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A CAUCHY-RIEMANN EQUATION FOR GENERALIZED ANALYTIC FUNCTIONS

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Abstract. We denote by T^2 the torus: $z = \exp i\theta, w = \exp i\phi$, and we fix a positive irrational number α. A_α denotes the space of continuous functions f on T^2 whose Fourier coefficient sequence is supported by the lattice half-plane $n + m\alpha \geq 0$. R. Arens and I. Singer introduced and studied the space A_α, and it turned out to be an interesting generalization of the disk algebra. Here we construct a differential operator X_Σ on a certain 3-manifold Σ_0 such that X_Σ characterizes A_α in a manner analogous to the characterization of the disk algebra by the Cauchy-Riemann equation in the disk.

1. Introduction

Let Γ be the unit circle. The disk algebra A on Γ is the space of all continuous functions f on Γ such that the Fourier expansion of f is:

$$\sum_{n=0}^{\infty} c_n \exp(in\theta);$$

i.e., the Fourier coefficient sequence of f is supported on the semi-group $n \geq 0$ of Z.

In [1], R. Arens and I.M. Singer studied the following generalization of the disk algebra: we replace Γ by the 2-torus T^2 and fix a positive irrational number α. The dual group of T^2 is $Z \oplus Z$. We replace the semi-group of nonnegative integers by the semi-group of all pairs of integers (n, m) with $n + m\alpha \geq 0$. We define the algebra A_α as the space of continuous functions on T^2 with Fourier expansion on the torus given by

$$\sum_{n+m\alpha \geq 0} c_{nm} \exp(in\theta) \exp(im\phi).$$

A_α is called a space of Generalized Analytic Functions. In [4], H. Helson and D. Lowdenslager made a detailed study of A_α and showed that many basic results of analytic function theory on the unit disk extend from A to A_α.

An alternative description to the disk algebra is the following: A consists of those functions f continuous on Γ which admit a continuous extension to the closed disk Δ, again denoted by f, such that f is smooth on the interior of Δ and there it

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satisfies the equation

$$\frac{\delta}{\delta \bar{z}}(f) = 0. \tag{1}$$

In [1] the disk Δ is replaced by the maximal ideal space Π of the Banach algebra A_α, taken in the Gelfand topology. It is shown in [1] that Π has a natural identification with the following compact subset of C^2:

The set of all points(z, w) in C^2 such that $|w| = |z|^\alpha$ and $|z| \leq 1$.

We denote this subset of C^2 by Σ. In this identification, T^2 turns into the set of all points $(z, w) \in C^2$ such that $|z| = |w| = 1$.

Our purpose is to give an equation analogous to (1) on the space $\Sigma \setminus (T^2 \cup 0)$, which characterizes A_α. To this end we define the differential operator X on C^2 by $X = \bar{z}\frac{\delta}{\delta \bar{z}} + \alpha \bar{w}\frac{\delta}{\delta \bar{w}}$. As is shown below, X restricts to a well-defined differential operator on the smooth manifold: $\Sigma \setminus T^2 \cup \{0\}$, which we denote by Σ_0.

Theorem 1.1. *A function $f \in C(T^2)$ lies in A_α if and only if f admits a continuous extension (denoted F) to Σ such that $XF = 0$, in the sense of distributions, on $\Sigma \setminus (T^2 \cup 0)$.*

Theorem 1.2. *Given a point $(z_0, w_0) \in \Sigma \setminus (T^2 \cup 0)$, there exists some function f in A_α such that the extension F is not differentiable on Σ at that point.*

2. Proof of Theorem 1.1

We put $\Sigma_0 = \Sigma \setminus T^2 \cup (0, 0)$.

Let ϕ be the function on $C^2 \setminus z = 0$ given by $\phi(z, w) = w\bar{w} - z^\alpha \bar{z}^\alpha$.

Σ has the equation: $\phi(z, w) = 0$. We write $D_{\bar{z}}$ for the derivative with respect to $\bar{z}$ and similarly for w. $X(\phi) = \alpha\phi$, by direct calculation. So $X(\phi) = 0$ on Σ.

