PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 5, May 2010, Pages 1803–1810
S 0002-9939(09)10145-4
Article electronically published on December 21, 2009

LOCATION OF NASH EQUILIBRIA: A RIEMANNIAN GEOMETRICAL APPROACH

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(Communicated by Peter A. Clarkson)

Abstract. Existence and location of Nash equilibrium points are studied for a large class of a finite family of payoff functions whose domains are not necessarily convex in the usual sense. The geometric idea is to embed these non-convex domains into suitable Riemannian manifolds regaining certain geodesic convexity properties of them. By using recent non-smooth analysis on Riemannian manifolds and a variational inequality for acyclic sets, an efficient location result of Nash equilibrium points is given. Some examples show the applicability of our results.

1. Introduction and main result

Nash equilibrium plays a central role in game theory; it is a concept strategy of a game involving n-players ($n \geq 2$) in which every player knows the equilibrium strategies of the other players, and by changing his/her own strategy alone, a player has nothing to gain.

Let $K_1, \ldots, K_n$ ($n \geq 2$) be the non-empty sets of strategies of the players and $f_i : K_1 \times \ldots \times K_n \to \mathbb{R}$ ($i \in \{1, \ldots, n\}$) be the payoff functions. A point $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ is a *Nash equilibrium point for* $(\mathbf{f}, \mathbf{K})$ if

$$f_i(\mathbf{p}; q_i) \geq f_i(\mathbf{p}) \quad \text{for all } q_i \in K_i, \ i \in \{1, \ldots, n\}.$$

Here and in the sequel, the following notation are used: $\mathbf{K} = \Pi_{i=1}^n K_i$; $\mathbf{p} = (p_1, \ldots, p_n)$; $(\mathbf{f}, \mathbf{K}) = (f_1, \ldots, f_n; K_1, \ldots, K_n)$; $(\mathbf{p}; q_i) = (p_1, \ldots, q_i, \ldots, p_n)$.

The most well-known existence result is due to Nash (see [12], [13]), which works for compact and *convex* subsets K_i of Hausdorff topological *vector spaces*, and the continuous payoff functions f_i are (quasi)convex in the i^{th}-variable, $i \in \{1, \ldots, n\}$. A natural question that arises at this point is:

> *How is it possible to guarantee the existence of Nash equilibrium points for a family of payoff functions without any convexity or, even more, when their domains are not convex in the usual sense?*

Although many extensions and applications of Nash's result can be found (see, for instance, Chang [5], Georgiev [6], Kulpa and Szymanski [9], Morgan and Scalzo [11], Yu and Zhang [18], and the references therein), to the best of our knowledge,

Received by the editors January 14, 2009.
2000 *Mathematics Subject Classification.* Primary 91A10, 58B20, 49J40, 49J52, 46N10.
Key words and phrases. Nash equilibrium point, Riemannian manifold, nonsmooth analysis.
This work was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and by PN II IDEI_527 of CNCSIS.

only a few works can be found in the literature dealing with the above question. Nessah and Kerstens [14] characterize the existence of Nash equilibrium points on non-convex strategy sets via a modified diagonal transfer convexity notion, Tala and Marchi [16] also treat games with non-convex strategies reducing the problem to the convex case via a certain homeomorphism, and Kassay, Kolumbán and Páles [7] and Ziad [19] considered Nash equilibrium points on convex domains for non-convex payoff functions having suitable regularity instead of their convexity.

In the present paper we propose a new approach to answer the above question. Our idea is geometrical: we assume that the strategy sets K_i (which are not convex in the usual sense) can be embedded into suitable Riemannain manifolds (M_i, g_i) in a *geodesic convex* way (i.e., for any two points of K_i there exists a unique geodesic in (M_i, g_i) joining them which belongs entirely to K_i). Note that the choice of such Riemannian structures does not influence the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$. Moreover, after fixing these manifolds if the payoff functions f_i become *convex on* K_i (i.e., $f_i \circ \gamma_i : [0, 1] \to \mathbb{R}$ is convex in the usual sense for every geodesic $\gamma_i : [0, 1] \to K_i$), the following existence result may be stated.

