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ON GROMOV'S SCALAR CURVATURE CONJECTURE

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Abstract. We prove the Gromov conjecture on the macroscopic dimension of the universal covering of a closed spin manifold with a positive scalar curvature under the following assumptions on the fundamental group.

0.1. **Theorem.** *Suppose that a discrete group π has the following properties:*

1. *The Strong Novikov Conjecture holds for π.*
2. *The natural map per : $ko_n(B\pi) \to KO_n(B\pi)$ is injective.*

Then the Gromov Macroscopic Dimension Conjecture holds true for spin n-manifolds M with the fundamental groups $\pi_1(M)$ that contain π as a finite index subgroup.

1. Introduction

In his study of manifolds with positive scalar curvature M. Gromov observed some large scale dimensional deficiency of their universal coverings: For an n-dimensional manifold M, its universal covering has to be at most $(n-2)$-dimensional from the macroscopic point of view. For example, the product of a closed $(n-2)$-manifold N and the standard 2-sphere is an n-manifold $M = N \times S^2$ that admits a metric of positive scalar curvature (by making the 2-sphere small). The universal covering $\widetilde{M} = \widetilde{N} \times S^2$ looks like an $(n-2)$-dimensional space $\widetilde{N}$. Gromov predicted similar behavior for all manifolds with positive scalar curvature. He stated it in [G1] as the following.

1.1. **Conjecture (Gromov).** *For every closed Riemannian n-manifold (M, g) with a positive scalar curvature there is the inequality*

$$\dim_{mc}(\widetilde{M}, \tilde{g}) \leq n - 2,$$

where $(\widetilde{M}, \tilde{g})$ is the universal cover of M with the pull-back metric.

Here $\dim_{\mathrm{mc}}$ stands for the *macroscopic dimension* [G1]. The first time this conjecture was stated was in the famous "filling" paper [G2] in a different language. In [GL] the conjecture was proved for 3-manifolds.

1.2. **Definition.** A map $f : X \to K$ of a metric space is called *uniformly cobounded* if there is $D > 0$ such that $\mathrm{diam}(f^{-1}(y)) \leq D$.

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A metric space X has the *macroscopic dimension* $\dim_{\mathrm{mc}} X \leq n$ if there is a uniformly cobounded proper continuous map $f : X \to K^n$ to an n-dimensional polyhedron.

In [G1] Gromov asked the following questions related to his conjecture which were stated in [B1], [B2] in the form of a conjecture:

1.3. **Conjecture (C1).** *Let (M, g) be a closed Riemannian n-manifold with torsion free fundamental group, and let $\widetilde{M}$ be the universal covering of M with the pull-back metric. Suppose that $\dim_{mc} \widetilde{M} < n$. Then*

(A) *If $\dim_{mc} \widetilde{M} < n$, then $\dim_{mc} \widetilde{M} \leq n-2$.*

(B) *If a classifying map for the universal covering map $f : M \to B\pi$ can be deformed to a map with $f(M) \subset B\pi^{(n-1)}$, then it can be deformed to a map with $f(M) \subset B\pi^{(n-2)}$.*

The Conjecture **C1** is proven for $n = 3$ by D. Bolotov in [B1]. In [B2] it was disproved for $n > 3$ by a counterexample. It turns out that Bolotov's example does not admit a metric of positive scalar curvature [B3], and hence it does not affect the Gromov Conjecture 1.1.

Perhaps the most famous conjecture on manifolds of positive scalar curvature is

The Gromov-Lawson Conjecture [GL]**.** *A closed spin n-manifold M admits a metric of positive scalar curvature if and only if $f_*([M]_{KO}) = 0$ in $KO_n(B\pi)$, where $f : M \to B\pi$ is a classifying map for the universal covering of M and $[M]_{KO}$ is the KO-theory fundamental class of M.*

J. Rosenberg connected the Gromov-Lawson conjecture with the Novikov conjecture. Namely, he proved [R] that $\alpha f_*([M]_{KO}) = 0$ in $KO_n(C^*(\pi))$ in the presence of positive scalar curvature where α is the assembly map.