Since Σ is given by the equation $\phi = 0$, it follows that the operator X is well-defined on $C^\infty(\Sigma)$. We denote this operator, which acts on functions defined on Σ, by X_Σ. We wish to express X_Σ in local coordinates on Σ. Fix a point (z_0, w_0) on Σ. Then $w_0 = z_0^\alpha \exp(i\theta_0)$ for some z_0, θ_0 with $|z_0| < 1, 0 \leq \theta_0 \leq 2\pi$. We define a neighborhood U of (z_0, w_0) on Σ by:

$$U = \text{ the set of } (t, exp(i\theta)t^\alpha), |t - z_0| < \delta, |\theta - \theta_0| < \delta.$$

We fix a single-valued branch of t^α near $t = z_0$.

We use $t, \bar{t}, \theta$ as local coordinates in U. Further, we denote the operator $\frac{\delta}{\delta \bar{t}}$ by $D_{\bar{t}}$.

Claim 1. $\bar{t}D_{\bar{t}} = X_\Sigma$ as a differential operator on U. Hence for a continuous function f on U, $tD_{\bar{t}}f = X_\Sigma f$, as a distribution on U.

Proof of Claim 1. We apply both sides to the functions $t, \bar{t}, \exp(i\theta), \exp(-i\theta)$. We note that $\bar{t}$ is the restriction of $\bar{z}$ to Σ. Since $X = \bar{z}D_{\bar{z}} + \alpha\bar{w}D_{\bar{w}}, X_\Sigma(\bar{t}) = \bar{t}$ on U. Next, t is the restriction of z to Σ. So $X_\Sigma(t) = 0$. Next, $X_\Sigma(\exp(i\theta)) = X(\frac{w}{z^\alpha}) = 0$. Similarly, $X_\Sigma(\exp(-i\theta)) = 0$. On the other hand, $\bar{t}D_{\bar{t}}(\bar{t}) = \bar{t}$, $\bar{t}D_{\bar{t}}(t) = 0$, $\bar{t}D_{\bar{t}}(\exp(i\theta)) = 0$, $\bar{t}D_{\bar{t}}(\exp(-i\theta)) = 0$.

So X_Σ and $\bar{t}D_{\bar{t}}$ agree on each of the functions $t, \bar{t}, \exp i\theta$ on U. Also, $0 = X_\Sigma(\exp i\theta) = i \exp i\theta X_\Sigma(\theta)$, so $X_\Sigma(\theta) = 0$. Similarly, $\bar{t}D_{\bar{t}}(\theta) = 0$. It follows that for all G in $C^\infty(U)$ we have $\bar{t}D_{\bar{t}}(G) = X_\Sigma(G)$. This proves our claim. □

We next follow the Arens-Singer paper in introducing a foliation of the 3-manifold Σ_0 by a one-parameter family of Riemann surfaces Λ_θ. We then shall prove

Theorem 2.1. *Fix f in $C^\infty(\Sigma)$. Then $Xf = 0$ on Σ if and only if the restriction of f to Λ_θ is holomorphic on Λ_θ for each θ.*

We denote by H^+ the right half-plane: $\mathrm{Re}\zeta > 0$. For each $\theta \in [0, 2\pi]$ we put $\chi_\theta(\zeta) = (\exp(-\zeta), \exp i\theta \exp(-\alpha\zeta))$, where ζ is in the closed right half-plane.

Definition. Fix θ. Λ_θ is the image in C^2 of H^+ under the map χ_θ.

Since $|\exp i\theta \exp(-\alpha\zeta)| = |\exp(-\zeta)|^\alpha$, Λ_θ is a subset of Σ. The map χ_θ is one-one from H^+ to Λ_θ. We use this map to give Λ_θ the structure of a Riemann surface. We verify that for θ and θ' distinct points in $[0, 2\pi]$, the sets Λ_θ and $\Lambda_{\theta'}$ are disjoint.

For a function g defined on Λ_θ, we say that "g is holomorphic on Λ_θ" if the composition $\zeta \to g(\chi_\theta(\zeta))$ is holomorphic on H^+.

Now fix f in $C(\Sigma)$ with $Xf = 0$ on Σ. Fix θ_0. We must show that f, restricted to Λ_{θ_0}, is holomorphic on Λ_0.

Let (z_0, w_0) be a point on Λ_{θ_0}. We fix a single-valued branch of the function z^α in a neighborhood $|z - z_0| < \delta$ and fix $\epsilon > 0$. Put $U_\epsilon = \{(z, z^\alpha \exp i\theta) | |z - z_0| < \delta, |\theta - \theta_0| < \epsilon\}$. Let D be the disk $\{(z, z^\alpha \exp i\theta_0), |z - z_0| < \delta\}$, so $D \subset \Lambda_0$.

