THE PINCHING CONSTANT OF MINIMAL HYPERSURFACES IN THE UNIT SPHERES

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ABSTRACT. In this paper, we prove that if $M^n$ ($n \leq 8$) is a closed minimal hypersurface in a unit sphere $S^{n+1}(1)$, then there exists a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $M$ is isometric to a Clifford torus, where $S$ is the squared norm of the second fundamental form of $M$.

1. INTRODUCTION

Let $M^n$ be an $n$-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n + 1$. Denote by $S$ the squared norm of the second fundamental form of $M^n$. Lawson [2], Simons [1] and Chern, do Carmo, Kobayashi [3] obtained independently the famous rigidity theorem, which says that if $S \leq n$, then $S \equiv 0$ or $S \equiv n$; i.e., $M^n$ is the great sphere $S^n(1)$ or the Clifford torus. Further discussions in this direction have been carried out by many other authors [6, 8, 9, 10, 11, 12, 13]. In [4], Peng and Terng proved that if the scalar curvature of $M^n$ is constant, then there exists a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. Later, Cheng and Yang [14] improved the pinching constant $\alpha(n)$ to $n/3$. More generally, Peng and Terng [5] proved that if $M^n(n \leq 5)$ is a closed minimal hypersurface in $S^{n+1}$, then there exists a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. So they proposed the following problem.

Let $M^n$ ($n \geq 6$) be a closed minimal hypersurface in $S^{n+1}$. Does there exist a positive constant $\alpha(n)$ depending only on $n$ such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$?

In [15], Cheng gives a positive answer under the additional condition that $M$ has only two distinct principal curvatures. Later, Cheng and Ishikawa [6] improved the result of Peng and Terng [5] when $n \leq 5$.

In this paper, we solve the problem proposed by Peng and Terng [5] for $n \leq 8$.

**Theorem 1.1.** Let $M^n$ ($n \leq 8$) be a closed minimal hypersurface in $S^{n+1}(1)$. If $n \leq S \leq n + \alpha(n)$, then $S \equiv n$ and $M^n$ is isometric to a Clifford torus $S^m(\sqrt{n}) \times S^{n-m}(\sqrt{n-m})$, where $\alpha(n) = \frac{2(n+4)(3-n\delta)}{9n+30}$, $\delta(3) = 0$, $\delta(4) = 0.16$, $\delta(5) = 0.23$, $\delta(6) = 0.28$, $\delta(7) = 0.32$ and $\delta(8) = 0.34$.
For $n \leq 5$, Cheng and Ishikawa [6] proved the following: Let $M$ be an $n$-dimensional ($n \leq 5$) closed minimal hypersurface of a unit sphere $S^{n+1}(1)$. If $n \leq S \leq n + \alpha(n)$, then $S = n$, where $\alpha(3) = \frac{42}{85}$, $\alpha(4) = \frac{8}{31}$ and $\alpha(5) = \frac{3(21 - 5\sqrt{17})}{28 + 3\sqrt{17}}$. It is obvious that our pinching constant is larger than theirs. Up to now, the open problem for $n \geq 9$ is still open and it is a very hard problem.

2. Fundamental formulas

Let $M^n$ be an $n$-dimensional hypersurface in an $(n + 1)$-dimensional unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in $S^{n+1}(1)$, restricted to $M^n$, so that $e_1, \ldots, e_n$ are tangent to $M^n$. Let $\omega_1, \ldots, \omega_{n+1}$ denote the dual coframe field in $S^{n+1}(1)$. Then in $M^n$, $\omega_{n+1} = 0$. It follows from Cartan’s Lemma that

\begin{equation}
\omega_{n+1} = \sum_j h_{ij} \omega_j.
\end{equation}

The second fundamental form $\alpha$ and the mean curvature $H$ of $M^n$ are defined by

\begin{equation}
\alpha = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1}, \quad nH = \sum_i h_{ii},
\end{equation}

respectively. If $M^n$ is a minimal hypersurface, then $\sum_i h_{ii} = 0$. The connection form $\omega_{ij}$ is characterized by the structure equations

\begin{equation}
d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,
\end{equation}

\begin{equation}
d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij},
\end{equation}

\begin{equation}
\Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,
\end{equation}

where $\Omega_{ij}$ (resp. $R_{ijkl}$) denotes the curvature form (resp. the components of the curvature tensor) of $M^n$. The Gauss equation is given by

\begin{equation}
R_{ijkl} = (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (h_{ik} h_{jl} - h_{il} h_{jk}).
\end{equation}