Theorem 1.1. *Let (M_i, g_i) be finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, compact, geodesic convex sets; and $f_i : \mathbf{K} \to \mathbb{R}$ be continuous functions such that $K_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is convex on K_i for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, \dots, n\}$. Then, there exists a Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$.*

Our further investigation is motivated by the following two questions:

- What about the case when some payoff functions f_i are *not* convex on K_i in spite of the geodesic convexity of K_i on (M_i, g_i)?
- Even for convex payoff functions f_i on K_i, how can we *localize* the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$?

The following concept is destined for simultaneously handling the above questions.

Let (M_i, g_i) be complete finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be non-empty, geodesic convex sets; and $f_i : (\mathbf{K}; D_i) \to \mathbb{R}$ be functions such that $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is locally Lipschitz for every $\mathbf{p} \in \mathbf{K}$, where $(\mathbf{K}; D_i) = K_1 \times ... \times D_i \times ... \times K_n$, with D_i open and geodesic convex, and $K_i \subseteq D_i \subseteq M_i$, $i \in \{1, \dots, n\}$.

A point $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{K}$ is a *Nash critical point for* $(\mathbf{f}, \mathbf{K})$ if

$$f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) \geq 0 \quad \text{for all } q_i \in K_i, \; i \in \{1, \dots, n\}.$$

Here, $f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$ denotes the i^{th} *partial Clarke derivative of f_i at point $p_i \in K_i$ in the direction* $\exp_{p_i}^{-1}(q_i) \in T_{p_i}M_i$; for details, see Section 2. A similar notion on vector spaces has been introduced by Kassay, Kolumbán and Páles [7].

A useful relation between Nash equilibrium points and Nash critical points is established by the following result.

Proposition 1.2. *Any Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$ is a Nash critical point for $(\mathbf{f}, \mathbf{K})$. In addition, if $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is convex for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, \dots, n\}$, the converse also holds.*

In order to state our existence result concerning Nash critical points, we consider the hypothesis

(H) $K_i \ni q_i \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i))$ *is convex for every* $\mathbf{p} \in \mathbf{K}$ *and* $i \in \{1, ..., n\}$.

Remark 1. Let $I_1, I_2 \subseteq \{1, ..., n\}$ be such that $I_1 \cup I_2 = \{1, ..., n\}$. Hypothesis (H) holds, for instance, when

- (M_i, g_i) is Euclidean, $i \in I_1$;
- $K_i = \text{Im}\gamma_i$, where $\gamma_i : [0,1] \to M_i$ is a minimal geodesic and for every $\mathbf{p} \in \mathbf{K}$ with $p_i = \gamma_i(t_i)$ $(0 \le t_i \le 1)$, $f_i^0(\mathbf{p}, \gamma_i'(t_i)) \ge -f_i^0(\mathbf{p}, -\gamma_i'(t_i))$, $i \in I_2$.

(a) In the first case, the problem reduces to the property that the Clarke generalized derivative $K_i \ni q_i \mapsto f_i^0(\mathbf{p}; q_i)$ is subadditive and positively homogeneous, thus convex; see Clarke [4, Proposition 2.1.1]. Note that in this case $\exp_{p_i} = p_i + \text{id}_{\mathbb{R}^{\dim M_i}}$.

