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NON-CROSSING LINKED PARTITIONS, THE PARTIAL ORDER $\ll$ ON $NC(n)$, AND THE S-TRANSFORM

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Abstract. The paper establishes a connection between two recent combinatorial developments in free probability: the non-crossing linked partitions introduced by Dykema in 2007 to study the S-transform, and the partial order $\ll$ on $NC(n)$ introduced by Belinschi and Nica in 2008 in order to study relations between free and Boolean probability. More precisely, one has a canonical bijection between $NCL(n)$ (the set of all non-crossing linked partitions of $\{1,\ldots,n\}$) and the set $\{(\alpha,\beta) \mid \alpha,\beta \in NC(n),\ \alpha \ll \beta\}$. As a consequence of this bijection, one gets an alternative description of Dykema's formula expressing the moments of a non-commutative random variable a in terms of the coefficients of the reciprocal S-transform $1/S_a$. Moreover, due to the Boolean features of $\ll$, this formula can be simplified to a form which resembles the moment-cumulant formula from c-free probability.

1. Introduction

This paper puts into evidence a connection between two seemingly unrelated combinatorial developments in free probability, which appeared recently in the papers [1] and [5].

The basic structure for combinatorial considerations in free probability is the poset $(NC(n), \leqslant)$, where $NC(n)$ is the set of all non-crossing partitions of $\{1,\ldots,n\}$, and "$\leqslant$" is the reverse refinement order for such partitions. $NC(n)$ enters free probability theory via the free cumulants introduced in [10]; a detailed exposition of how this happens can be found in Part II of the monograph [7]. Both papers [1], [5] build on the basic combinatorics of $NC(n)$, but appear to go in different directions, following different goals.

On the one hand, in 2007, Dykema's paper [5] introduced the concept of "non-crossing linked partition" and used it to study the S-transform (an important transform introduced by Voiculescu [12] in order to treat the operation of multiplying free random variables). In particular, it was shown in [5] how the moment of order n of a non-commutative random variable a can be expressed in terms of the coefficients of the series $1/S_a(z)$ via a summation over $NCL(n)$, where $S_a(z)$ is the S-transform of a and where $NCL(n)$ denotes the set of all non-crossing linked

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partitions of $\{1, \ldots, n\}$. Following up [5], further work related to $NCL(n)$ was done in [4], [8]. A review of $NCL(n)$ and of its connection to the S-transform is given in Section 2B and Section 4 below.

On the other hand, in 2008, the paper [1] by Belinschi and Nica introduced a partial order "$\ll$" on $NC(n)$, coarser than the reverse refinement order. The partial order $\ll$ was used in [1] and subsequently in [2] in order to study connections between free probability and Boolean probability (another brand of non-commutative probability, where instead of the $NC(n)$'s one works with Boolean posets). A useful fact which illustrates how $\ll$ mixes together features from the non-crossing and the Boolean worlds is the following: for a fixed $\beta \in NC(n)$, the set $\{\alpha \in NC(n) \mid \alpha \ll \beta\}$ can be identified as an interval with respect to the reverse refinement order, and is consequently counted by a product of Catalan numbers. But for a fixed $\alpha \in NC(n)$, the set $\{\beta \in NC(n) \mid \beta \gg \alpha\}$ is isomorphic to a Boolean poset and is thus counted by a power of 2. (See the review in Propositions 2.4 and 2.5 below.)

The present paper puts into evidence a canonical bijection from $NCL(n)$ onto the set $\{(\alpha, \beta) \mid \alpha, \beta \in NC(n),\ \alpha \ll \beta\}$. If $\pi \in NCL(n)$ is mapped by this bijection to (α, β), then:

• β is the partition generated by the blocks of π. (Note that the blocks of a linked partition are allowed, under certain special circumstances, to not be disjoint. Hence π does not have to be itself a partition, and it may happen that $\beta \neq \pi$.)

• α is obtained by applying a certain "cycling" procedure to a partition $\breve{\pi} \in NC(n)$, which is defined in [5] and is called the "unlinking of π".

The precise details of how this works are given in Theorem 3.4 of this paper.

As a consequence of the above bijection, one gets an alternative description of Dykema's formula expressing the moments of a non-commutative random variable a in terms of the coefficients of the reciprocal S-transform $1/S_a$. Moreover, due to the Boolean features of $\ll$, this formula can be simplified to a form which expresses the moment of order n of a by using a summation over $NC(n)$ (instead of $NCL(n)$), and where the term indexed by $\alpha \in NC(n)$ in the summation can still be canonically written as a product over the set of blocks of α. The precise description of how this goes is given in Theorem 4.5 below. It is worth noting that equation (4.27) from Theorem 4.5 has a close resemblance to the moment-cumulant formula developed in [3] for the framework of conditionally free random variables (another framework where the free and Boolean probability worlds interact).

Besides the introduction, the paper has three other sections. Section 2 contains a review of the background and notation, Section 3 presents the canonical bijection advertised above, and Section 4 examines the application of this bijection to the reciprocal S-transform.

2. Background and notation

2A. Non-crossing partitions and the partial order $\ll$. We will use the standard conventions of notation for non-crossing partitions (as in [9], or in Lecture 9 of [7]). The partial order given by reverse refinement on $NC(n)$ will be simply denoted as "$\leqslant$"; in other words, for $\alpha, \beta \in NC(n)$, we write "$\alpha \leqslant \beta$" to mean that every block of β is a union of blocks of α. The minimal and maximal elements of $(NC(n), \leqslant)$ are denoted by 0_n (the partition of $\{1, \ldots, n\}$ into n blocks of 1 element each) and respectively 1_n (the partition of $\{1, \ldots, n\}$ into 1 block of n elements).

The coarser partial order $\ll$ on $NC(n)$ was defined in [1] as follows.