Denote by $h_{ijk}, h_{ijkl}, h_{ijklm}$ the components of the first, second and third covariant derivatives of the second fundamental form, respectively. Then

\begin{equation}
h_{ijk} = h_{ikj} = h_{jik},
\end{equation}

\begin{equation}
h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl},
\end{equation}

\begin{equation}
h_{ijklm} - h_{ijkml} = \sum_r h_{rjk} R_{rilm} + \sum_r h_{irk} R_{rjlm} + \sum_r h_{ijr} R_{rklm}.
\end{equation}

For any fixed point $p$ in $M^n$, we take a local orthonormal frame field $e_1, \cdots, e_n$ such that

\begin{equation}
h_{ij} = \begin{cases}
\lambda_i, & i = j, \\
0, & i \neq j.
\end{cases}
\end{equation}
Let \( S := \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2 \). The following formulas can be obtained by a direct computation (cf. [7]):

\[
\begin{align*}
(2.11) & \quad \frac{1}{2} \Delta S = \sum_{i,j,k} h_{ijk}^2 - S(S - n), \\
(2.12) & \quad \frac{1}{2} \sum_{i,j,k} h_{ijk}^2 = \sum_{i,j,k,l} h_{ijkl}^2 + (2n + 3 - S) \sum_{i,j,k} h_{ijk}^2 + 3(2B - A) - \frac{3}{2} |\nabla S|^2,
\end{align*}
\]

where \( A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2, B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2 \).

### 3. PROOF OF THEOREM [1.1]

At first we give a proposition and some lemmas which will play a crucial role in the proof of our theorem. For convenience, we let

\[
b_i = h_{iii}, b = \sum_{i \neq 1} b_i^2 + \frac{1}{3} b_1^2, f = \sum_{i \neq 1} (\lambda_i^2 - 4 \lambda_1 \lambda_i) b_i^2 - \lambda_1^2 b_1^2.
\]

**Proposition 3.1.** Let \( M^n \) be a closed minimal hypersurface in \( S^{n+1}(1) \). Suppose that

\[
3(A - 2B) \leq [2 + \delta(n)] \sum_{i,j,k} h_{ijk}^2,
\]

where \( \delta(n) \) is a number depending only on \( n \) such that \( 0 \leq \delta(n) < \min \{ \frac{1}{2}, \frac{3}{n} \} \).

Then there exists a positive constant \( \alpha(n) \) depending only on \( n \) such that if \( n \leq S \leq n + \alpha(n) \), then \( S \equiv n \); i.e., \( M^n \) is isometric to a Clifford torus \( S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}}) \). Here, \( \alpha(n) = \frac{2(n+4)(3-n\delta)}{9n+30} \).

**Proof.** Since \( M^n \) is a minimal hypersurface in \( S^{n+1}(1) \), from (2.11) and (2.12) we have

\[
\begin{align*}
(3.1) & \quad \int_M \sum_{i,j,k} h_{ijk}^2 dM = \int_M S(S-n)dM, \\
(3.2) & \quad -\frac{1}{2} \int_M |\nabla S|^2 dM = \int_M \left[ S \sum_{i,j,k} h_{ijk}^2 - S^2(S-n) \right] dM, \\
(3.3) & \quad \int_M \sum_{i,j,k,l} h_{ijkl}^2 dM = \int_M \left[ (S-2n-3) \sum_{i,j,k} h_{ijk}^2 + 3(A-2B) + \frac{3}{2} |\nabla S|^2 \right] dM.
\end{align*}
\]

Letting \( f_3 = \sum_i \lambda_i^3 \) and \( f_4 = \sum_i \lambda_i^4 \), we have (cf. [6])

\[
\begin{align*}
(3.4) & \quad \sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2} (Sf_4 - f_3^2 - 2S^2 + nS) + \frac{3S(S-n)^2}{2(n+4)}, \\
(3.5) & \quad \int_M (A-2B) dM = \int_M \left[ Sf_4 - f_3^2 - S^2 - \frac{1}{4} |\nabla S|^2 \right] dM.
\end{align*}
\]

