PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 5, May 2010, Pages 1693–1699
S 0002-9939(09)10226-5
Article electronically published on December 30, 2009

ON MEMS EQUATION WITH FRINGING FIELD

JUNCHENG WEI AND DONG YE

(Communicated by Matthew J. Gursky)

Abstract. We consider the MEMS equation with fringing field

$$-\Delta u = \lambda(1+\delta|\nabla u|^2)(1-u)^{-2} \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,$$

where $\lambda, \delta > 0$ and $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain. We show that when the fringing field exists (i.e. $\delta > 0$), given any $\mu > 0$, we have a uniform upper bound of classical solutions u away from *the rupture level 1* for all $\lambda \geq \mu$. Moreover, there exists $\overline{\lambda}^*_\delta > 0$ such that there are at least two solutions when $\lambda \in (0, \overline{\lambda}^*_\delta)$; a unique solution exists when $\lambda = \overline{\lambda}^*_\delta$; and there is no solution when $\lambda > \overline{\lambda}^*_\delta$. This represents a dramatic change of behavior with respect to the zero fringing field case (i.e., $\delta = 0$) and confirms the simulations in a paper by Pelesko and Driscoll as well as a paper by Lindsay and Ward.

1. Introduction

We consider the elliptic equation

$$(E_\lambda) \qquad -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^2} \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega,$$

where δ, λ are positive constants, and Ω is a bounded smooth domain in $\mathbb{R}^n$ ($n \geq 2$).

Problem (E_λ) arises in the study of electrostatic Micro-Electromechanical System (MEMS) devices. We refer to [5] and the book [13] for detailed discussions on MEMS devices modeling. The parameter λ is called the voltage and the term $\delta|\nabla u|^2$ is called a fringing field (cf. [14, 11]). The eventual singular set $\{x \in \Omega, u(x) = 1\}$ is called a *rupture set*. When $\delta = 0$, problem (E_λ) becomes

$$(S_\lambda) \qquad -\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega.$$

Recently there have been many studies on (S_λ). We summarize some of the results here:

- There exists a critical number $\overline{\lambda}^* > 0$ such that for $0 < \lambda < \overline{\lambda}^*$ problem (S_λ) has a minimal stable solution $\overline{u}_\lambda$, while for $\lambda > \overline{\lambda}^*$ there are no solutions to (S_λ) (see [6]).

Received by the editors August 13, 2009.
2010 *Mathematics Subject Classification.* Primary 35B45; Secondary 35J15.
Key words and phrases. MEMS, rupture, fringing field, bifurcation.
The research of the first author is supported by the General Research Fund from the Research Grant Council of Hong Kong.
The second author is supported by the French ANR project ANR-08-BLAN-0335-01.

- Either the solution branch stops at $\overline{\lambda}^*$ and $\lim_{\lambda\to\overline{\lambda}^*}\|\overline{u}_\lambda\|_\infty = 1$ (if Ω is a ball in $\mathbb{R}^n$ with $n \geq 8$ for example) or the solution branch bends back and we could have another critical parameter $0 < \overline{\lambda}_* < \overline{\lambda}^*$ (when Ω is a ball in $\mathbb{R}^n$ with $2 \leq n \leq 7$ or a convex domain with two axes of symmetry in $\mathbb{R}^2$) such that the solution branch takes infinitely many turns and converges to a *rupture* solution of $(S_{\overline{\lambda}_*})$ (see [4, 9, 10]).
- For general strictly convex domains with $n \geq 2$, it can be shown that for $\lambda > 0$ small, the minimal solution is the unique one for (S_λ) (see [3, 16]). So we must have a family of solutions (u^k, λ^k) such that $\lim_{k\to\infty}\lambda^k = \overline{\lambda} > 0$ and $\lim_{k\to\infty}\|u^k\|_\infty = 1$.

In this short paper, we show that the fringing field dramatically changes the structure of solutions of (E_λ) (see Theorem 5 below): we prove that there exists a critical parameter $\overline{\lambda}^*_\delta$ such that for $\lambda > \overline{\lambda}^*_\delta$ there are no solutions to (E_λ), for $0 < \lambda < \overline{\lambda}^*_\delta$ there are at least *two* solutions, and when $\lambda = \overline{\lambda}^*_\delta$ there exists a unique solution. Furthermore, for any fixed $\mu > 0$, all solutions to (E_λ) with $\lambda \geq \mu$ are below $C_\mu < 1$; i.e., *no ruptures* can occur by using solutions with λ tending to some $\overline{\lambda} > 0$. Our study holds for *any dimension* and confirms the numerical results obtained in [14, 11]. Here all solutions considered are *classical* solutions.