1.4. **Conjecture (Strong Novikov Conjecture).** *The analytic assembly map*

$$\alpha : KO_*(B\pi) \longrightarrow KO_*(C^*(\pi))$$

is a monomorphism.

Then Rosenberg and Stolz proved the Gromov-Lawson conjecture for manifolds with the fundamental group π which satisfies the Strong Novikov conjecture and has the natural transformation map

$$per : ko_*(B\pi) \to KO_*(B\pi)$$

injective ([RS], Theorem 4.13).

The main goal of this paper is to prove the Gromov Conjecture 1.1 under the Rosenberg-Stolz conditions.

2. Connective spectra and n-connected complexes

We refer to the textbook [Ru] on the subject of spectra. We recall that for every spectrum $\mathbf{E}$ there is a connective cover $\mathbf{e} \to \mathbf{E}$, i.e., the spectrum $\mathbf{e}$ with the morphism $\mathbf{e} \to \mathbf{E}$ that induces the isomorphisms for $\pi_i(\mathbf{e}) = \pi_i(\mathbf{E})$ for $i \geq 0$ and with $\pi_i(\mathbf{e}) = 0$ for $i < 0$. By KO we denote the spectrum for real K-theory, by ko its connective cover, and by $per : ko \to KO$ the corresponding transformation (morphism of spectra). We will use both the old and new notation for an $\mathbf{E}$-homology of a space X: old-fashioned $\mathbf{E}_*(X)$ and modern $H_*(X; \mathbf{E})$. We recall

that $KO_n(pt) = \mathbb{Z}$ if $n = 0$ or $n = 4$ mod 8, $KO_n(pt) = \mathbb{Z}_2$ if $n = 1$ or $n = 2$ mod 8, and $KO_n(pt) = 0$ for all other values of n. By $\mathbf{S}$ we denote the spherical spectrum. Note that for any ring spectrum $\mathbf{E}$ there is a natural morphism $\mathbf{S} \to \mathbf{E}$ which leads to the natural transformation of the stable homotopy to $\mathbf{E}$-homology $\pi_*^s(X) \to H_*(X; \mathbf{E})$.

2.1. **Proposition.** *Let X be an $(n-1)$-connected $(n+1)$-dimensional CW complex. Then X is homotopy equivalent to the wedge of spheres of dimensions n and $n+1$ together with the Moore spaces $M(\mathbb{Z}_m, n)$.*

Proof. This is a partial case of the Minimal Cell Structure Theorem (see Proposition 4C.1 and Example 4C.2 in [Ha]). □

2.2. **Proposition.** *The natural transformation $\pi_*^s(pt) \to ko_*(pt)$ induces an isomorphism $\pi_n^s(K/K^{(n-2)}) \to ko_n(K/K^{(n-2)})$ for any CW complex K.*

Proof. Since π^s and ko are both connective, it suffices to show that

$$\pi_n^s(K^{(n+1)}/K^{(n-2)}) \to ko_n(K^{(n+1)}/K^{(n-2)})$$

is an isomorphism. Consider the diagram generated by exact sequences of the pair $(K^{(n+1)}/K^{(n-2)}, K^{(n)}/K^{(n-2)})$

$$\begin{array}{ccccccc}
\oplus\mathbb{Z} & \longrightarrow & \pi_n^s(K^{(n)}/K^{(n-2)}) & \longrightarrow & \pi_n^s(K^{(n+1)}/K^{(n-2)}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\oplus\mathbb{Z} & \longrightarrow & ko_n(K^{(n)}/K^{(n-2)}) & \longrightarrow & ko_n(K^{(n+1)}/K^{(n-2)}) & \longrightarrow & 0.
\end{array}$$

Since the left vertical arrow is an isomorphism and the right vertical arrow is an isomorphism of zero groups, it suffices to show that $\pi_n^s(K^{(n)}/K^{(n-2)}) \to ko_{(n)}(K^{(n)}/K^{(n-2)})$ is an isomorphism.

Note that $\pi_n(S^k) \to ko_n(S^k)$ is an isomorphism for $k = n, n-1$. In view of Proposition 2.1 it suffices to show that $\pi_n^s(M(\mathbb{Z}_m, n-1)) \to ko_n(M(\mathbb{Z}_m, n-1))$ is an isomorphism for any m and n. This follows from the Five Lemma applied to the cofibration $S^{n-1} \to S^{n-1} \to M(\mathbb{Z}_m, n-1)$. □

3. Inessential manifolds

We recall the following definition, which is due to Gromov.