Definition 2.1. For $\alpha, \beta \in NC(n)$ we write "$\alpha \ll \beta$" to mean that $\alpha \leqslant \beta$ and that, in addition, the following condition is fulfilled:

$$\left\{ \begin{array}{l} \text{For every block } W \text{ of } \beta \text{ there exists a block} \\ V \text{ of } \alpha \text{ such that } \min(W), \max(W) \in V. \end{array} \right. \tag{2.1}$$

It is immediate that if $\alpha \leqslant \beta$ in $NC(n)$ and if V, W are as in (2.1), then one must have $V \subseteq W$ and $\min(V) = \min(W)$, $\max(V) = \max(W)$. It will be convenient to give a special name to the blocks V of α that can be matched to a block W of β in this way.

Definition 2.2. Let α, β be partitions in $NC(n)$ such that $\alpha \ll \beta$. A block V of α will be said to be *β-special* when there exists a block W of β such that $\min(V) = \min(W)$ and $\max(V) = \max(W)$.

Remark 2.3. Let $\alpha, \beta \in NC(n)$ be such that $\alpha \ll \beta$.

1. The correspondence $V \mapsto W$ from Definition 2.2 clearly gives a bijection from the set $\{V \text{ block of } \alpha \mid V \text{ is } \beta\text{-special}\}$ onto the set of all blocks of β.

2. Let us recall that a block V of α is said to be *inner* (respectively *outer*) when there exists (respectively when there does not exist) another block V' of α such that $\min(V') < \min(V)$ and $\max(V') > \max(V)$. It is easily seen that every outer block of α is β-special; moreover, the correspondence $V \mapsto W$ from Definition 2.2 induces a bijection between the outer blocks of α and those of β; cf. Remarks 2.9 and 2.12 in [1].

The next two propositions state in more detail the facts mentioned in the introduction about sets of the form $\{\alpha \in NC(n) \mid \alpha \ll \beta\}$ and $\{\beta \in NC(n) \mid \beta \gg \alpha\}$.

Proposition 2.4. *Let $\beta = \{W_1, \ldots, W_q\}$ be a partition in $NC(n)$. Consider the partition $\beta_0 \in NC(n)$ obtained by refining β as follows: every block W_j with $|W_j| \leqslant 2$ is left intact, while every block W_j with $|W_j| \geqslant 3$ is broken into the doubleton $\{\min(W_j), \max(W_j)\}$ and $|W_j| - 2$ singletons. (Thus every block of β_0 has either 1 or 2 elements.) Then*

$$\{\alpha \in NC(n) \mid \alpha \ll \beta\} = \{\alpha \in NC(n) \mid \beta_0 \leqslant \alpha \leqslant \beta\}. \tag{2.2}$$

As a consequence, one has that

$$\mid \{\alpha \in NC(n) \mid \alpha \ll \beta\} \mid \; = \prod_{j=1}^{q} Cat_{|W_j|-1}, \tag{2.3}$$

where, for every $k \geqslant 0$, we denote $Cat_k := (2k)!/\big(k!(k+1)!\big)$ (the kth Catalan number).

Proof. The equality in (2.2) follows immediately from how the partial order $\ll$ is defined. The right-hand side of (2.2) is the interval $[\beta_0, \beta]$ with respect to the reverse refinement order, and every interval of $(NC(n), \leqslant)$ is known to be canonically isomorphic (as a poset) to a direct product of lattices $NC(m)$, $2 \leqslant m \leqslant n$; see the detailed discussion on pp. 149-150 of [7], which also gives a concrete algorithm for how to obtain the canonical factorization of the interval. By following this

algorithm it is immediately found that

$$[\beta_0, \beta] \simeq \prod_{\substack{1 \leqslant j \leqslant q \text{ such} \\ \text{that } |W_j| \geqslant 3}} NC(\,|W_j| - 1\,), \tag{2.4}$$

and (2.3) follows by taking cardinalities in (2.4). □

The next proposition uses the abbreviations $V \in \alpha$ for "V is a block of α" and $\mathcal{V} \subseteq \alpha$ for "$\mathcal{V}$ is a set of blocks of α", where α is a partition in $NC(n)$. For the proof of this proposition, the reader is referred to Proposition 2.13 and Remark 2.14 of [1].

Proposition 2.5. *Let α be in $NC(n)$ and consider the set of partitions*

$$\{\beta \in NC(n) \mid \beta \gg \alpha\}. \tag{2.5}$$

Then $\beta \mapsto \{V \in \alpha \mid V$ is β-special$\}$ is a one-to-one map from the set (2.5) *to the set of subsets of α. The image of this map is equal to $\{\mathcal{V} \subseteq \alpha \mid \mathcal{V}$ contains all outer blocks of $\alpha\}$.*

2B. Linked partitions and $NCL(n)$. Following [5], we will use the term "*linked partition* of $\{1, \ldots, n\}$" for a set $\pi = \{A_1, \ldots, A_p\}$ of non-empty subsets of $\{1, \ldots, n\}$ such that $A_1 \cup \cdots \cup A_p = \{1, \ldots, n\}$, and where for every $i \neq j$ $(1 \leqslant i, j \leqslant p)$ one has that either $A_i \cap A_j = \varnothing$ or that the following holds:

$$\begin{cases} |A_i| \geqslant 2,\ |A_j| \geqslant 2,\ |A_i \cap A_j| = 1,\ \min(A_i) \neq \min(A_j), \text{ and} \\ \text{the unique element of } A_i \cap A_j \text{ is one of } \min(A_i), \min(A_j). \end{cases} \tag{2.6}$$

Obviously, every partition of $\{1, \ldots, n\}$ (in the usual sense of the term) is a linked partition of $\{1, \ldots, n\}$, but the converse is not true. Throughout this paper we use the letters $\alpha, \beta, \ldots$ to denote partitions, and $\pi, \rho, \ldots$ to denote linked partitions (which may or may not be partitions). A few more terms and basic facts from Section 5 of [5] are reviewed next.