The results of this paper are also true for the generalized MEMS equation

$$(E_{\lambda,p}) \qquad -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $p > 1$.

2. A key transformation

To study the structure of solutions for (E_λ), we present a suitable transformation, which leads to considering a semilinear equation. More precisely, we have

Lemma 1. *Let*

$$(1) \qquad v = \zeta_\lambda(u) = \int_0^u e^{\frac{\lambda\delta}{1-s}}\,ds, \quad \forall\, u \in\ [0,1).$$

Then $u : \Omega \to [0,1)$ is a solution (resp. supersolution, subsolution) of (E_λ) if and only if v is a solution (resp. supersolution, subsolution) for

$$(F_\lambda) \qquad -\Delta v = \rho_\lambda(v), \ \ v > 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,$$

where ρ_λ is a smooth increasing function from $\mathbb{R}_+$ into $(0,\infty)$, defined by

$$(2) \qquad \rho_\lambda(v) = \xi_\lambda \circ \zeta_\lambda^{-1} \quad \text{with} \quad \xi_\lambda(u) = \frac{\lambda e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2}.$$

Proof. As ξ_λ, ζ_λ are increasing in $[0,1)$ and $\lim_{u\to 1^-}\zeta_\lambda(u) = \infty$, so ρ_λ is also increasing in $\mathbb{R}_+$. By direct calculus, $v = \zeta_\lambda(u)$ satisfies

$$-\Delta v = -e^{\frac{\lambda\delta}{1-u}}\Delta u - \frac{\lambda\delta e^{\frac{\lambda\delta}{1-u}}}{(1-u)^2}|\nabla u|^2;$$

all conclusions are straightforward. □

Otherwise, it is not difficult to prove the following.

Theorem 1. *Fixing $\delta > 0$, there exists $\overline{\lambda}^*_\delta \in (0, \infty)$ such that for any $\lambda < \overline{\lambda}^*_\delta$, the equation (E_λ) has a minimal solution u_λ, while for any $\lambda > \overline{\lambda}^*_\delta$, no solution exists for (E_λ). Moreover $\lambda \mapsto u_\lambda$ is increasing for $\lambda \in (0, \overline{\lambda}^*_\delta)$.*

Here the minimal solution means that for any solution u to (E_λ), we have $u_\lambda \leq u$ in Ω.

Proof. The result is a direct consequence of the following claims:

(i) If (E_λ) is solvable with $\lambda > 0$, then $(E_{\lambda'})$ is solvable for any $\lambda' \in (0, \lambda)$.
(ii) The equation (E_λ) has no solution for λ sufficiently large.
(iii) For $\lambda > 0$ small enough, we have a solution to (E_λ).
(iv) If (E_λ) is solvable, then there exists a minimal solution u_λ.

If u is a solution to (E_λ), it is clearly a supersolution to $(E_{\lambda'})$, so $v = \zeta_\lambda(u)$ is a supersolution to $(F_{\lambda'})$ by Lemma 1. As $\rho_{\lambda'}(0) = \lambda' e^{\lambda'\delta} > 0$, 0 is always a subsolution. Moreover $\rho_{\lambda'}$ is locally Lipschitz in $\mathbb{R}_+$, so we have a solution to $(F_{\lambda'})$, which yields the claim (i).

The claim (ii) comes from the fact that any solution of (E_λ) is a supersolution for the equation (S_λ), which has no solution for large λ. Let $-\Delta\xi = 1$ in Ω and $\xi = 0$ on $\partial\Omega$, and fix $c > 0$ such that $c\|\xi\|_\infty < 1$. We can check that $c\xi$ is a supersolution of (E_λ) if $\lambda > 0$ is small enough; this leads to the claim (iii).

The last claim is due to the monotonicity of ρ_λ (cf. (4) below), ζ_λ and the monotone iteration for (F_λ) as $-\Delta v^{n+1} = \rho_\lambda(v^n)$ with Dirichlet boundary condition and $v^0 \equiv 0$. □

Remark 1. Of course, the transformation $v = \zeta_\lambda$ is not really necessary for the above proof. Thanks to the monotonicity of the function $g(u) = (1-u)^{-2}$, we can consider directly the iteration operator $w = Tu$, the unique solution of

$$-\Delta w = \frac{\lambda(1 + \delta|\nabla u|^2)}{(1-u)^2} \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