(b) In the second case, if $\sigma_i : [0,1] \to M_i$ is a geodesic segment joining the points $\sigma_i(0) = \gamma_i(\tilde{t}_0)$ with $\sigma_i(1) = \gamma_i(\tilde{t}_1)$ $(0 \le \tilde{t}_0 < \tilde{t}_1 \le 1)$, then $\text{Im}\sigma_i \subseteq \text{Im}\gamma_i = K_i$. Fix $p_i = \gamma_i(t_i) \in K_i$ $(0 \le \tilde{t}_i \le 1)$. Let $a_0, a_1 \in \mathbb{R}$ $(a_0 < a_1)$ be such that $\exp_{p_i}(a_0\gamma_i'(t_i)) = \gamma_i(\tilde{t}_0)$ and $\exp_{p_i}(a_1\gamma_i'(t_i)) = \gamma_i(\tilde{t}_1)$. Then, $\sigma_i(t) = \exp_{p_i}((a_0 + (a_1 - a_0)t)\gamma_i'(t_i))$. The claim follows if $t \mapsto f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(\sigma_i(t))) = f_i^0(\mathbf{p}, (a_0 + (a_1 - a_0)t)\gamma_i'(t_i)) := g(t)$ is convex. If $a_0 \ge 0$ or $a_1 \le 0$, then g is affine. If $a_0 < 0 < a_1$, then

$$g(t) = \begin{cases} -(a_0 + (a_1 - a_0)t)f_i^0(\mathbf{p}, -\gamma_i'(t_i)), & t \in [0, -a_0/(a_1 - a_0)], \\ (a_0 + (a_1 - a_0)t)f_i^0(\mathbf{p}, \gamma_i'(t_i)), & t \in (-a_0/(a_1 - a_0), 1]. \end{cases}$$

Therefore, g is convex if and only if $-f_i^0(\mathbf{p}, -\gamma_i'(t_i)) \le f_i^0(\mathbf{p}, \gamma_i'(t_i))$.

(c) We finally emphasize that the inequality in the second case holds automatically whenever $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is either convex (see Udrişte [17, Theorem 4.2, pp. 71-72]) or is of class C^1, for every $\mathbf{p} \in \mathbf{K}$, $i \in I_2$.

Theorem 1.3. *Let (M_i, g_i) be complete finite-dimensional Riemannian manifolds; $K_i \subset M_i$ be nonempty, compact, geodesic convex sets; and $f_i : (\mathbf{K}; D_i) \to \mathbb{R}$ be continuous functions such that f_i^0 is upper semicontinuous and $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is locally Lipschitz for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, \ldots, n\}$. If* (H) *holds, there exists a Nash critical point for* $(\mathbf{f}, \mathbf{K})$.

Remark 2. Proposition 1.2 and Theorem 1.3 together give a possible answer to the location of Nash equilibrium points. Indeed, because of Theorem 1.3 we are able to find explicitly the Nash critical points for $(\mathbf{f}, \mathbf{K})$. Then, due to Proposition 1.2, among Nash critical points for $(\mathbf{f}, \mathbf{K})$ we may choose the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$.

Existence results of Nash equilibria are often derived from fixed point theorems, minimax/variational inequalities or KKM-type intersection theorems. In Section 2 we recall a Fan-type variational inequality proved by McClendon [10] concerning compact acyclic ANR sets, which is the main tool in the proof of Theorems 1.1 and 1.3. In order to prove Theorem 1.3, we also recall some basic elements from non-smooth analysis on Riemannian manifolds developed by Azagra, Ferrera and Lópes-Mesas [2]. In Section 3 we prove our results, while in Section 4 we give two applications which illustrate the applicability of the developed method.

2. Preliminaries

A non-empty set X is *acyclic* if it is connected and its Čech homology (coefficients in a fixed field) is zero in dimensions greater than zero. Note that every contractible set is acyclic (but the converse need not hold in general). The following result is the main tool in the proof of our existence results:

Theorem 2.1 ([10, Theorem 3.1]). *Suppose that X is a compact acyclic finite-dimensional ANR. Suppose $h : X \times X \to \mathbb{R}$ is a function such that $\{(x, y) : h(y, y) > h(x, y)\}$ is open and $\{x : h(y, y) > h(x, y)\}$ is contractible or empty for all $y \in X$. Then there is a $y_0 \in X$ with $h(y_0, y_0) \leq h(x, y_0)$ for all $x \in X$.*

In the rest of this section, let (M, g) be a complete, finite-dimensional Riemannian manifold. The following result is probably known, but since we have not found an explicit reference, we give its proof.