3.1. **Definition.** An n-manifold M is called *inessential* if it admits a map $f : M \to K^{n-1}$ to an $(n-1)$-dimensional complex that induces an isomorphism on fundamental groups. Note that one can always take K^{n-1} to be the $(n-1)$-skeleton $B\pi^{(n-1)}$ of the classifying space $B\pi$ of the fundamental group $\pi = \pi_1(M)$.

The following is well-known to experts.

3.2. **Proposition.** *An orientable n-manifold M is inessential if and only if $f_*([M]) \in H_n(B\pi)$ is zero for a map $f : M \to B\pi_1(M)$ classifying the universal covering of M.*

Proof. If M admits a classifying map $f : M \to B\pi^{(n-1)}$, then clearly $f_*([M]) = 0$.

Let $f_*([M]) = 0$ for some map $f : M \to B\pi_1(M)$ that induces an isomorphism of the fundamental groups. Let $o_n(f) \in H^n(M; \pi_{n-1}(F))$ be the primary obstruction to deform f to the $(n-1)$-dimensional skeleton $B\pi^{(n-1)}$, and let $o_n(1_{B\pi}) \in H^n(B\pi; \pi_{n-1}(F))$ be the primary obstruction to retraction of $B\pi$ to the

$(n-1)$-skeleton. Here F denotes the homotopy fiber of the inclusion $B\pi^{(n-1)} \to B\pi$ and $\pi_{n-1}(F)$ is considered as a π-module. Since f_* induces an isomorphism of the fundamental groups, $f_* : H_0(M;\pi_{n-1}(F)) = \pi_{n-1}(F)_\pi \to H_0(B\pi;\pi_{n-1}(F)) = \pi_{n-1}(F)_\pi$ is an isomorphism. Then $f_*([M]\cap o_n(f)) = f_*([M])\cap o_n(1_{B\pi}) = 0$. By the Poincaré duality, $o_n(f) = 0$. □

3.3. **Proposition.** *An orientable spin n-manifold M is inessential if $f_*([M]_{ko}) \in ko_n(B\pi)$ is zero for a map $f : M \to B\pi$ classifying the universal covering.*

Proof. By the definition, a ko-fundamental class $[M]_{ko}$ of an n-manifold goes to the generator of $ko_n(S^n) = \mathbb{Z}$ under any degree-one map $g : M \to S^n$.

There is a natural transformation T of ko to the Eilenberg-MacLane spectrum $H(\mathbb{Z})$ which is induced by the 0-dimensional cohomology of the spectrum ko (see [Ru]). Since on the n-sphere this transformation induces an isomorphism of n-homology groups $ko_n(S^n) \to H_n(S^n)$, it sends the ko-fundamental class $[M]_{ko}$ to the fundamental class $[M]$. From the commutativity of the diagram

$$\begin{array}{ccc} ko_n(M) & \xrightarrow{f_*} & ko_n(B\pi) \\ {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow \\ H_n(M) & \xrightarrow{f_*} & H_n(B\pi), \end{array}$$

it follows that $f_*([M]) = 0$. Proposition 3.2 completes the proof. □

There are many ways to detect essentiality of manifolds. One of them deals with the Lusternik-Schnirelmann category of X, $\operatorname{cat}_{\mathrm{LS}} X$, which is the minimal m such that X admits an open cover $U_0, \dots, U_m$ contractible in X.