From (3.3), (3.4) and (3.5), we have

\[
\int_M \left[ (S-2n-3) \sum_{i,j,k} h_{ijk}^2 + \frac{3}{2} (A-2B) + \frac{3}{2} S(S-n) + \frac{9}{8} |\nabla S|^2 - \frac{3S(S-n)^2}{2(n+4)} \right] dM \geq 0.
\]
Noticing that \( S^2 = S(S - n) + nS \), from (3.1), (3.2) and (3.6), we have

\[
(3.7) \quad \int_M \left[ \frac{3}{2}(A - 2B) + \frac{9n + 30}{4(n + 4)} S(S - n)^2 - \left( \frac{5}{4} S - \frac{n}{4} + \frac{3}{2} \right) \sum_{i,j,k} h_{ijk}^2 \right] dM \geq 0.
\]

Suppose \( 3(A - 2B) \leq [2 + \delta(n)] S \sum_{i,j,k} h_{ijk}^2 \) and \( n \leq S \leq n + \alpha(n) \). From the above inequality, we have

\[
(3.8) \quad \int_M \left\{ \frac{9n + 30}{4(n + 4)} \alpha(n) - \frac{1 - 2\delta(n)}{4} (S - n) - \frac{3 - n\delta(n)}{2} \right\} \sum_{i,j,k} h_{ijk}^2 dM \geq 0.
\]

Since \( \alpha(n) = \frac{2(n + 4)(3 - n\delta)}{9n + 30} \) and \( \delta(n) < \min \left\{ \frac{1}{2}, \frac{1}{4} \right\} \), we have

\[
(3.9) \quad - \int_M (S - n) \sum_{i,j,k} h_{ijk}^2 dM \geq 0.
\]

Hence, \( S \equiv n \). This completes the proof of Proposition 3.1.

**Lemma 3.2.** Let \( M^n \) be a closed minimal hypersurface in \( S^{n+1}(1) \). If \( \lambda_1^2 - 4\lambda_1 \lambda_2 \geq tS \) \( (t \geq 2) \), then \( (\lambda_1^2 - 4\lambda_1 \lambda_i) - (\lambda_2^2 - 4\lambda_1 \lambda_i) \geq rS \) \( (i \neq 1, 2) \). Here, \( r = \frac{16t - 8 - 12\sqrt{-2t^2 + 2t + 8}}{17} \).

**Proof.** Let \( \lambda_1^2 = x^2S, \lambda_2^2 = y^2S \) \( (x, y > 0) \). Since \( \lambda_1^2 - 4\lambda_1 \lambda_2 \geq tS \) \( (t \geq 2) \), we have \( x^2 + 4xy \geq t \), that is, \( y \geq \frac{t - x^2}{4x} \). Hence, we have

\[
\lambda_1^2 - 4\lambda_1 \lambda_i \leq \left( x^2 + 4x\sqrt{1 - x^2 - y^2} \right) S
\leq \left\{ x^2 + \sqrt{16x^2 - 16x^4 - (t - x^2)^2} \right\} S
= \left\{ x^2 + \sqrt{-17x^4 + (16 + 2t)x^2 - t^2} \right\} S.
\]

Let \( g(z) = z + \sqrt{-17z^2 + (16 + 2t)z - t^2} \) \( (0 < z < 1) \). Then

\[
g'(z) = 1 - \frac{17z - (8 + t)}{\sqrt{-17z^2 + (16 + 2t)z - t^2}}.
\]

Letting \( g'(z_0) = 0 \), we have

\[
z_0 = \frac{3(8 + t) + 2\sqrt{-2t^2 + 2t + 8}}{51}.
\]

Hence we have

\[
g(z) \leq g(z_0) = \frac{t + 8 + 12\sqrt{-2t^2 + 2t + 8}}{17},
\]

which implies that \( \lambda_1^2 - 4\lambda_1 \lambda_i \leq \frac{t + 8 + 12\sqrt{-2t^2 + 2t + 8}}{17} \).

Since \( \lambda_1^2 - 4\lambda_1 \lambda_2 \geq tS \), we have

\[
(\lambda_2^2 - 4\lambda_1 \lambda_2) - (\lambda_1^2 - 4\lambda_1 \lambda_i) \geq \frac{16t - 8 - 12\sqrt{-2t^2 + 2t + 8}}{17}.
\]

This completes the proof of Lemma 3.2.
Lemma 3.3. Let \( f_n(t) = 17[t - 2 - \delta(n)][3(n - 2)t + (n + 2)\delta(n) + 10 - 4n] \) and \( g_n(t) = [8 + 16\delta(n)][4t - 2 - 3\sqrt{-2t^2 + 2t + 8}] \). Then \( h_n(t) = f_n(t) - g_n(t) \leq 0 \) (\( t \geq 2, 4 \leq n \leq 8 \)).