Proposition 2.2. *Any geodesic convex set $K \subset M$ is contractible.*

Proof. Let us fix $p \in K$ arbitrarily. Since K is geodesic convex, every point $q \in K$ can be connected to p uniquely by the geodesic segment $\gamma_q : [0, 1] \to K$, i.e., $\gamma_q(0) = p$, $\gamma_q(1) = q$. Moreover, the map $K \ni q \mapsto \exp_p^{-1}(q) \in T_pM$ is well-defined and continuous. Note actually that $\gamma_q(t) = \exp_p(t \exp_p^{-1}(q))$. We define the map $F : [0, 1] \times K \to K$ by $F(t, q) = \gamma_q(t)$. It is clear that F is continuous, $F(1, q) = q$ and $F(0, q) = p$ for all $q \in K$; i.e., the identity map id_K is homotopic to the constant map p. □

Let $D \subseteq M$ be open and $f : D \to \mathbb{R}$ be a locally Lipschitz function. That is, for every $p \in D$, there exist $r_p > 0$ and $K_p > 0$ such that $|f(p_1) - f(p_2)| \leq K_p d(p_1, p_2)$ for every $p_1, p_2 \in B(p, r_p)$, where $B(p, r_p)$ is the geodesic ball with radius r_p around p, while $d(\cdot, \cdot)$ is the metric function generated by the Riemannian metric g.

Fix $p \in D$, $v \in T_pM$, and let $U_p \subset D$ be the totally normal neighborhood of p. If $q \in U_p$, following Azagra, Ferrera and Lópes-Mesas [2, Section 5], for small values of $|t|$, we may introduce

$$\sigma_{q,v}(t) = \exp_q(tw),\ w = d(\exp_q^{-1} \circ \exp_p)_{\exp_p^{-1}(q)} v. \tag{2.1}$$

The *Clarke derivative of f at point $p \in D$ in the direction $v \in T_pM$* is defined by

$$f^0(p, v) = \limsup_{q \to p,\ t \to 0^+} \frac{f(\sigma_{q,v}(t)) - f(q)}{t}.$$

One can easily prove that the function $f^0(\cdot, \cdot)$ is upper-semicontinuous on $TD = \bigcup_{p \in D} T_pM$ and $f^0(p, \cdot)$ is positive homogeneous.

In addition, if $D \subseteq M$ is geodesic convex and $f : D \to \mathbb{R}$ is convex, then

$$f^0(p, v) = \lim_{t \to 0^+} \frac{f(\exp_p(tv)) - f(p)}{t}; \tag{2.2}$$

see Claim 5.4 and the first relation on p. 341 of [2].

3. Proofs

Proof of Theorem 1.1. Let $X = \mathbf{K} = \Pi_{i=1}^n K_i$ and $h : X \times X \to \mathbb{R}$ be defined by $h(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n [f_i(\mathbf{p}; q_i) - f_i(\mathbf{p})]$. First of all, note that the sets K_i are ANRs, due to Hanner's theorem; see Bessage and Pelczyński [3, Theorem 5.1]. Moreover, since a product of a finite family of ANRs is an ANR (see [3, Corollary 5.5]), it follows that X is an ANR. Due to Proposition 2.2, X is contractible, thus acyclic.

Note that the function h is continuous and that $h(\mathbf{p}, \mathbf{p}) = 0$ for every $\mathbf{p} \in X$. Consequently, the set $\{(\mathbf{q}, \mathbf{p}) \in X \times X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is open.