Review 2.6. 1. If $\pi = \{A_1, \ldots, A_p\}$ is a linked partition of $\{1, \ldots, n\}$, then $A_1, \ldots, A_p$ are called the *blocks* of π. It is easy to see that every $m \in \{1, \ldots, n\}$ belongs either to exactly one or to exactly two blocks A_i. In the first case one says that m is *singly-covered* by π, and in the second case one says that m is *doubly-covered* by π.

2. Let π be a linked partition of $\{1, \ldots, n\}$. The partition of $\{1, \ldots, n\}$ which is generated by π will be denoted as $\widehat{\pi}$. In other words, $\widehat{\pi}$ is the smallest (with respect to the reverse refinement order) among all partitions β of $\{1, \ldots, n\}$ which have the following property: "for every block A of π there exists a block V of β such that $A \subseteq V$".

3. Let $\pi = \{A_1, \ldots, A_p\}$ be a linked partition of $\{1, \ldots, n\}$. For every $1 \leqslant i \leqslant p$ define

$$V_i = \begin{cases} A_i, & \text{if } \min(A_i) \text{ is singly-covered by } \pi, \\ A_i \backslash \{\min(A_i), & \text{if } \min(A_i) \text{ is doubly-covered by } \pi. \end{cases} \tag{2.7}$$

Then $\{V_1, \ldots, V_p\}$ is a partition of $\{1, \ldots, n\}$ called the *unlinking* of π and denoted as $\check{\pi}$.

4. A linked partition π of $\{1, \ldots, n\}$ is said to be *non-crossing* if it is not possible to find two distinct blocks A, B of π and elements $a, a' \in A$, $b, b' \in B$ such that $a < a' < b < b'$.

5. Let n be a positive integer. The set of all non-crossing linked partitions of $\{1,\dots,n\}$ is denoted by $NCL(n)$. It is not hard to see that if $\pi \in NCL(n)$, then the partitions $\widehat{\pi}$ and $\check{\pi}$ defined in 2 and 3 above belong to $NC(n)$.

Remark 2.7 (Restrictions of linked partitions). It is immediate that the above discussion about linked partitions could be carried out without any modifications in the larger framework where instead of $\{1,\dots,n\}$ one uses an abstract finite totally ordered set F, and one considers the set $NCL(F)$ of non-crossing linked partitions of F (instead of just sticking to the $NCL(n)$). This doesn't really bring anything new, since it is obvious that $NCL(F)$ is canonically identified to $NCL(\,|F|\,)$ via the map $\{A_1,\dots,A_p\} \mapsto \{f(A_1),\dots,f(A_p)\}$, where f is the unique order-preserving bijection from F onto $\{1,\dots,|F|\}$. But in the subsequent discussion it will be nevertheless convenient to allow linked partitions for a slightly more general kind of set F (specifically, for F being a non-empty finite subset of $\mathbb{N}$), in order to simplify the notation for restrictions of linked partitions.

So let $E \subseteq F$ be non-empty subsets of $\mathbb{N}$, and let $\pi \in NCL(F)$ be such that E is saturated with respect to π (which means that whenever A is a block of π and $A \cap E \neq \varnothing$, it follows that $A \subseteq E$). Then π is of the form $\{A_1,\dots,A_p,A'_1,\dots,A'_q\}$ with $A_1,\dots,A_p \subseteq E$ and $A'_1,\dots,A'_q \subseteq F\backslash E$, and one defines the *restriction* of π to E to be

$$\pi\,|_{\,E} := \{A_1,\dots,A_p\}. \tag{2.8}$$

It is immediate that $\pi\,|_{\,E} \in NCL(E)$. Moreover, it is easily seen that the operation of restriction is well-behaved with respect to the maps $\pi \mapsto \widehat{\pi}$ and $\pi \mapsto \check{\pi}$ from 2 and 3 of Review 2.6, in the sense that one has

$$\Big(\pi\,|_{\,E}\Big)^{\wedge} = \big(\widehat{\pi}\,\big)\,|_{\,E} \quad \text{and} \quad \Big(\pi\,|_{\,E}\Big)^{\vee} = \big(\check{\pi}\,\big)\,|_{\,E}. \tag{2.9}$$

Let us next record the observation (cf. [5], Corollary 5.13 and its proof) that non-crossing linked partitions can be broken into "irreducible" pieces, as follows.

Proposition 2.8. *Let n be a positive integer and let $\beta = \{W_1,\dots,W_q\}$ be a partition in $NC(n)$. The map*

$$\pi \mapsto \Big(\pi\,|_{\,W_1},\dots,\pi\,|_{\,W_q}\Big)$$

is a bijection from $\{\pi \in NCL(n) \mid \widehat{\pi} = \beta\}$ onto $\prod_{j=1}^{q}\{\pi_j \in NCL(W_j) \mid \widehat{\pi}_j = 1_{W_j}\}$ (where $1_{W_j} \in NC(W_j)$ is the partition of W_j into only one block).

So, after suitably renumbering every W_j from the preceding proposition as $\{1,\dots,|W_j|\}$, one is reduced in the end to looking at sets of non-crossing linked partitions of the form $\{\pi \in NCL(m) \mid \widehat{\pi} = 1_m\}$. In Proposition 5.11 of [5] it is pointed out that the latter sets can be identified with sets of usual non-crossing partitions, as follows.

Proposition 2.9. *For every $n \geqslant 2$, the map $\pi \mapsto \check{\pi}$ is a bijection from $\{\pi \in NCL(n) \mid \widehat{\pi} = 1_n\}$ onto $\{\alpha \in NC(n) \mid 1$ and 2 belong to the same block of $\alpha\}$.*

3. The canonical bijection relating $NCL(n)$ to $\ll$

This section is devoted to proving the bijection announced in the introduction. The main result is Theorem 3.4. We start by recalling some connections between set-partitions and permutations and by defining precisely what is the "cycled unlinking" of a linked partition $\pi \in NCL(n)$.