3. Stability of minimal solutions

The minimal solution for (E_λ) will ensure some stability properties, even though the equation (E_λ) does not have a variational structure. First, for the linearized operator of (E_λ),

$$L_\lambda\varphi = -\Delta\varphi - \frac{2\lambda(1 + \delta|\nabla u|^2)}{(1-u)^3}\varphi - \frac{2\lambda\delta\nabla u\nabla\varphi}{(1-u)^2},$$

we can define the principal eigenvalue μ_1 of L_λ, associated to the Dirichlet boundary condition (cf. [12]). Then a solution u of (E_λ) is said to be stable if and only if $\mu_1(L_\lambda) \geq 0$. Another idea is to use the transformation $v = \zeta_\lambda(u)$ and the corresponding linearized operator. Following the ideas in [1], we obtain

Theorem 2. *Letting $\lambda \in (0, \overline{\lambda}^*_\delta)$, the minimal solution v_λ of (F_λ) satisfies*

$$\int_\Omega |\nabla\varphi|^2 \geq \int_\Omega \rho'_\lambda(v_\lambda)\varphi^2 dx, \quad \forall\, \varphi \in H^1_0(\Omega). \tag{3}$$

Furthermore, v_λ is the unique solution of (F_λ) satisfying (3), and u_λ is the unique stable solution of (E_λ).

Moreover, $u = \zeta_\lambda^{-1}(v)$ implies that

$$\rho'_\lambda(v) = \left(\xi_\lambda \circ \zeta_\lambda^{-1}\right)'(v) = \frac{\xi'_\lambda}{\zeta'_\lambda} \circ \zeta_\lambda^{-1}(v) = \frac{\lambda^2\delta}{(1-u)^4} + \frac{2\lambda}{(1-u)^3} > 0. \tag{4}$$

As the minimal solution u_λ of (E_λ) is just $\zeta_\lambda^{-1}(v_\lambda)$, we conclude then

Theorem 3. *For $\lambda \in (0, \overline{\lambda}_\delta^*)$, the minimal solution u_λ is the unique solution of (E_λ) satisfying the stability condition*

$$\int_\Omega |\nabla\varphi|^2 \geq \int_\Omega \left[\frac{\lambda^2\delta}{(1-u_\lambda)^4} + \frac{2\lambda}{(1-u_\lambda)^3}\right]\varphi^2 dx, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{5}$$

4. Bifurcation and uniform estimate

Using the equation (F_λ) and the standard bifurcation theory of Rabinowitz (section 3 of [15]), we can say that a solution curve (λ, v) exists in $\mathbb{R}_+ \times C(\overline{\Omega})$; it goes from $(0,0)$ to "infinity". By Theorem 1, the only possibility is that $\|v\|_\infty$ tends to ∞. For (F_λ), when $\|v\|_\infty \to \infty$, we show that λ must tend to 0 by the following result.

Theorem 4. *For any $\mu > 0$, there exists a constant $C_\mu > 0$ such that any solution of (F_λ) with $\lambda \geq \mu$ satisfies $\|v\|_\infty < C_\mu$. Consequently, there exists $c_\mu \in (0,1)$ such that any solution u of (E_λ) with $\lambda \geq \mu$ is such that $u \leq c_\mu < 1$.*

Proof. In fact, using integration by parts, we can see that

$$v = \zeta_\lambda(u) \sim \frac{(1-u)^2}{\lambda\delta} e^{\frac{\lambda\delta}{1-u}} \quad \text{as } u \to 1^-.$$

Hence for $\mu \in (0, \overline{\lambda}_\delta^*)$ fixed, there exist positive constants C, C' such that

$$Cv(\ln v)^4 \leq \rho_\lambda(v) \leq C'v(\ln v)^4 \quad \forall\, (\lambda, v) \in [\mu, \overline{\lambda}_\delta^*) \times [2, \infty).$$

We also have the uniform estimate $\rho_\lambda(v) \geq Cv + \mu$ for $(\lambda, v) \in [\mu, \overline{\lambda}_\delta^*) \times \mathbb{R}_+$. The proof of Theorem 2.1 in [2] holds and shows that there exists $C_\mu > 0$ such that $\|v\|_\infty < C_\mu < \infty$. The conclusion for u is an immediate consequence. □

An important consequence is just the uniqueness of a solution for $(E_{\overline{\lambda}_\delta^*})$. We shall use the problem (F_λ). Now $v^* = \lim_{\lambda \to \overline{\lambda}_\delta^*} v_\lambda$ is a smooth solution for the limit problem $(F_{\overline{\lambda}_\delta^*})$; we claim that $\mu_1\big[-\Delta - \rho'_{\overline{\lambda}_\delta^*}(v^*)\big] = 0$. In fact, the stability of v^* (in the sense of (3)) means that $\mu_1\big[-\Delta - \rho'_{\overline{\lambda}_\delta^*}(v^*)\big] \geq 0$, while the definition of $\overline{\lambda}_\delta^*$ prevents having $\mu_1\big[-\Delta - \rho'_{\overline{\lambda}_\delta^*}(v^*)\big] > 0$. Hence we get a positive eigenfunction φ_1 satisfying $-\Delta\varphi_1 - \rho'_{\overline{\lambda}_\delta^*}(v^*)\varphi_1 = 0$ in Ω and $\varphi_1 = 0$ on $\partial\Omega$.