It remains to prove that $S_{\mathbf{p}} = \{\mathbf{q} \in X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is contractible or empty for all $\mathbf{p} \in X$. Assume that $S_{\mathbf{p}} \neq \emptyset$ for some $\mathbf{p} \in X$. Then, there exists $i_0 \in \{1, ..., n\}$ such that $f_{i_0}(\mathbf{p}; q_{i_0}) - f_{i_0}(\mathbf{p}) < 0$ for some $q_{i_0} \in K_{i_0}$. Therefore, $\mathbf{q} = (\mathbf{p}; q_{i_0}) \in S_{\mathbf{p}}$;

i.e., $\text{pr}_i S_{\mathbf{p}} \neq \emptyset$ for every $i \in \{1, ..., n\}$. Now, we fix $\mathbf{q}^j = (q_1^j, ..., q_n^j) \in S_{\mathbf{p}}$, $j \in \{1, 2\}$ and let $\gamma_i : [0, 1] \to K_i$ be the unique geodesic joining the points $q_i^1 \in K_i$ and $q_i^2 \in K_i$ (note that K_i is geodesic convex), $i \in \{1, ..., n\}$. Let $\gamma : [0, 1] \to \mathbf{K}$ be defined by $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Due to the convexity of the function $K_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$, for every $t \in [0, 1]$ we have

$$\begin{aligned} h(\gamma(t), \mathbf{p}) &= \sum_{i=1}^{n} [f_i(\mathbf{p}; \gamma_i(t)) - f_i(\mathbf{p})] \\ &\leq \sum_{i=1}^{n} [t f_i(\mathbf{p}; \gamma_i(1)) + (1 - t) f_i(\mathbf{p}; \gamma_i(0)) - f_i(\mathbf{p})] \\ &= t h(\mathbf{q}^2, \mathbf{p}) + (1 - t) h(\mathbf{q}^1, \mathbf{p}) \\ &< 0. \end{aligned}$$

Consequently, $\gamma(t) \in S_{\mathbf{p}}$ for every $t \in [0, 1]$. That is, $S_{\mathbf{p}}$ is a geodesic convex set in the product manifold $\mathbf{M} = \Pi_{i=1}^n M_i$ endowed with its natural (warped-)product metric (with the constant weight functions 1); see O'Neill [15, p. 208]. Now, Proposition 2.2 implies that $S_{\mathbf{p}}$ is contractible. Alternatively, we may exploit the fact that the projections $\text{pr}_i S_{\mathbf{p}}$ are geodesic convex, thus contractible sets, $i \in \{1, ..., n\}$.

We are in the position to apply Theorem 2.1. Therefore, there exists $\mathbf{p} \in \mathbf{K}$ such that $0 = h(\mathbf{p}, \mathbf{p}) \leq h(\mathbf{q}, \mathbf{p})$ for every $\mathbf{q} \in \mathbf{K}$. In particular, putting $\mathbf{q} = (\mathbf{p}; q_i)$, $q_i \in K_i$ fixed, we obtain that $f_i(\mathbf{p}; q_i) - f_i(\mathbf{p}) \geq 0$ for every $i \in \{1, ..., n\}$; i.e., $\mathbf{p}$ is a Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$. □

Proof of Proposition 1.2. Let $\mathbf{p} \in \mathbf{K}$ be a Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$. In particular, for every $q_i \in K_i$ with $i \in \{1, \ldots, n\}$ fixed, the geodesic segment $t \mapsto \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))$, $t \in [0, 1]$ joining the points p_i and q_i, belongs entirely to K_i; thus,

$$f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p}) \geq 0 \quad \text{for all } t \in [0, 1]. \tag{3.1}$$

Note that

$$\lim_{\tilde{q}_i \to p_i} f_i(\mathbf{p}; \tilde{q}_i) = f_i(\mathbf{p}), \tag{3.2}$$

and taking into account the notation from (2.1), for every $t \in [0, 1]$, we have that

$$\lim_{\tilde{q}_i \to p_i} \sigma^i_{\tilde{q}_i, \exp_{p_i}^{-1}(q_i)}(t) = \exp_{p_i}(t \exp_{p_i}^{-1}(q_i)). \tag{3.3}$$

Since

$$f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) = \limsup_{\tilde{q}_i \to p_i,\ t \to 0^+} \frac{f_i(\mathbf{p}; \sigma^i_{\tilde{q}_i, \exp_{p_i}^{-1}(q_i)}(t)) - f_i(\mathbf{p}; \tilde{q}_i)}{t},$$

combining relations (3.2) and (3.3) with (3.1), we obtain that $f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) \geq 0$, which proves that $\mathbf{p} \in \mathbf{K}$ is a Nash critical point for $(\mathbf{f}, \mathbf{K})$.