Notation 3.1. 1. If τ is a permutation of $\{1,\ldots,n\}$ and if $\alpha = \{V_1,\ldots,V_p\}$ is a partition of $\{1,\ldots,n\}$, then we denote

$$\tau\cdot\alpha := \{\tau(V_1),\ldots,\tau(V_p)\} \quad \text{(a new partition of } \{1,\ldots,n\}). \tag{3.10}$$

It is clear that formula (3.10) defines an action of the symmetric group $\mathcal{S}_n$ on the set of all partitions of $\{1,\ldots,n\}$.

2. Every partition $\alpha\in NC(n)$ has associated to it a permutation of $\{1,\ldots,n\}$, which is denoted by P_α and is defined by the following prescription: for every block $V=\{i_1,\ldots,i_m\}$ of π, with $i_1<\cdots<i_m$, one creates a cycle of P_α by putting

$$P_\alpha(i_1)=i_2,\ldots,P_\alpha(i_{m-1})=i_m, P_\alpha(i_m)=i_1.$$

Definition 3.2. Let π be in $NCL(n)$. Consider the partitions $\widehat{\pi},\breve{\pi}\in NC(n)$ (as in Review 2.6) and form the new partition

$$\overset{\circ}{\pi} := P_{\widehat{\pi}}^{-1}\cdot\breve{\pi} \tag{3.11}$$

(where the permutation $P_{\widehat{\pi}}$ is defined as in Notation 3.1.2). The partition $\overset{\circ}{\pi}$ will be called the *cycled unlinking* of π.

Example 3.3 (A concrete example)**.** Say for instance that $n=11$ and that

$$\pi=\big\{\{1,2,4\},\{2,3\},\{4,5,6\},\{6,7\},\{8,9,11\},\{9,10\}\big\}\in NCL(11).$$

Then one has

$$\widehat{\pi}=\big\{\{1,2,3,4,5,6,7\},\{8,9,10,11\}\big\},\quad \breve{\pi}=\big\{\{1,2,4\},\{3\},\{5,6\},\{7\},\{8,9,11\},\{10\}\big\}.$$

The permutation associated to $\widehat{\pi}$ is

$$P_{\widehat{\pi}}=\big(1,2,3,4,5,6,7\big)\ \big(8,9,10,11\big)\quad \text{(written in cycle notation);}$$

hence the cycled unlinking of π is

$$\overset{\circ}{\pi}=P_{\widehat{\pi}}^{-1}\cdot\breve{\pi}\ =\big\{\{1,3,7\},\{2\},\{4,5\},\{6\},\{8,10,11\},\{9\}\big\}.$$

Note that $\overset{\circ}{\pi}\in NC(11)$ and $\overset{\circ}{\pi}\ll\widehat{\pi}$ (we will see that this must generally be the case).

Theorem 3.4. *Let n be a positive integer. The map $\pi\mapsto\big(\overset{\circ}{\pi},\widehat{\pi}\big)$ is a bijection from $NCL(n)$ onto the set $\{(\alpha,\beta)\mid\alpha,\beta\in NC(n),\ \alpha\ll\beta\}$.*

For the proof of Theorem 3.4, it is convenient to first establish some lemmas.

Lemma 3.5. *Let β be a partition in $NC(n)$. The map $\alpha\mapsto P_\beta^{-1}\cdot\alpha$ sends the set $\{\alpha\in NC(n)\mid\alpha\leqslant\beta\}$ into itself.*

Proof. Let us fix an $\alpha\in NC(n)$ such that $\alpha\leqslant\beta$, and let us consider the partition $\alpha':=P_\beta^{-1}\cdot\alpha$. Observe that $\alpha'\leqslant\beta$ in reverse refinement order, i.e. that every block V' of α' is contained in a block of β. Indeed, V' can be written as $V'=P_\beta^{-1}(V)$, where V is a block of α, and the block W of β which contains V must contain V' as well (since $V'\subseteq P_\beta^{-1}(W)=W$).

It remains to show that the partition α' is non-crossing. Let us fix two distinct blocks V_1' and V_2' of α', and let us prove that V_1' and V_2' do not cross. Consider the blocks W_1,W_2 of β such that $W_1\supseteq V_1'$ and $W_2\supseteq V_2'$, and distinguish the following two cases.

Case 1. $W_1\neq W_2$. In this case, W_1 and W_2 do not cross, so the smaller sets V_1' and V_2' can't cross either.

Case 2. $W_1 = W_2 =: W$. In this case, consider the sets $V_1 := P_\beta(V_1')$ and $V_2 := P_\beta(V_2')$. These are two distinct blocks of α, which are still contained in W. We know that V_1 and V_2 don't cross (since α is in $NC(n)$), and we will conclude the proof by arguing that if V_1' and V_2' would cross, then so would V_1 and V_2. Indeed, suppose that $i_1, j_1 \in V_1'$ and $i_2, j_2 \in V_2'$ are such that $i_1 < i_2 < j_1 < j_2$. If j_2 is not the maximal element of W, then it follows that $P_\beta(i_1) < P_\beta(i_2) < P_\beta(j_1) < P_\beta(j_2)$, which is a crossing between V_1 and V_2. In the opposite situation when $j_2 = \max(W)$, it follows that $P_\beta(j_2) < P_\beta(i_1) < P_\beta(i_2) < P_\beta(j_1)$, and we get a crossing between V_2 and V_1 (since $P_\beta(j_2), P_\beta(i_2) \in V_2$ and $P_\beta(i_1), P_\beta(j_1) \in V_1$). □

Lemma 3.6. *Let π be in $NCL(n)$. Then the partition $\overset{\circ}{\pi}$ is in $NC(n)$, and $\overset{\circ}{\pi} \ll \widehat{\pi}$.*

Proof. Since $\check{\pi}, \widehat{\pi} \in NC(n)$ and $\check{\pi} \leqslant \widehat{\pi}$, the preceding lemma gives us that $\overset{\circ}{\pi} = P_{\widehat{\pi}}^{-1} \cdot \check{\pi}$ is in $NC(n)$ and that $\overset{\circ}{\pi} \leqslant \widehat{\pi}$.

