq 0},\xi\in V(n\lambda)^*,$$

is an H-equivariant surjective algebra homomorphism with kernel

$$\{c^{-n\lambda}_w c^{n\lambda}_{\xi,v_{n\lambda}}\mid n\in\mathbb{Z}_{>0},\xi\in V(n\lambda)^*,\xi\perp\mathcal{U}_+T_wv_{n\lambda}\}.$$

Here H acts on the the first algebra by the induced right action from (2.1) *and on the second algebra by* (3.15)*.*

The homomorphism ψ_w induces an H-equivariant algebra isomorphism between $R^H_{\lambda,w}$ and $\mathcal{U}^w_-$.

Invoking Theorem 3.7, one obtains:

Theorem 3.12. *For all $\lambda\in Q^{++}_I$, the H-invariant prime ideals of $R_q[G/P_I]^\lambda$ not containing $\mathcal{J}^\lambda_+$ are parametrized by the set*

$$\{(w,v)\in W^I\times W\mid v\leq w\}.$$

All such ideals are completely prime.

Denote by $\mathcal{I}^\lambda_{w,v}$ the ideal of $R_q[G/P_I]^\lambda$ which corresponds to the ideal $I_w(v)$ of $\mathcal{U}^w_-$ of [19, Theorem 3.8] under the bijections of (3.19) and Proposition 3.11. We conjecture:

Conjecture 3.13. *Let $(w,v),(w',v')\in S_{W,I}$; see* (1.1)*. One has $\mathcal{I}^\lambda_{w,v}\subseteq\mathcal{I}^\lambda_{w',v'}$ if and only if there exists $z\in W_I$ such that*

$$w\geq w'z\ \text{ and }\ v\leq v'z.$$

Analogously to (3.18) one uses the poset part of the statement of Theorem 3.7, the order preserving bijections (3.19) and Proposition 3.11 to prove the case of Conjecture 3.13 when $w=w'$.

Remark 3.14. Similarly to Remark 3.9 one can define the algebras $R_q[G/P_I]^\lambda$, over any field $\mathbb{K}$ (not necessarily of characteristic 0), for $q\in\mathbb{K}$ which is not a root of unity. The above arguments and the Mériaux–Cauchon [15] result parametrizing H-invariant prime ideals of $\mathcal{U}^w_-$ prove that the parametrization of H-primes of $R_q[G/P_I]^\lambda$ from Theorem 3.12 is valid in this more general situation.

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