Here, \( \delta(4) = 0.16, \delta(5) = 0.23, \delta(6) = 0.28, \delta(7) = 0.32 \) and \( \delta(8) = 0.34 \).

Proof. By a direct computation, we obtain

\[
h''_n(t) = 51\left\{2(n - 2) - [8 + 16\delta(n)](-2t^2 + 2t + 8)^{-\frac{3}{2}}\right\}
\]

and

\[
h''_n(t) = -153[8 + 16\delta(n)][2t - 1)(-2t^2 + 2t + 8)^{-\frac{3}{2}} < 0 \quad (t \geq 2).
\]

On the other hand, \( h''_n(2) > 0 \) and \( h_n(2) < 0 \). Hence, if there exist real numbers \( t_i > 2 (i = 1, 2) \) such that \( h''_n(t_1) > 0; h''_n(t_2) < 0 \) and \( h_n(t) \leq 0 \) (\( t_1 \leq t \leq t_2 \)), then \( h_n(t) \leq 0 \) (\( \forall t \geq 2 \)).

In the case \( n = 4 \), since \( \delta(4) = 0.16 \), we have

\[
\begin{align*}
f_4(t) &= 17(6t^2 - 18t + 10.8864), \\
g_4(t) &= 10.56(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}), \\
h'_4(t) &= 204t - 348.24 - 31.68(2t - 1)(-2t^2 + 2t + 8)^{-\frac{3}{2}}.
\end{align*}
\]

By a direct computation, we obtain

\[ h'_4(2.48) > 0; h'_4(2.5) < 0. \]

On the other hand, when \( 2.48 \leq t \leq 2.5 \), we have

\[ f_4(t) \leq f_4(2.5) \leq 57.6, \quad g_4(t) \geq g_4(2.48) \geq 57.8. \]

This implies that

\[ h_4(t) \leq 0 \quad (2.48 \leq t \leq 2.5). \]

From [3.10] and [3.11], we know that Lemma 3.3 is true in the case \( n = 4 \).

In the case \( n = 5 \), since \( \delta(5) = 0.23 \), we have

\[
\begin{align*}
f_5(t) &= 17(9t^2 - 28.46t + 18.7097), \\
g_5(t) &= 11.68(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}), \\
h'_5(t) &= 306t - 530.54 - 35.04(2t - 1)(-2t^2 + 2t + 8)^{-\frac{3}{2}}.
\end{align*}
\]

By a direct computation, we obtain

\[ h'_5(2.51) > 0; h'_5(2.52) < 0. \]

On the other hand, when \( 2.51 \leq t \leq 2.52 \), we have

\[ f_5(t) \leq f_5(2.52) \leq 70.5, \quad g_5(t) \geq g_5(2.51) \geq 71. \]

This implies that

\[ h_5(t) \leq 0 \quad (2.51 \leq t \leq 2.52). \]

From [3.12] and [3.13], we know that Lemma 3.3 is true in the case \( n = 5 \).

In the case \( n = 6 \), since \( \delta(6) = 0.28 \), we have

\[
\begin{align*}
f_6(t) &= 17(12t^2 - 39.12t + 26.8128), \\
g_6(t) &= 12.48(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}), \\
h'_6(t) &= 408t - 714.96 - 37.44(2t - 1)(-2t^2 + 2t + 8)^{-\frac{3}{2}}.
\end{align*}
\]
By a direct computation, we obtain

\[(3.14) \quad h_6'(2.53) > 0, \quad h_6'(2.535) < 0.\]

On the other hand, when \(2.53 \leq t \leq 2.535\), we have

\[f_6(t) \leq f_6(2.535) \leq 81, \quad g_6(t) \geq g_6(2.51) \geq 82.\]

This implies that

\[(3.15) \quad h_6(t) \leq 0 (2.53 \leq t \leq 2.535) .\]

From (3.14) and (3.15), we know that Lemma 3.3 is true in the case \(n = 6\).