It remains to show that, for every block W of $\widehat{\pi}$, the numbers $\min(W)$ and $\max(W)$ belong to the same block of $\overset{\circ}{\pi}$. For the rest of the proof we fix such a W, with $|W| =: m \geqslant 2$ (the case $|W| = 1$ is obvious). Let us write explicitly $W = \{i_1, i_2, \ldots, i_m\}$, with $i_1 < i_2 < \cdots < i_m$; our goal is then to prove that i_1 and i_m belong to the same block of $\overset{\circ}{\pi}$.

Consider the linked partition $\pi \mid_W \in NCL(W)$, which has $\left(\pi \mid_W\right)^\wedge = \widehat{\pi} \mid_W = 1_W$. Proposition 2.9 gives us (after going through the suitable identification of $i_1, i_2, \ldots, i_m$ with $1, 2, \ldots, m$) that i_1 and i_2 belong to the same block of $\left(\pi \mid_W\right)^\vee$. But $\left(\pi \mid_W\right)^\vee = \check{\pi} \mid_W$ (cf. equation (2.9)); hence it follows that i_1 and i_2 belong to the same block V of $\check{\pi}$. This implies in turn that $P_{\widehat{\pi}}^{-1}(i_1)$ and $P_{\widehat{\pi}}^{-1}(i_2)$ belong to the same block $P_{\widehat{\pi}}^{-1}(V)$ of $\overset{\circ}{\pi}$, but $P_{\widehat{\pi}}^{-1}(i_1) = i_m$ and $P_{\widehat{\pi}}^{-1}(i_2) = i_1$, so we are done. □

Lemma 3.7. *Let $\pi, \rho \in NCL(n)$ be such that $\widehat{\pi} = \widehat{\rho}$ and $\check{\pi} = \check{\rho}$. Then $\pi = \rho$.*

Proof. Let us denote $\widehat{\pi} = \widehat{\rho} =: \beta \in NC(n)$, and let us write explicitly $\beta = \{W_1, \ldots, W_q\}$. In order to prove that $\pi = \rho$ it suffices to verify that they have the same image by the bijection from Proposition 2.8, i.e. that $\pi \mid_{W_j} = \rho \mid_{W_j} \in NCL(W_j)$ for every $1 \leqslant j \leqslant q$.

So let us fix j ($1 \leqslant j \leqslant q$). Observe that $\left(\pi \mid_{W_j}\right)^\wedge = \widehat{\pi} \mid_{W_j} = \beta \mid_{W_j} = 1_{W_j}$, and similarly $\left(\rho \mid_{W_j}\right)^\wedge = 1_{W_j}$. Now, by invoking Proposition 2.9 (and by using the suitable renumbering of W_j into $\{1, \ldots, |W_j|\}$) we see that the map $\theta \mapsto \check{\theta}$ is one-to-one on $\{\theta \in NCL(W_j) \mid \widehat{\theta} = 1_{W_j}\}$. Thus the required fact that $\pi \mid_{W_j} = \rho \mid_{W_j}$ will follow if we can prove that $\left(\pi \mid_{W_j}\right)^\vee = \left(\rho \mid_{W_j}\right)^\vee$. But the latter equality amounts (in view of (2.9)) to $\check{\pi} \mid_{W_j} = \check{\rho} \mid_{W_j}$, and thus it follows from the hypothesis that $\check{\pi} = \check{\rho}$. □

Proof of Theorem 3.4. From Lemma 3.6 it follows that the map $\pi \mapsto (\overset{\circ}{\pi}, \widehat{\pi})$ is well-defined (with target set as described in the theorem). In order to prove the injectivity of this map, consider two linked partitions $\pi, \rho \in NCL(n)$ such that $\overset{\circ}{\pi} = \overset{\circ}{\rho} =: \alpha$ and $\widehat{\pi} = \widehat{\rho} =: \beta$. Then $\check{\pi} = \check{\rho} = P_\beta \cdot \alpha$; hence π and ρ satisfy the hypotheses of Lemma 3.7, and it follows that $\pi = \rho$.

In order to complete the proof of the theorem, it now suffices to verify that the sets $NCL(n)$ and $\{(\alpha, \beta) \mid \alpha, \beta \in NC(n),\ \alpha \ll \beta\}$ have the same cardinality. For

$NCL(n)$ we can invoke Corollary 5.13 of [5], which gives us the formula

$$|\,NCL(n)\,| \;=\; \sum_{\beta\in NC(n)} \Big(\prod_{W\in\beta} \mathrm{Cat}_{|W|-1}\Big). \tag{3.12}$$

But on the other hand we have

$$|\,\{(\alpha,\beta)\mid \alpha,\beta\in NC(n),\ \alpha\ll\beta\}\,| \;=\; \sum_{\beta\in NC(n)} |\,\{\alpha\mid \alpha\in NC(n),\ \alpha\ll\beta\}\,|\ ,$$

which is indeed equal to the right-hand side of (3.12), due to Proposition 2.4. □

Remark 3.8. Theorem 3.4 allows one to transfer enumeration properties in between the two sets involved in the bijection of the theorem. In particular, as it is known that $NCL(n)$ is counted by the $(n-1)$th Schröder number r_{n-1} (Theorem 8.3 in [5]; see also the discussion in Section 2 of [4]), it follows that the same is true for $\{(\alpha,\beta)\mid \alpha,\beta\in NC(n),\ \alpha\ll\beta\}$. By writing the latter set as $\bigcup_\alpha\{\beta\in NC(n)\mid \beta\gg\alpha\}$ and by invoking Proposition 2.5, one thus comes to the following amusing (possibly already known) enumerative interpretation of the Schröder number r_{n-1}: it counts coloured non-crossing partitions $\alpha\in NC(n)$, where every block of α is coloured in either red or blue, and all the outer blocks are red. For a nice collection of other enumerative interpretations of the Schröder numbers, see Example 6.2.8 and Exercise 6.39 of the monograph [11].