Now, let us assume that $\mathbf{p} \in \mathbf{K}$ is a Nash critical point for $(\mathbf{f}, \mathbf{K})$. Since $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is convex, $i \in \{1, \ldots, n\}$, due to (2.2), for every $q_i \in K_i$, we have

$$0 \leq f_i^0(\mathbf{p}, \exp_{p_i}^{-1}(q_i)) = \lim_{t \to 0^+} \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p})}{t}. \tag{3.4}$$

The function

$$g(t) = \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p})}{t}$$

is well-defined on the whole interval $(0,1]$. Indeed, $t \mapsto \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))$ is the minimal geodesic joining the points $p_i \in K_i$ and $q_i \in K_i$ which belongs to $K_i \subset D_i$. Moreover, a standard computation based on convexity shows that $t \mapsto g(t)$ is non-decreasing on $(0,1]$. Consequently, (3.4) implies that

$$0 \leq \lim_{t \to 0^+} g(t) \leq g(1) = f_i(\mathbf{p}; \exp_{p_i}(\exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p}) = f_i(\mathbf{p}; q_i) - f_i(\mathbf{p}),$$

which completes the proof. □

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.1; we show only the differences. Let $X = \mathbf{K} = \Pi_{i=1}^n K_i$ and $h : X \times X \to \mathbb{R}$ be defined by $h(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i))$. It is clear that $h(\mathbf{p}, \mathbf{p}) = 0$ for every $\mathbf{p} \in X$.

First of all, the upper-semicontinuity of $h(\cdot, \cdot)$ on $X \times X$ implies the fact that the set $\{(\mathbf{q}, \mathbf{p}) \in X \times X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is open.

Now, let $\mathbf{p} \in X$ be such that $S_\mathbf{p} = \{\mathbf{q} \in X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is not empty. Then, there exists $i_0 \in \{1, ..., n\}$ such that $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_{i_0})) < 0$ for some $q_{i_0} \in K_{i_0}$. Consequently, $\mathbf{q} = (\mathbf{p}; q_{i_0}) \in S_\mathbf{p}$; i.e., $\text{pr}_i S_\mathbf{p} \neq \emptyset$ for every $i \in \{1, ..., n\}$. Now, we fix $\mathbf{q}^j = (q_1^j, ..., q_n^j) \in S_\mathbf{p}$, $j \in \{1, 2\}$, and let $\gamma_i : [0,1] \to K_i$ be the unique geodesic joining the points $q_i^1 \in K_i$ and $q_i^2 \in K_i$. Also let $\gamma : [0,1] \to \mathbf{K}$ be defined by $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Due to hypotheses (H), the convexity of the function $[0,1] \ni t \mapsto h(\gamma(t), \mathbf{p})$, $t \in [0,1]$, easily follows. Therefore, $\gamma(t) \in S_\mathbf{p}$ for every $t \in [0,1]$; i.e., $S_\mathbf{p}$ is a geodesic convex set, thus contractible.

Theorem 2.1 implies the existence of $\mathbf{p} \in \mathbf{K}$ such that $0 = h(\mathbf{p}, \mathbf{p}) \leq h(\mathbf{q}, \mathbf{p})$ for every $\mathbf{q} \in \mathbf{K}$. In particular, if $\mathbf{q} = (\mathbf{p}; q_i)$, $q_i \in K_i$ fixed, we obtain that $f_i^0(\mathbf{p}; \exp_{p_i}^{-1}(q_i)) \geq 0$ for every $i \in \{1, ..., n\}$; i.e., $\mathbf{p}$ is a Nash critical point for $(\mathbf{f}, \mathbf{K})$. The proof is completed. □

Remark 3. It seems that the class of locally Lipschitz functions (in the appropriate variable) is the optimal one among not necessarily convex functions for which the existence of Nash critical points can be guaranteed. The proximal calculus for lower semicontinuous functions (see Azagra and Ferrera [1], Ledyaev and Zhu [8]) could be another good candidate for developing a similar concept as Nash critical points. Unfortunately, the lack of a suitable regularity of the proximal subdifferential leaves this question open.