In the case \(n = 7\), since \(\delta(7) = 0.32\), we have

\[f_7(t) = 17(15t^2 - 49.92t + 35.0784),\]
\[g_7(t) = 13.12(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}),\]
\[h_7'(t) = 510t - 901.12 - 39.36(2t - 1)(-2t^2 + 2t + 8)^{-\frac{1}{2}} .\]

By a direct computation, we obtain

\[(3.16) \quad h_7'(2.54) > 0, \quad h_7'(2.544) < 0.\]

On the other hand, when \(2.54 \leq t \leq 2.544\), we have

\[f_7(t) \leq f_7(2.544) \leq 88, \quad g_7(t) \geq g_7(2.54) \geq 90.\]

This implies that

\[(3.17) \quad h_7(t) \leq 0 (2.54 \leq t \leq 2.544) .\]

From (3.16) and (3.17), we know that Lemma 3.3 is true in the case \(n = 7\).

In the case \(n = 8\), since \(\delta(8) = 0.34\), we have

\[f_8(t) = 17(18t^2 - 60.72t + 43.524),\]
\[g_8(t) = 13.44(4t - 2 - 3\sqrt{-2t^2 + 2t + 8}),\]
\[h_8'(t) = 612t - 1086 - 40.32(2t - 1)(-2t^2 + 2t + 8)^{-\frac{1}{2}} .\]

By a direct computation, we obtain

\[(3.18) \quad h_8'(2.5465) > 0, \quad h_8'(2.5468) < 0.\]

On the other hand, when \(2.5465 \leq t \leq 2.5468\), we have

\[f_8(t) \leq f_8(2.5468) \leq 95.78, \quad g_8(t) \geq g_8(2.5465) \geq 95.8.\]

This implies that

\[(3.19) \quad h_8(t) \leq 0 (2.5465 \leq t \leq 2.5468) .\]

From (3.18) and (3.19), we know that Lemma 3.3 is true in the case \(n = 8\). This completes the proof of Lemma 3.3.

\[\square\]

**Lemma 3.4.** Let \(M^n\) be a closed minimal hypersurface in \(S^{n+1}(1)\). Then

\[f \leq [2 + \delta(n)]Sb, \quad 3 \leq n \leq 8.\]

Here, \(\delta(3) = 0, \delta(4) = 0.16, \delta(5) = 0.23, \delta(6) = 0.28, \delta(7) = 0.32\) and \(\delta(8) = 0.34\).
Proof. In the case $n = 3$, if $b_1 = 0$, then $b_2^2 = b_3^2 = \frac{1}{2}b$. Hence
\[
 f = \frac{1}{2}(\lambda_1^2 - 4\lambda_1\lambda_2 + \lambda_2^2 - 4\lambda_1\lambda_3)b
 = \left\{\lambda_1^2 - 2\cdot \frac{\lambda_1}{\sqrt{2}} \cdot \sqrt{2}\lambda_2 - 2\cdot \frac{\lambda_1}{\sqrt{2}} \cdot \sqrt{2}\lambda_3\right\}b
 \leq 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)b \leq 2Sb.
\]
When $b_1 \neq 0$, we let $b_2 = (x - \frac{1}{2})b_1$ and $\lambda_2 = (y - \frac{1}{2})\lambda_1$. Then
\[
 f = \frac{36x^2 + 48xy + 3}{48x^2y^2 + 36x^2 + 20y^2 + 15} \cdot 2Sb
 = \frac{36x^2 + 48xy + 3 + 48(xy - 1/2)^2 + 20y^2}{2Sb}
 \leq 2Sb.
\]
From the above discussion, we know that Lemma 3.4 is true in the case $n = 3$.

In the case $4 \leq n \leq 8$, if $\lambda_1^2 - 4\lambda_1\lambda_2 \leq [2 + \delta(n)]S \, (\forall i \neq 1)$, then Lemma 3.4 is true. Otherwise, without loss of generality, we suppose that $\lambda_1^2 - 4\lambda_1\lambda_2 = tS \,(t \geq 2)$ and $b_1 = x b_2$. Then
\[
 \sum_{i \neq 1, 2} b_i^2 \geq \frac{(1 + x)^2}{n - 2} b_2^2, \quad \lambda_1^2 \geq (t - 2)S.
\]
From the above inequalities and Lemma 3.2, we have
\[
 f - [2 + \delta(n)]Sb \leq \left[ t - 2 - \delta(n) \right]Sb_2^2 + \left[ t - r - 2 - \delta(n) \right]S \sum_{i \neq 1, 2} b_i^2
 - \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\}Sb_2^2
 \leq \left[ t - 2 - \delta(n) \right]Sb_2^2 + \frac{t - r - 2 - \delta(n)}{n - 2}(1 + x)^2 Sb_2^2
 - \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\}x^2 Sb_2^2.
\]
Here, $r = \frac{16t - 8 - 12\sqrt{-2t^2 + 2t + 8}}{17}$.