4. Application to the reciprocal S-transform

Let $(\mathcal{A},\varphi)$ be an algebraic non-commutative probability space (which simply means that $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi:\mathcal{A}\to\mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}})=1$), and let $a\in\mathcal{A}$ be such that $\varphi(a)=1$. The *S-transform* of a is the series

$$S_a(z) := \frac{1+z}{z} M_a^{\langle -1\rangle}(z) \in \mathbb{C}\,[[z]], \tag{4.13}$$

where $M_a(z):=\sum_{n=1}^\infty \varphi(a^n)z^n$ (the *moment series* of a), and where $M_a^{\langle -1\rangle}$ is the inverse of M_a under composition. The S-transform plays a fundamental role in the study of multiplication of free random variables (see Section 3.6 of [13] or Lecture 18 of [7] for the details of how this goes).

It is immediate that the series $S_a(z)$ from (4.13) has constant term equal to 1. Hence one can consider its reciprocal (i.e. inverse under multiplication) $1/S_a(z)$, which is another series with constant term equal to 1:

$$1/S_a(z) = \sum_{n=0}^\infty t_n z^n, \qquad \text{with } t_0=1. \tag{4.14}$$

In the paper [5] the series $1/S_a$ goes under the name of "*T-transform of a*". It is observed there that the formula giving back the family of moments $\{\varphi(a^n)\mid n\geqslant 1\}$ of a in terms of the family $\{t_n\mid n\geqslant 0\}$ of coefficients of $1/S_a$ only uses positive integer coefficients:

$$\begin{cases} \varphi(a)=1,\ \varphi(a^2)=t_1+1,\ \varphi(a^3)=t_2+t_1^2+3t_1+1, \\ \varphi(a^4)=t_3+3t_2t_1+t_1^3+4t_2+6t_1^2+6t_1+1,\ldots. \end{cases} \tag{4.15}$$

Moreover, [5] identifies precisely the combinatorial structure which indexes the sums in (4.15): this is nothing but $NCL(n)$, and the formula generalizing the special cases

from (4.15) is

$$\varphi(a^n) = \sum_{\pi \in NCL(n)} \Big(\prod_{A \in \pi} t_{|A|-1} \Big), \quad n \geqslant 1, \tag{4.16}$$

where "$A \in \pi$" is an abbreviation for "A is a block of π"; see Proposition 8.1 in [5].

In view of Theorem 3.4 of the present paper, the formula giving $\varphi(a^n)$ in terms of the coefficients of $1/S_a$ can be equivalently understood as a summation over the set $\{(\alpha, \beta) \mid \alpha, \beta \in NC(n),\ \alpha \ll \beta\}$. This alternative version of the formula will be stated in Proposition 4.3 below. In preparation of that statement, we prove two lemmas.

Lemma 4.1. *Let π be a linked partition in $NCL(n)$ such that $\widehat{\pi} = 1_n$ and let A be a block of π such that $A \not\ni 1$. Then* $\min(A)$ *is doubly-covered by π.*

Proof. Denote $\min(A) =: m$. Since 1 and m belong to the same (unique) block of $\widehat{\pi}$, there have to exist $p \geqslant 1$, some $m_0, m_1, \ldots, m_p \in \{1, \ldots, n\}$ and some blocks $A_1, \ldots, A_p$ of π such that $m_0 = 1$, $m_p = m$, and such that

$$m_0, m_1 \in A_1;\ m_1, m_2 \in A_2;\ \ldots, m_{p-1}, m_p \in A_p. \tag{4.17}$$

Let us suppose moreover that in (4.17) p is picked to be as small as possible. It is then immediate that $m_{j-1} \neq m_j$ for every $1 \leqslant j \leqslant p$, and that $A_{j-1} \neq A_j$ for every $2 \leqslant j \leqslant p$.

Observe that $m_1 \in A_1 \cap A_2$ and m_1 is not the minimum of A_1 (since $m_0 \in A_1$, and $m_0 = 1 < m_1$); from the definition of a linked partition it follows that $m_1 = \min(A_2)$. We next observe that $m_2 \in A_2 \cap A_3$ and $m_2 \neq \min(A_2)$ (since $\min(A_2) = m_1 \neq m_2$), so the same argument as above applies to give us that $m_2 = \min(A_3)$. Continuing like this by induction we find that $m_j = \min(A_{j+1})$ for every $1 \leqslant j \leqslant p-1$, and in particular that $m_{p-1} = \min(A_p)$.[1] Thus the block A_p of π contains $m_p = m$, and A_p is different from A (because $\min(A_p) = m_{p-1} \neq m_p = \min(A)$). This shows that m is doubly-covered by π, as required. □

Lemma 4.2. *Let π be a linked partition in $NCL(n)$, let A be a block of π, and let W be the unique block of $\widehat{\pi}$ such that $W \supseteq A$. Then*

$$\left(\begin{array}{c} \min(A) \textit{ is} \\ \textit{singly-covered by } \pi \end{array} \right) \Leftrightarrow \Big(\min(A) = \min(W) \Big). \tag{4.18}$$

Proof. By replacing π with $\pi\,|_W$ and by redenoting the elements of W as $\{1, 2, \ldots, |W|\}$ in increasing order, we may assume without loss of generality that $\widehat{\pi} = 1_n$ and hence that $W = \{1, \ldots, n\}$. The statement on the right-hand side of equivalence (4.18) becomes "$\min(A) = 1$". The implication "$\Leftarrow$" in this equivalence is then immediate (as the definition of a linked partition implies that 1 always is singly-covered), while the implication "$\Rightarrow$" follows from Lemma 4.1. □

Proposition 4.3. *Let $(\mathcal{A}, \varphi)$ be an algebraic non-commutative probability space, let $a \in \mathcal{A}$ be such that $\varphi(a) = 1$, and consider the reciprocal S-transform $1/S_a(z) =$*

[1] This argument was run by assuming that $p \geqslant 2$. If $p = 1$, then the conclusion that $m_{p-1} = \min(A_p)$ still holds, as we must have that $m_0 = 1 = \min(A_1)$.