4. Examples

Example 4.1. Let $K_1 = [-1, 1]$, $K_2 = \{(\cos t, \sin t) : t \in [\pi/4, 3\pi/4]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ be defined for every $x \in K_1$, $(y_1, y_2) \in K_2$ by

$$f_1(x, (y_1, y_2)) = |x| y_1^2 - y_2, \quad f_2(x, (y_1, y_2)) = (1 - |x|)(y_1^2 - y_2^2).$$

Note that $K_1 \subset \mathbb{R}$ is convex in the usual sense, but $K_2 \subset \mathbb{R}^2$ is not. However, if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_\mathbb{H})$, the set $K_2 \subset \mathbb{H}^2$ is geodesic convex with respect to the metric $g_\mathbb{H}$ being the image of a geodesic segment from $(\mathbb{H}^2, g_\mathbb{H})$. It is clear that $f_1(\cdot, (y_1, y_2))$ is a convex function on K_1 in the usual sense for every $(y_1, y_2) \in K_2$. Moreover, $f_2(x, \cdot)$ is also a convex function on $K_2 \subset \mathbb{H}^2$ for every $x \in K_1$. Indeed, the latter fact reduces to the convexity of the function $t \mapsto (1 - |x|)\cos(2t)$, $t \in [\pi/4, 3\pi/4]$. Therefore, Theorem 1.1 guarantees the existence of at least one Nash equilibrium point for $(f_1, f_2; K_1, K_2)$. Using Proposition 1.2, a simple calculation shows that the set of Nash equilibrium (= critical) points for $(f_1, f_2; K_1, K_2)$ is $K_1 \times \{(0, 1)\}$.

Example 4.2. Let $K_1 = [-1,1]^2$, $K_2 = \{(y_1, y_2) : y_2 = y_1^2,\ y_1 \in [0,1]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ be defined for every $(x_1, x_2) \in K_1$, $(y_1, y_2) \in K_2$ by

$$f_1((x_1,x_2),(y_1,y_2)) = -x_1^2 y_2 + x_2 y_1, \quad f_2((x_1,x_2),(y_1,y_2)) = x_1 y_2^2 + x_2 y_1^2.$$

The set $K_1 \subset \mathbb{R}^2$ is convex, but $K_2 \subset \mathbb{R}^2$ is not in the usual sense. However, K_2 may be considered as the image of a geodesic segment on the paraboloid of revolution $p_{\text{rev}}(u,v) = (v\cos u, v\sin u, v^2)$ endowed with its natural Riemannian structure having the coefficients

$$g_{11}(u,v) = v^2,\ g_{12}(u,v) = g_{21}(u,v) = 0,\ g_{22}(u,v) = 1 + 4v^2. \tag{4.1}$$

More precisely, K_2 becomes geodesic convex on $\text{Im}p_{\text{rev}}$, being actually identified with $\{(y,0,y^2) : y \in [0,1]\} \subset \text{Im}p_{\text{rev}}$. Note that neither $f_1(\cdot,(y_1,y_2))$ nor $f_2((x_1,x_2),\cdot)$ is convex (the convexity of the latter function being considered on $K_2 \subset \text{Im}p_{\text{rev}}$); thus, Theorem 1.1 is not applicable. In view of Remark 1 (c), Theorem 1.3 can be applied in order to determine the set of Nash critical points. This set is nothing but the set of solutions in the form $((\tilde{x}_1,\tilde{x}_2),(\tilde{y},\tilde{y}^2)) \in K_1 \times K_2$ of the system

$$(\text{System}_{\text{NCP}}) \qquad \begin{cases} -2\tilde{x}_1\tilde{y}^2(x_1 - \tilde{x}_1) + \tilde{y}(x_2 - \tilde{x}_2) \geq 0, & \forall (x_1,x_2) \in K_1, \\ \tilde{y}(2\tilde{y}^2\tilde{x}_1 + \tilde{x}_2)(y - \tilde{y}) \geq 0, & \forall y \in [0,1]. \end{cases}$$