Choose a test function ϕ in $C_0^\infty(D)$, and extend ϕ to a smooth function $\tilde{\phi}_\epsilon$ in $C_0^\infty(U_\epsilon)$.

Since $Xf = 0$, by hypothesis, $X_\Sigma f = 0$ on U (where we suppress the subscript ϵ). So by Claim 1, $\bar{t}D_{\bar{t}}(f) = 0$ as a distribution on U. Therefore, $D_{\bar{t}}(f) = 0$ as a distribution on U. So $\langle D_{\bar{t}}(f), \tilde{\phi}_\epsilon\rangle = -\int_U f D_{\bar{t}}\tilde{\phi}_\epsilon = -\int_D dt \wedge d\bar{t} \int_{\theta_0-\epsilon}^{\theta_0+\epsilon} f(t, \exp i\theta t^\alpha) D_{\bar{t}}\tilde{\phi}_\epsilon d\theta$.

Since $D_{\bar{t}}(f) = 0$ on U, we get for each $\epsilon > 0$: $0 = \int_D dt d\bar{t} \epsilon^{-1} \int_{\theta_0-\epsilon}^{\theta_0+\epsilon} f D_{\bar{t}}\tilde{\phi}_\epsilon d\theta$, where the integrand of the inner integral is evaluated at $(t, \exp i\theta t^\alpha)$. As ϵ approaches zero, we get in the limit

$$0 = \int_D f(t, \exp i\theta_0 t^\alpha) D_{\bar{t}}\phi(t, \exp i\theta_0 t^\alpha) dt \wedge d\bar{t} = \langle f, D_{\bar{t}}\phi\rangle.$$

So $D_{\bar{t}}(f) = 0$, since this holds for every test function ϕ on D. Since D is an arbitrary small disk on $\Lambda_{\theta_0}, D_{\bar{t}}(f) = 0$ as a distribution on Λ_0. By Weyl's Lemma, then, f, restricted to Λ_{θ_0}, is holomorphic on Λ_{θ_0}.

Conversely, fix $f \in C(\Sigma_0)$ such that f restricted to Λ_θ is holomorphic on Λ_θ for each θ. We must show that $Xf = 0$ on U, where we write X for X_Σ.

Fix (z_0, w_0) in Σ_0. Thus $w_0 = z_0^\alpha \exp i\theta_0$, for some θ_0. We choose a branch of the function z^α and also fix $b > 0$, and form the set

$$U_b = \{(z, z^\alpha \exp i\theta) | |z - z_0| < b, |\theta - \theta_0| < b\}.$$

We claim that $D_{\bar{t}} f = 0$ on U_b. Choose a test function ϕ on U_b. We define

$$I = \int_{U_b} f D_{\bar{t}}\phi dt d\bar{t} d\theta = \langle D_{\bar{t}} f, \phi\rangle.$$

We choose a sequence of smooth functions $\{f_n\}$ on U_b such that for each n the restriction of f_n to Λ_θ is holomorphic on Λ_θ for each θ in $[\theta_0 - b, \theta_0 + b]$ and f_n converges to f uniformly on U_b as $n \to \infty$ We fix n. Put

$$I_n = \int_{U_b} f_n D_{\bar{t}}\phi dt d\bar{t} d\theta = -\int_{U_b} D_{\bar{t}} f_n \phi d_{\bar{t}} d\theta.$$

Since f_n is holomorphic on Λ_θ for each θ, I_n vanishes. Letting $n \to \infty$, we have $I_n \to I$. So $I = 0$.

This holds for all test functions ϕ on U_b. So $D_{\bar{t}} f = 0$ as a distribution on U_b. Since $Xf = \bar{t} D_{\bar{t}} f$ on U_b, then $Xf = 0$ on U_b. Since (z_0, w_0) is an arbitrary point on Σ_0, $Xf = 0$ on Σ_0.

Theorem 2.1 is proved. We now proceed to the proof of Theorem 1.1.