If we have a solution v of $(F_{\overline{\lambda}_\delta^*})$ such that $v \neq v^*$, we know that $v \geq v^*$ as $v \geq v_\lambda$ for any $\lambda < \overline{\lambda}_\delta^*$. Letting $\phi = v - v^*$, so that $-\Delta\phi = \rho_{\overline{\lambda}_\delta^*}(v) - \rho_{\overline{\lambda}_\delta^*}(v^*) \geq 0$ by (4), the strong maximum principle implies that $\phi > 0$ in Ω. Remarking also that $\rho''_\lambda > 0$ in $\mathbb{R}_+$ for any $\lambda > 0$, then $-\Delta\phi - \rho'_{\overline{\lambda}_\delta^*}(v^*)\phi > 0$ in Ω. By multiplying with φ_1 and integrating by parts, we immediately get a contradiction.

Another consequence is that v^* is a bifurcation point for the solution curve, which will continue with $\|v\|_\infty$ tending to ∞ and the associated λ must go to zero. So we get at least two solutions to (F_λ) for any $\lambda \in (0, \overline{\lambda}_\delta^*)$. Coming back to u, we obtain the main theorem of this paper.

Theorem 5. *If a family of solutions* $\{u^k\}$ *of* (E_{λ^k}) *satisfies* $\lim_{k\to\infty}\|u^k\|_\infty = 1$, *then* $\lim_{k\to\infty}\lambda^k = 0$. *Furthermore,* $u^* = \lim_{\lambda\to\overline{\lambda}^*_\delta} u_\lambda$ *is the unique solution of the limit equation* $(E_{\overline{\lambda}^*_\delta})$ *while for any* $\lambda \in (0, \overline{\lambda}^*_\delta)$, *the equation* (E_λ) *has at least two solutions.*

5. Estimate of $\overline{\lambda}^*_\delta$

Here we compare $\overline{\lambda}^*_\delta$ with $\overline{\lambda}^*$ in the lower dimension situation.

Theorem 6. *For* $n < 8$ *and* $\delta > 0$, *we have*

$$\frac{\overline{\lambda}^*}{1+\delta\|\nabla\overline{u}_*\|^2_\infty} \le \overline{\lambda}^*_\delta \le \overline{\lambda}^*, \tag{6}$$

where $\overline{\lambda}^*$ *is the critical value for the problem* (S_λ) *and* $\overline{u}_*$ *is the unique solution of* $(S_{\overline{\lambda}^*})$.

Proof. As any solution of (E_λ) is a supersolution of (S_λ), it is clear that $\overline{\lambda}^*_\delta \le \overline{\lambda}^*$. On the other hand, when $n < 8$, $\overline{u}_*$ is a smooth function with $\|\overline{u}_*\|_\infty < 1$ (see [4]). Obviously $\overline{u}_*$ is a supersolution for (E_λ) with

$$\lambda = \frac{\overline{\lambda}^*}{1+\delta\|\nabla\overline{u}_*\|^2_\infty},$$

so we get the lower bound. □

Therefore $\overline{\lambda}^*_\delta = \overline{\lambda}^* + O(\delta)$ in dimension two. This confirms somehow the formal result in [11] (see also another bound of $\overline{\lambda}^*_\delta$ in section 5 of [14]).

6. Remarks and open questions

As we have seen in Theorem 5, the introduction of a fringing field basically destroys the infinite fold point structure of the basic membrane problem (S_λ) for any smooth domain.