3.4. **Theorem.** *A closed n-manifold is essential if and only if its Lusternik-Schnirelmann category equals n.*

We refer to [CLOT] for the proof and more facts about the Lusternik-Schnirelmann category. Note that $\operatorname{cat}_{\mathrm{LS}} X$ is estimated from below by the cup-length of X possibly with twisted coefficient and it's estimated from above by the dimension of X. The definition of the Lusternik-Schnirelmann category can be reformulated in terms of existence of a section of some universal fibration (called Ganea's fibration). The characteristic class arising from the universal Ganea fibration over the classifying space $B\pi$ is called the Berstein-Švarc class $\beta_\pi \in H^1(\pi; I(\pi))$ of π, where $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}(\pi)$ (see [Ber], [Sv], [CLOT]. Formally, β_π is the image of the generator under the connecting homomorphism $H^0(\pi;\mathbb{Z}) \to H^1(\pi; I(\pi))$ in the long exact sequence generated by the short exact sequence of coefficients

$$0 \to I(\pi) \to \mathbb{Z}(\pi) \to \mathbb{Z} \to 0.$$

The main property of β_π is universality: Every cohomology class $\alpha \in H^k(\pi; L)$ is the image of $(\beta_\pi)^k$ under a suitable coefficients homomorphism $I(\pi)^k = I(\pi)\otimes\cdots\otimes I(\pi) \to L$. We refer to [DR] (see also [Sv]) for more details.

3.5. **Lemma.** *Let M be a closed inessential n-manifold, $n \geq 4$, supplied with a CW complex structure and let $\pi = \pi_1(M)$. Then M admits a classifying map $f : M \to B\pi$ of the universal covering such that $f(M) \subset B\pi^{(n-1)}$ and $f(M^{(n-1)}) \subset B\pi^{(n-2)}$.*

Proof. Let $f : M \to B\pi$ be the classifying map for the universal covering of M. We may assume that $f|_{M^{(2)}:M^{(2)}} \to B\pi^{(2)}$ is the identity map. First we show that $f^*(\beta^{n-1}) = 0$, where $\beta = \beta_\pi$ is the Berstein-Švarc class of π. Assume that $f^*(\beta^{n-1}) \neq 0$. Then $a = [M] \cap f^*(\beta^{n-1}) \neq 0$ by the Poincaré Duality Theorem [Br]. There is $u \in H^1(X; \mathcal{A})$ such that $a \cap u \neq 0$ for some local system $\mathcal{A}$ (see Proposition 2.3 in [DKR]). Then $f^*(\beta)^{n-1} \cup u \neq 0$. Thus the twisted cup-length of M is at least n, and hence $\mathrm{cat}_{\mathrm{LS}}\, M = n$. This contradicts Theorem 3.4.

Let $g : M \to B\pi^{(n-1)}$ be a cellular classifying map. Consider the obstruction $o^{n-1} \in H^{n-1}(M; \pi_{n-2}(B\pi^{(n-2)}))$ for the extension of $g|_{M^{(n-2)}}$ to the $(n-1)$-skeleton $M^{(n-1)}$. Here $\pi_{n-2}(B\pi^{(n-2)})$ is considered as a π-module. Note that $o^{n-1} = g^*(o_B^{n-1})$, where o_B^{n-1} is the obstruction for retraction of $B\pi^{(n-1)}$ to $B\pi^{(n-2)}$. In view of the universality of the Berstein-Švarc class there is a morphism of π-modules $I(\pi)^{n-1} \to \pi_{n-2}(B\pi^{(n-2)})$ such that o_B^{n-1} is the image of β^{n-1} under the induced cohomology homomorphism. The square diagram induced by g and the fact that $g^*(\beta_\pi^{n-1}) = 0$ imply that $o^{n-1} = 0$. Therefore there is a map $f' : M^{(n-1)} \to B\pi^{(n-2)}$ that coincides with g on $M^{(n-3)}$. Clearly, for $n \geq 5$, this map induces an isomorphism of the fundamental groups. It is still the case for $n = 4$, since g is the identity on the 1-skeleton, and the 2-skeletons of M and $B\pi$ are taken to be the same.

We show that there is an extension $f : M \to B\pi^{(n-1)}$. It suffices to show that the inclusion homomorphism $\pi_{n-1}(B\pi^{(n-2)}) \to \pi_{n-1}(B\pi^{(n-1)})$ is trivial. This homomorphism coincides with the homomorphism

$$\pi_{n-1}(E\pi^{(n-2)}) \to \pi_{n-1}(E\pi^{(n-1)}).$$

Since $E\pi^{(n-2)}$ is contractible in $E\pi$, by the Cellular Approximation Theorem it is contractible in $E\pi^{(n-1)}$. This implies that the inclusion homomorphism is zero. □