Proof. For f in $C(T^2)$, we put $||f|| = \max |f|$, taken over T^2. We define $\mathcal{A} = \{f \in C(T^2)\}$ such that f has a continuous extension to Σ, denoted F, with $XF = 0$ on Σ_0, in the sense of distributions. Fix f in $\mathcal{A}$. By Theorem 2.1, then, F, restricted to Λ_θ, is holomorphic on Λ_θ for each θ; i.e., $F(\chi_\theta)$ is holomorphic on H^+, where χ_θ was defined above. Also, since F is continuous on the compact set Σ, $F(\chi_\theta)$ is bounded on H^+. Finally, for $\zeta = it$, t real, where $\chi_\theta(\zeta) = (\exp it, \exp i\theta \exp -i\alpha t) \in T^2$, $|F(\chi_\theta(\zeta))| \leq ||f||$.

By the Phragmén-Lindelöf theorem, then, $|F(\chi_\theta(\zeta))| \leq ||f||$ for all $\zeta \in H^+$, so $|F| \leq ||f||$ on Σ. Thus the functions in $\mathcal{A}$, viewed on Σ, satisfy the maximum principle relative to T^2. We note that $\mathcal{A}$ is a linear space of functions.

Claim 2. $\mathcal{A}$ is closed under uniform convergence on T^2.

Proof of Claim 2. Let $\{f_n\}$ be a sequence of functions in $\mathcal{A}$ which converges uniformly on T^2 to a function f. Fix $(z_0, w_0) \in \Sigma$. For each of the indices n, m, we have

$$|F_n(z_0, w_0) - F_m(z_0, w_0)| \leq ||f_n - f_m||,$$

since $f_n - f_m \in \mathcal{A}$. Hence as n, m tend to ∞, F_n converges, uniformly on Σ, to some continuous function F, and $F = f$ on T^2. Furthermore, for each of the Riemann surfaces Λ_θ, each F_n is holomorphic. Hence F is holomorphic on Λ_θ. By Theorem 2.1, then, F satisfies $XF = 0$ on Σ_0. So f again belongs to $\mathcal{A}$. This was the claim. $\square$

Claim 3. $\mathcal{A}$ is an algebra of functions on T^2.

Proof of Claim 3. Let $f, g \in \mathcal{A}$, and let F, G be their corresponding extensions to Σ. Since F and G are continuous on Σ, so is FG, and since F and G are each holomorphic on Λ_θ for every θ, so is FG. Hence by Theorem 2.1, $X(FG) = 0$ on Σ_0. Also FG is a continuous extension of fg from T^2 to Σ. So fg lies in $\mathcal{A}$. Claim 3 is proved. $\square$

Claim 4. $\mathcal{A}$ contains A_α.

Proof of Claim 4. By Fejér's theorem, A_α is the closed span in $C(T^2)$ of the set of functions $\phi_{n,m} = \exp in\theta \exp im\phi, n + m\alpha > 0$. Fix n, m with $n + m\alpha > 0$. We claim that $\phi_{n,m}$ lies in $\mathcal{A}$. With z, w the complex coordinates in C^2, we consider the extension $z^n w^m$ of $\phi_{n,m}$ to Σ. The continuity is clear except at the origin. For $z, w \epsilon \Sigma_0$,

$$|z^n w^m| = |z|^n |z|^{m\alpha} = |z|^{n+m\alpha}.$$

As $(z, w) \to (0, 0)$, this tends to 0. So $z^n w^m$ provides a continuous extension of $\phi_{n,m}$ to Σ. Further, $X(z^n w^m) = 0$ on Σ_0, since $z^n w^m$ extends to be holomorphic in a neighborhood of Σ_0 in C^2. So $z^n w^m$ provides the desired extension of $\phi_{n,m}$, and so $\phi_{n,m} \in \mathcal{A}$. Since A_α is the closed span of the $\phi_{n,m}$ in $C(T^2)$, A_α is contained in $\mathcal{A}$. Claim 4 is proved. $\square$

By Claims 1 and 2, we know that $\mathcal{A}$ is closed under uniform convergence on T^2 and is an algebra of functions on T^2. By Claim 3, $\mathcal{A}$ contains A_α. Theorem 2.3 in Chapter 7 of T.W. Gamelin's book [2] gives that A_α is a maximal subalgebra of $C(T^2)$; i.e., no closed subalgebra of $C(T^2)$ lies properly between A_α and $C(T^2)$. So $A_\alpha = \mathcal{A}$. Theorem 1.1 is proved. □

We proceed to the proof of Theorem 1.2.