$\sum_{n=0}^{\infty} t_n z^n$. *Then for every* $n \geqslant 1$ *one has*

$$\varphi(a^n) = \sum_{\substack{\alpha,\beta \in NC(n) \\ \text{such that } \alpha \ll \beta}} \Big(\prod_{\substack{U \in \alpha, \\ \beta\text{-special}}} t_{|U|-1} \Big) \cdot \Big(\prod_{\substack{V \in \alpha, \\ \text{not } \beta\text{-special}}} t_{|V|} \Big) \tag{4.19}$$

(where the concept of a β*-special block of* α *is as in Definition* 2.2*).*

Proof. We will verify that the sums on the right-hand sides of (4.16) and (4.19) are identified term by term when one uses the bijection $\pi \leftrightarrow (\alpha, \beta)$ from Theorem 3.4. We thus fix for the whole proof $\pi \in NCL(n)$ and $\alpha, \beta \in NC(n)$ such that $\pi \leftrightarrow (\alpha, \beta)$ in Theorem 3.4, and our goal is to show that

$$\prod_{A \in \pi} t_{|A|-1} = \Big(\prod_{\substack{U \in \alpha, \\ \beta\text{-special}}} t_{|U|-1} \Big) \cdot \Big(\prod_{\substack{V \in \alpha, \\ \text{not } \beta\text{-special}}} t_{|V|} \Big). \tag{4.20}$$

The fact that $\pi \leftrightarrow (\alpha, \beta)$ in Theorem 3.4 means of course that $\alpha = \overset{\circ}{\pi}$ and $\beta = \widehat{\pi}$. Let us write explicitly $\beta = \{W_1, \ldots, W_q\}$. In view of Lemma 4.2 we see that π can then be written in the form $\pi = \{A_1, \ldots, A_q, B_1, \ldots, B_r\}$, where $\min(A_1) = \min(W_1), \ldots, \min(A_q) = \min(W_q)$ and where $\min(A_1), \ldots, \min(A_q)$ are singly-covered by π, while $\min(B_1), \ldots, \min(B_r)$ are doubly-covered by π. From how the unlinking $\breve{\pi}$ is defined (cf. Review 2.6.3) we next infer that

$$\begin{cases} \breve{\pi} = \{U_1, \ldots, U_q, V_1, \ldots, V_r\}, \quad \text{where} \\ U_j = A_j, \ \forall\, 1 \leqslant j \leqslant q \text{ and } V_k = B_k \backslash \{\min(B_k)\}, \ \forall\, 1 \leqslant k \leqslant r. \end{cases}$$

From Definition 3.2 we further infer that the partition $\alpha = \overset{\circ}{\pi}$ can be written in the form

$$\begin{cases} \overset{\circ}{\pi} = \{U'_1, \ldots, U'_q, V'_1, \ldots, V'_r\}, \quad \text{where} \\ U'_j = P_\beta^{-1}(U_j), \ \forall\, 1 \leqslant j \leqslant q \text{ and } V'_k = P_\beta^{-1}(V_k), \ \forall\, 1 \leqslant k \leqslant r. \end{cases}$$

We next observe that

$$\min(W_j), \max(W_j) \in U'_j, \quad \forall\, 1 \leqslant j \leqslant q. \tag{4.21}$$

Indeed, this statement is clear in the case $|W_j| = 1$ (when one has $W_j = U_j = U'_j$). In the case when $|W_j| =: m \geqslant 2$ we write $W_j = \{i_1, i_2, \ldots, i_m\}$ with $i_1 < i_2 < \cdots < i_m$ and we observe that the argument used in the proof of Lemma 3.6 applies, giving us that $i_1, i_2 \in U_j$, and hence that $i_1, i_m \in U'_j$.

From (4.21) it follows that the β-special blocks of α are precisely $U'_1, \ldots, U'_q$. The right-hand side of (4.20) thus takes the form

$$\Big(\prod_{j=1}^{q} t_{|U'_j|-1} \Big) \cdot \Big(\prod_{k=1}^{r} t_{|V'_k|} \Big). \tag{4.22}$$

But it is clear that $|U'_j| = |U_j| = |A_j|$, $1 \leqslant j \leqslant q$, and that $|V'_k| = |V_k| = |B_k| - 1$, $1 \leqslant k \leqslant r$. Hence the product from (4.22) equals $\big(\prod_{j=1}^{q} t_{|A_j|-1} \big) \cdot \big(\prod_{k=1}^{r} t_{|B_k|-1} \big)$, and (4.20) follows. □

It is in fact fairly easy, in hindsight, to give a direct proof of Proposition 4.3 by using the R-transform R_a (another important transform of free probability) as an intermediate for passing from the series $1/S_a$ to the moments of a.

Second proof of Proposition 4.3. Let us consider the R-transform of a. This is the series $R_a(z) = \sum_{n=1}^{\infty} \kappa_n z^n$ whose coefficients $(\kappa_n)_{n=1}^{\infty}$, called the *free cumulants* of a, are uniquely determined by the fact that they satisfy the relations

$$\varphi(a^n) = \sum_{\beta \in NC(n)} \Big(\prod_{W \in \beta} \kappa_{|W|} \Big), \quad \forall\, n \geqslant 1. \tag{4.23}$$

(For instance $\kappa_1 = \varphi(a)\big(= 1\big)$, $\kappa_2 = \varphi(a^2) - \big(\varphi(a)\big)^2$ and $\kappa_3 = \varphi(a^3) - 3\varphi(a)\,\varphi(a^2) + 2\big(\varphi(a)\big)^3$. For a more detailed discussion of equation (4.23), see e.g. Lecture 12 of [7].) It was recently observed in [6] that the formula expressing the free cumulants of a in terms of the coefficients of $1/S_a(z)$ is merely a shifted version of equation (4.23):

$$\kappa_n = \sum_{\gamma \in NC(n-1)} \Big(\prod_{V \in \gamma} t_{|V|} \Big), \quad \forall\, n \geqslant 2. \tag{4.24}$$

Formula (4.24) arises when comparing the various functional equations satisfied by the series M_a, R_a and S_a; see Lemma 6.2 of [6].