4. The main theorem

4.1. **Lemma.** *Suppose that a classifying map $f : M \to B\pi$ of a closed spin n-manifold, $n > 3$, takes the ko_* fundamental class to 0, $f_*([M]_{ko}) = 0$. Then f is homotopic to a map $g : M \to B\pi^{(n-2)}$.*

Proof. In view of Proposition 3.3 we may assume that $f(M) \subset B\pi^{(n-1)}$. In view of Lemma 3.5 we may additionally assume that $f(M^{(n-1)}) \subset B\pi^{(n-2)}$. Also we assume that M has one n-dimensional cell. As in the proof of Proposition 3.3 we can say that the primary obstruction for moving f into the $(n-2)$-skeleton is defined by the cocycle $c_f : \pi_n(M, M^{(n-1)}) \to \pi_n(B\pi, B\pi^{n-2})$ which defines the cohomology class $o_f = [c_f]$ that lives in the group of coinvariants $\pi_n(B\pi, B\pi^{(n-2)})_\pi = \pi_n(B\pi/B\pi^{(n-2)})$ and is represented by $\bar{f}_*(1)$ for the homomorphism

$$\bar{f}_* : \pi_n(M/M^{(n-1)}) = \mathbb{Z} \to \pi_n(B\pi/B\pi^{(n-2)})$$

induced by the map of quotient spaces $\bar{f} : M/M^{(n-1)} \to B\pi/B\pi^{(n-2)}$.

We assume that the obstruction $[c_f]$ is nonzero. We show that $\bar{f}_* : ko_n(S^n) \to ko_n(B\pi/B\pi^{n-2})$ is nontrivial to obtain a contradiction as in the proof of Proposition 3.3. Thus, $\bar{f}_*(1)$ defines a nontrivial element of $\pi_n(B\pi/B\pi^{(n-2)})$. The restriction $n > 3$ implies that $\bar{f}_*(1)$ survives in the stable homotopy group. In view of Proposition 2.2, the element $\bar{f}_*(1)$ survives in the composition

$$\pi_n(B\pi/B\pi^{(n-2)}) \to \pi_n^s(B\pi/B\pi^{(n-2)}) \to ko_n(B\pi/B\pi^{(n-2)}).$$

The commutative diagram

$$\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{\bar{f}_*} & \pi_n(B\pi/B\pi^{(n-2)}) \\
\cong\big\downarrow & & \cong\big\downarrow \\
\pi_n^s(S^n) & \xrightarrow{\bar{f}_*} & \pi_n^s(B\pi/B\pi^{(n-2)}) \\
\cong\big\downarrow & & \cong\big\downarrow \\
ko_n(S^n) & \xrightarrow{\bar{f}_*} & ko_n(B\pi/B\pi^{(n-2)})
\end{array}$$

implies that $f_*(1) \neq 0$ for ko_n, a contradiction. □

The Strong Novikov Conjecture is connected to the Gromov Conjecture by means of the following theorem, which is due to J. Rosenberg.

4.2. **Theorem** ([R]). *Suppose M^n is a spin manifold with a fundamental group π. Let f be the classifying map $f : M^n \to B\pi$. If M^n is a positive scalar curvature manifold, then $\alpha f_*([M^n]_{KO}) = 0$, where $\alpha : KO_*(B\pi) \longrightarrow KO_*(C_r^*(\pi))$ is the analytic assembly map.*

4.3. **Theorem.** *Suppose that a discrete group π contains a finite index subgroup π' that has the following properties:*

1. *The Strong Novikov Conjecture holds for π'.*
2. *The natural map $per : ko_n(B\pi') \to KO_n(B\pi')$ is injective.*

Then the Gromov Conjecture holds for spin n-manifolds M with fundamental group $\pi_1(M) = \pi$.