Proof. We use the earlier notation.

Claim. There exist integers $p_j, q_j \in Z^+, j = 1, 2, ...,$ such that
(1) $-p_j + \alpha q_j > 0$ for all j, and
(2) $-p_j + \alpha q_j \to 0$, as $j \to \infty$.

Proof of Claim. A classical fact from the theory of continued fractions (see Hardy and Wright [3], Chapter X) gives the existence of a sequence of rational numbers $\frac{p_j}{q_j}$ such that

(3) $|\alpha - \frac{p_j}{q_j}| < \frac{1}{q_j^2}, j = 1, 2, ...$ such that p_j and q_j are positive integers tending to ∞ as $j \to \infty$, and $\frac{p_j}{q_j} < \alpha$ for each j.

Thus for each j, we have $\alpha = \frac{p_j}{qj} + \delta_j$, with $0 < \delta_j < \frac{1}{q_j^2}$. It follows that we have $-p_j + \alpha q_j = q_j \delta_j$. In view of the bound on δ_j, then, we have (1) and (2). So the Claim is proved. □

Let $\{\epsilon_n\}$ be a sequence of real numbers tending to 0. Fix a point (z_0, z_0^α) in Σ. We now define a sequence of bounded linear functionals L_n on A_α, as follows:

For f in A_α, and F denoting the extension of f to Σ, we put

$$L_n f = (\epsilon_n)^{-1}(F(z_0, \exp i\epsilon_n z_0^\alpha) - F(z_0, z_0^\alpha)).$$

Let p_j, q_j be as in the Claim. Define $f_j = \exp -ip_j\theta \exp iq_j\theta$ on T^2. Since $-p_j + \alpha q_j > 0$, by (1), $f_j \in A_\alpha$. Further, for $(z, w) \in \Sigma, F_j(z, w) = z^{-p_j} w^{q_j}$. So

$$\begin{aligned} L_n f_j = (\epsilon_n^{-1})(z_0^{-p_j}(\exp i\epsilon_n z_0^\alpha)^{q_j} - z_0^{-p_j}(z_0^\alpha)^{q_j}) &= (\epsilon_n^{-1})(z_0^{-p_j} z_0^{\alpha q_j})(\exp i\epsilon_n q_j - 1) \\ &= (z_0^{-p_j + \alpha q_j})(\frac{\exp i\epsilon_n q_j - 1}{\epsilon_n}). \end{aligned}$$

We now take $j = n$ and take absolute values. We get

$$|L_n(f_n)| = (|z_0|^{-p_n + \alpha q_n})(\epsilon_n^{-1})|(\exp i\epsilon_n q_n - 1)|.$$

We next take $\epsilon_n = \frac{\pi}{q_n}$. This gives $|L_n(f_n)| = |z_0|^{-p_n + \alpha q_n}(\frac{2}{\pi})q_n$. Since $-p_n + \alpha q_n \to 0$ and $q_n \to \infty$ as $n \to \infty, |L_n(f_n)| \to \infty$ as n approaches ∞. Also, $||f_n|| = 1$ for each n. So the norm $||L_n||$, as a functional on A_α, becomes unbounded as n grows.

Next, we fix $f \in A_\alpha$ and the point $x_0 = (z_0, z_0^\alpha)$. F denotes the extension of f to Σ. We put $\Psi(\theta) = F(z_0, \exp i\theta z_0^\alpha), -\pi \le \theta \le \pi$. Then

$$L_n(f) = (\epsilon_n^{-1})(F((z_0, \exp i\epsilon_n z_0^\alpha)) - F(z_0, z_0^\alpha)) = (\epsilon_n^{-1})(\Psi(\epsilon_n) - \Psi(0)).$$

Suppose now that F is differentiable on Σ at x_0. Then Ψ is differentiable at $\theta = 0$. Hence by the preceding equality, the sequence $\{L_n(f)\}$ converges as n approaches ∞. Since L_n converges pointwise on the Banach space A_α, the uniform boundedness theorem yields that the sequence $\{L_n\}$ is bounded. This contradicts our earlier result. So for some f in A_α, F fails to be differentiable at the given point. Theorem 1.2 is proved. □

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