Note that the second inequality is obtained by (4.1) and relations

$$\begin{aligned} &\partial_2 f_2((x_1,x_2),(y_1,y_2)) = \left(2x_2y_1y_2^{-2}, 2x_1y_2(1+4y_2^2)^{-1}\right), \\ &\exp^{-1}_{(\tilde{y},0,\tilde{y}^2)}(y,0,y^2) = (y - \tilde{y}, 2\tilde{y}(y-\tilde{y})), \quad \tilde{y}, y \in [0,1]. \end{aligned}$$

We distinguish three cases: (a) $\tilde{y} = 0$; (b) $\tilde{y} = 1$; and (c) $0 < \tilde{y} < 1$.

(a) $\tilde{y} = 0$. Then, any $((\tilde{x}_1,\tilde{x}_2),(0,0)) \in K_1 \times K_2$ solves $(\text{System}_{\text{NCP}})$.

(b) $\tilde{y} = 1$. After an easy computation, we obtain that $((-1,-1),(1,1)) \in K_1 \times K_2$ and $((0,-1),(1,1)) \in K_1 \times K_2$ solve $(\text{System}_{\text{NCP}})$.

(c) $0 < \tilde{y} < 1$. The unique situation when $(\text{System}_{\text{NCP}})$ is solvable is $\tilde{y} = \sqrt{2}/2$. In this case, $(\text{System}_{\text{NCP}})$ has a unique solution $((1,-1),(\sqrt{2}/2,1/2)) \in K_1 \times K_2$. Consequently, the set of Nash critical points for $(f_1, f_2; K_1, K_2)$, denoted in the sequel by S_{NCP}, is the union of the points from (a), (b) and (c), respectively.

Let us denote by S_{NEP} the set of Nash equilibrium points for $(f_1, f_2; K_1, K_2)$. Due to Proposition 1.2, we may select the elements of S_{NEP} from S_{NCP}. Therefore, the elements of S_{NEP} are the solutions $((\tilde{x}_1,\tilde{x}_2),(\tilde{y},\tilde{y}^2)) \in S_{\text{NCP}}$ of the system

$$(\text{System}_{\text{NEP}}) \qquad \begin{cases} -x_1^2\tilde{y}^2 + x_2\tilde{y} \geq -\tilde{x}_1^2\tilde{y}^2 + \tilde{x}_2\tilde{y}, & \forall (x_1,x_2) \in K_1, \\ \tilde{x}_1 y^4 + \tilde{x}_2 y^2 \geq \tilde{x}_1\tilde{y}^4 + \tilde{x}_2\tilde{y}^2, & \forall y \in [0,1]. \end{cases}$$

We consider again the above three cases.

(a) $\tilde{y} = 0$. Among the elements $((\tilde{x}_1,\tilde{x}_2),(0,0)) \in K_1 \times K_2$ which solve $(\text{System}_{\text{NCP}})$, only those which fulfill the condition $\tilde{x}_2 \geq \max\{-\tilde{x}_1, 0\}$ are solutions for $(\text{System}_{\text{NEP}})$.

(b) $\tilde{y} = 1$. We have $((-1,-1),(1,1)) \in S_{\text{NEP}}$, but $((0,-1),(1,1)) \notin S_{\text{NEP}}$.

(c) $0 < \tilde{y} < 1$. We have $((1,-1),(\sqrt{2}/2,1/2)) \in S_{\text{NEP}}$.

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