There are still some interesting questions:

- Do we have some *weak* solutions with $\|u\|_\infty = 1$ for (E_λ)? We turn to conjecture that no *weak* solution exists for the fringing field model. In fact, using Sobolev embedding and a boot-strap argument, any weak solution of (F_λ) satisfying $\rho_\lambda(v) \in L^1(\Omega)$ is indeed smooth. However, if u is just a *weak* solution for (E_λ), it is not clear that $v = \rho_\lambda(u)$ is then a weak solution for (F_λ).
- In [11], Lindsay and Ward derived the following asymptotic behavior of $\overline{\lambda}^*_\delta$:

$$\overline{\lambda}^*_\delta = \lambda^* - C\delta + O(\delta^2) \tag{7}$$

  in the case of a unit disk or a slab in $\mathbb{R}^2$, where $C > 0$ is a constant depending on $\overline{u}_*$ of the unit disk or slab without the fringing field. Can we prove rigorously this first-order expansion (7)? A key point seems to prove a uniform upper bound for v^* as δ tends to zero.
- In nice domains (disks, convex domains with two axes of symmetry in $\mathbb{R}^2$), it has been shown that for the problem (S_λ), there exists a $\overline{\lambda}_* > 0$ such that

the solution branch has infinitely many turns as λ crosses $\overline{\lambda}_*$ (see [9, 10]). On the other hand, in the presence of a fringing field, there are at most finitely many turns. What is the asymptotic behavior of the solutions near $\overline{\lambda}_*$ as $\delta \to 0^+$?

- It seems that there are no studies on the corresponding parabolic equation

$$u_t - \Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^2}. \tag{8}$$

What is the effect of the fringing field on (8)? Can we establish results similar to [1, 7, 8]?

Acknowledgments

The authors thank the Department of Mathematics, East China Normal University, for its kind hospitality. The first author thanks Prof. M. Ward for introducing this problem and Prof. Z. M. Guo for useful discussions.

References

[1] Brezis H., Cazenave T., Martel Y. and Ramiandrisoa A., *Blow-up for $u_t - \Delta u = g(u)$ revisited*, Advances in Differential Equations **1**, 73-90 (1996). MR1357955 (96i:35063)

[2] Brezis H. and Turner R.E.L., *On a class of superlinear elliptic problems*, Comm. Partial Differential Equations **2(6)**, 601–614 (1977). MR0509489 (58:23044)

[3] Esposito P. and Ghoussoub N., *Uniqueness of solutions for an elliptic equation modeling MEMS*, Methods Appl. Anal. **15(3)**, 341–353 (2008). MR2500851

[4] Esposito P., Ghoussoub N. and Guo Y.J., *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60**, 1731-1768 (2007). MR2358647 (2009c:35119)

[5] Flores G., Mercado G.A. and Pelesko J.A., *Dynamics and touchdown in electrostatic MEMS*, Proceedings of ASME DETC'03, 1-8, IEEE Computer Soc. (2003).

[6] Ghoussoub N. and Guo Y.J., *On the partial differential equations of electrostatic MEMS devices: Stationary case*, SIAM. J. Math. Anal. **38(5)**, 1423-1449 (2007). MR2286013 (2007m:35063)

[7] Ghoussoub N. and Guo Y.J., *Estimates for the quenching time of a parabolic equation modeling electrostatic MEMS*, Methods Appl. Anal. **15(3)**, 361-376 (2008). MR2500853

[8] Guo Y.J., Pan Z. and Ward M.J., *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties*, SIAM J. Appl. Math. **166**, 309-338 (2006). MR2179754 (2006f:35130)

[9] Guo Z.M. and Wei J.C., *Asymptotic behavior of touch-down solutions and global bifurcations for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Anal. **7**, 765-786 (2008). MR2393396

[10] Guo Z.M. and Wei J.C., *Infinitely many turning points for an elliptic problem with a singular nonlinearity*, J. Lond. Math. Soc. **78(1)**, 21-35 (2008). MR2427049

[11] Lindsay A.E. and Ward M.J., *Asymptotics of nonlinear eigenvalue problems modeling a MEMS capacitor: Part I: Fold point asymptotics*, Methods Appl. Anal. **15**, 297-326 (2008). MR2500849

[12] López-Gomez J., *The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems*, J. Differential Equations **127(1)**, 263-294 (1996). MR1387266 (97b:35037)

[13] Pelesko J.A. and Bernstein D.H., *Modeling MEMS and NEMS*, Chapman Hall and CRC Press (2002). MR1955412 (2003m:74004)

[14] Pelesko J.A. and Driscoll T.A., *The effect of the small-aspect-ratio approximation on canonical electrostatic MEMS models*, J. Eng. Math. **53**, 239–252 (2005). MR2230109 (2006m:74113)

[15] Rabinowitz P., *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7**, 487-513 (1971). MR0301587 (46:745)

[16] Schaaf R., *Uniqueness for semilinear elliptic problems: Supercritical growth and domain geometry,* Adv. Differential Equations **5**, 1201-1220 (2000). MR1785673 (2002b:35058)

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk

LMAM, UMR 7122, Université de Metz, 57045 Metz, France

E-mail address: dong.ye@univ-metz.fr