Proof. Let M be a closed spin n-manifold that admits a metric with positive scalar curvature. Then its finite cover M' corresponding to the subgroup π' is a spin manifold with positive scalar curvature. By Theorem 4.2, $\alpha \circ per \circ f_*([M']_{ko}) = 0$. The conditions on π' imply that $f_*([M']_{ko}) = 0$ for the classifying map $f : M' \to B\pi'$. Then by Lemma 4.1, f is homotopic to $g : M' \to B\pi'^{(n-2)}$. In view of the fact that $\widetilde{M'} = \widetilde{M}$, the induced map of the universal covering spaces $\widetilde{M'} \to E\pi'^{(n-2)}$ produces the inequality $\dim_{\mathrm{mc}} \widetilde{M} \leq n-2$. □

We recall that the virtual cohomological dimension $vcd(\pi)$ of a discrete group π is the cohomological dimension of a torsion free finite index subgroup. It is known that $vcd(\pi)$ is well-defined if it is finite; i.e., it does not depend on the choice of a torsion free finite index subgroup.

4.4. **Corollary.** *The Gromov Conjecture holds for spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$ having $vcd(\pi) \leq n+4$ and satisfying the Strong Novikov Conjecture.*

Proof. We consider a finite index subgroup π' with $cd(\pi') \leq n+4$. We show that per is an isomorphism in dimension n for π'. Let $\mathbf{F} \to ko \to KO$ be the fibration of spectra induced by the morphism $ko \to KO$. Since $\pi_k(ko) \to \pi_k(KO)$ is an isomorphism for $k \geq 0$ and $\pi_{-1}(\mathbf{F}) = 0$, we have $\pi_k(\mathbf{F}) = 0$ for $k \geq -1$. Also $\pi_k(\mathbf{F}) = \pi_{k+1}(KO) = KO_{k+1}(pt) = 0$ if $k = -2, -3, -4 \bmod 8$. The Atiyah-Hirzebruch F-homology spectral sequence for $B\pi'$ implies that $H_n(B\pi'; \mathbf{F}) = 0$

since all entries on the n-diagonal in the E^2-term are 0. Then the coefficient exact sequence for homology

$$H_n(B\pi';\mathbf{F}) \to ko_n(B\pi') \to KO_n(B\pi') \to \dots$$

implies that $per : ko_n(B\pi') \to KO_n(B\pi')$ is a monomorphism. □

We note that this corollary for $cd(\pi) \leq n-1$ was first proven in [B3].

4.5. **Corollary.** *The Gromov Conjecture holds for spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$ having a finite index subgroup π' with finite complex $B\pi'$ and with asymptotic dimension* $\operatorname{asdim}\pi \leq n+4$.

Proof. This is a combination of the fact that the Strong Novikov conjecture holds true for such groups π' ([Ba], [DFW]), the above corollary, and the inequality $vcd(\pi) \leq \operatorname{asdim}\pi$ proven in [Dr]. □

4.6. **Corollary.** *The Gromov conjecture holds for spin n-manifolds M with the fundamental group $\pi_1(M)$ equal to the product of free groups $F_1 \times \dots \times F_n$. In particular, it holds for free abelian groups.*

Proof. For reduced homology with coefficients in a spectrum $\mathbf{E}$, the formula

$$H_i(X \times S^1;\mathbf{E}) \cong H_i(X;\mathbf{E}) \oplus H_{i-1}(X;\mathbf{E}) \oplus H_i(S^1)$$

implies that if $ko_*(X) \to KO_*(X)$ is a monomorphism, then $ko_*(X \times S^1) \to KO_*(X \times S^1)$ is a monomorphism. By induction on m using the Mayer-Vietoris sequence this formula can be generalized to the following:

$$H_i(X \times (\bigvee_m S^1);\mathbf{E}) \cong H_i(X;\mathbf{E}) \oplus \bigoplus_m (H_{i-1}(X;\mathbf{E}) \oplus H_i(S^1).$$

Therefore,

$$ko_*(X \times (\bigvee_m S^1)) \to KO_*(X \times (\bigvee_m S^1))$$

is a monomorphism. □

As we noted in the proof of Theorem 4.3, the Gromov Conjecture for spin manifolds with fundamental group π' implies the Gromov Conjecture for spin manifolds with fundamental groups π containing π' as a finite index subgroup. Since every finitely generated abelian group has the torsion subgroup finite, in view of the above corollary, we obtain the following.

4.7. **Corollary.** *The Gromov Conjecture holds true for spin n-manifolds M with virtually abelian fundamental groups $\pi_1(M)$.*

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