Now, for every $n \geqslant 2$ one has a natural identification between $NC(n-1)$ and $\{\alpha \in NC(n) \mid \alpha \ll 1_n\}$. Indeed, the latter set consists of the partitions α in $NC(n)$ such that 1 and n belong to the same block of α; and any such partition is uniquely obtained from a $\gamma \in NC(n-1)$ by adjoining the number n to the block of γ which contains 1. By using this identification, the summation on the right-hand side of (4.24) can be turned into a summation over $\{\alpha \in NC(n) \mid \alpha \ll 1_n\}$. It is suggestive to write the ensuing equation in the form

$$\kappa_n = \sum_{\substack{\alpha \in NC(n), \\ \alpha \ll 1_n}} \Big(\prod_{\substack{U \in \alpha, \\ 1_n\text{-special}}} t_{|U|-1} \Big) \cdot \Big(\prod_{\substack{V \in \alpha, \\ \text{not } 1_n\text{-special}}} t_{|V|} \Big), \tag{4.25}$$

where the product over U has in fact only one factor (a partition $\alpha \ll 1_n$ has a unique 1_n-special block, the one which contains 1 and n). It is a straightforward exercise, left to the reader, to verify that (4.25) can be upgraded to the statement that for every $n \geqslant 1$ and every $\beta \in NC(n)$ one has

$$\prod_{W \in \beta} \kappa_{|W|} = \sum_{\substack{\alpha \in NC(n), \\ \alpha \ll \beta}} \Big(\prod_{\substack{U \in \alpha, \\ \beta\text{-special}}} t_{|U|-1} \Big) \cdot \Big(\prod_{\substack{V \in \alpha, \\ \text{not } \beta\text{-special}}} t_{|V|} \Big). \tag{4.26}$$

Finally, we sum over $\beta \in NC(n)$ on both sides of (4.26) and we invoke (4.23) on the left-hand side, and (4.19) follows. □

Remark 4.4. The double sum over α and β from equation (4.19) can be treated as an iterated sum in two ways. One of them (sum first over α, then over β) has in fact just been invoked at the end of the preceding proof. But actually it is the other order of summation (with β first) which brings the double sum to a simpler form, because it allows one to take advantage of the Boolean features of the partial order $\ll$; specifically, one arrives at equation (4.27) stated in the next theorem. It is remarkable that (4.27) closely resembles the formula used to define the concept of "c-free cumulants" in the theory of conditionally free convolution; compare e.g. to the third displayed equation on p. 366 of [3].

Theorem 4.5. *Let $(\mathcal{A},\varphi)$ be an algebraic non-commutative probability space, let $a\in\mathcal{A}$ be such that $\varphi(a)=1$, and consider the reciprocal S-transform $1/S_a(z)=\sum_{n=0}^{\infty}t_nz^n$. Then for every $n\geqslant 1$ one has*

$$\varphi(a^n)=\sum_{\alpha\in NC(n)}\Big(\prod_{\substack{U\in\alpha,\\ outer}}t_{|U|-1}\Big)\cdot\Big(\prod_{\substack{V\in\alpha,\\ inner}}t_{|V|-1}+t_{|V|}\Big). \tag{4.27}$$

Proof. Let α be a partition in $NC(n)$, and let $\mathcal{I},\mathcal{O}$ denote the set of inner blocks and respectively the set of outer blocks of α. Proposition 2.5 implies that

$$\begin{aligned}&\sum_{\substack{\beta\in NC(n),\\ \beta\gg\alpha}}\Big(\prod_{\substack{U\in\alpha,\\ \beta\text{-special}}}t_{|U|-1}\Big)\cdot\Big(\prod_{\substack{V\in\alpha,\\ \text{not }\beta\text{-special}}}t_{|V|}\Big)\\ &=\sum_{\substack{\mathcal{V}\subseteq\alpha\text{ such}\\ \text{that }\mathcal{V}\supseteq\mathcal{O}}}\Big(\prod_{U\in\mathcal{V}}t_{|U|-1}\Big)\cdot\Big(\prod_{V\in\alpha\setminus\mathcal{V}}t_{|V|}\Big).\end{aligned} \tag{4.28}$$

By performing the substitution $\mathcal{V}\setminus\mathcal{O}=:\mathcal{U}$ in the sum on the right-hand side of (4.28), this can be continued with

$$=\Big(\prod_{U\in\mathcal{O}}t_{|U|-1}\Big)\cdot\Big[\sum_{\mathcal{U}\subseteq\mathcal{I}}\Big(\prod_{U\in\mathcal{U}}t_{|U|-1}\Big)\cdot\Big(\prod_{V\in\mathcal{I}\setminus\mathcal{U}}t_{|V|}\Big)\Big].$$

But it is clear that the latter sum over $\mathcal{U}\subseteq\mathcal{I}$ is precisely the expansion of the product $\prod_{V\in\mathcal{I}}\big(t_{|V|-1}+t_{|V|}\big)$. So, altogether, what we have obtained is that

$$\begin{aligned}&\sum_{\substack{\beta\in NC(n),\\ \beta\gg\alpha}}\Big(\prod_{\substack{U\in\alpha,\\ \beta\text{-special}}}t_{|U|-1}\Big)\cdot\Big(\prod_{\substack{V\in\alpha,\\ \text{not }\beta\text{-special}}}t_{|V|}\Big)\\ &=\Big(\prod_{U\in\mathcal{O}}t_{|U|-1}\Big)\cdot\Big(\prod_{V\in\mathcal{I}}t_{|V|-1}+t_{|V|}\Big).\end{aligned} \tag{4.29}$$

Equation (4.29) holds for every $\alpha\in NC(n)$. Let us sum over α on both its sides. Then on the left-hand side we obtain exactly the double sum over α and β which is known from Proposition 4.3 to be equal to $\varphi(a^n)$, and the required formula (4.27) follows. □

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