Let $F(n, t, x) = t - 2 - \delta(n) + \frac{t - r - 2 - \delta(n)}{n - 2} (1 + x)^2 - \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\} x^2$.
The above inequality becomes
\[
 (3.20) \quad f - [2 + \delta(n)]Sb \leq F(n, t, x)Sb_2^2.
\]
Since
\[
 \frac{1}{2} \frac{\partial F(n, t, x)}{\partial x} = \frac{t - r - 2 - \delta(n)}{n - 2}(1 + x) - \left\{ t - 2 + \frac{2 + \delta(n)}{3} \right\} x,
\]
we have
\[
 (3.21) \quad F(n, t, x) \leq F(n, t, x_0)
 = \frac{t - 2 - \delta(n)}{G} \left\{ (n - 2)t + \frac{(n + 2)\delta(n) + 10 - 4n}{3} \right\} - \frac{2 + 4\delta(n)}{3G} r.
\]
Here, \(-x_0 = \frac{3r + 2 + \delta(n) - t}{3r + 3(n - 3)t + 14 + (n + 1)\delta(n) - 4n^2} \), \(\frac{\partial F(n, t, x_0)}{\partial x} = 0\), and \(G = r + 2 + \delta(n) - t + (n - 2)\left(t - 2 + \frac{2 + \delta(n)}{3}\right)\).

Notice that \(h_n(t) = 51F(n, t, x_0)G\), where \(h_n(t)\) is defined as in Lemma 3.3. From (3.20), (3.21) and Lemma 3.3, we have

\[
f \leq [2 + \delta(n)]S_b.
\]

This completes the proof of Lemma 3.4. \(\square\)

Now we are in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** From Lemma 3.4, we have obtained

\[
f = \sum_{i \neq 1}(\lambda_i^2 - 4\lambda_i\lambda_j)h_{i1i}^2 - \lambda_j^2h_{i111}^2 \leq (2 + \delta)S\left(\sum_{i \neq 1}h_{i1i}^2 + \frac{1}{3}h_{i111}^2\right).
\]

In general,

\[
f_j = \sum_{i \neq j}(\lambda_i^2 - 4\lambda_i\lambda_j)h_{i1j}^2 - \lambda_j^2h_{i1jj}^2 \leq (2 + \delta)S\left(\sum_{i \neq j}h_{i1j}^2 + \frac{1}{3}h_{i1jj}^2\right), \forall j.
\]

Hence we get

\[
3(A - 2B) = \sum_{i \neq j \neq k \neq i} [2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2]h_{ijk}^2
\]

\[
- 3 \sum_i \lambda_j^2h_{iij}^2 + 3 \sum_{i \neq j}(\lambda_i^2 - 4\lambda_i\lambda_j)h_{iij}^2
\]

\[
\leq 2S \sum_{i \neq j}h_{i1j}^2 + 3 \sum_{i \neq j}[(\lambda_i^2 - 4\lambda_i\lambda_j)h_{iij}^2 - \lambda_j^2h_{i1jj}^2]
\]

\[
\leq (2 + \delta)S \left\{ \sum_{i \neq j \neq k \neq i} h_{ijk}^2 + 3 \sum_{i \neq j} h_{iij}^2 + 3 \sum_j h_{ijj}^2 \right\}
\]

\[
= (2 + \delta)S \sum_{i, j, k} h_{ijk}^2.
\]

Notice that \(\delta(3) = 0, \delta(4) = 0.16, \delta(5) = 0.23, \delta(6) = 0.28, \delta(7) = 0.32, \delta(8) = 0.34\) and \(\alpha(n) = \frac{2(n + 4)(3 - n\delta)}{9n + 30}\). We conclude from Proposition 3.1 that \(S \equiv n\). This completes the proof of Theorem 1.1. \(\square\)

**References**


15. Q. M. Cheng, *The rigidity of Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$*, Comment. Math. Helvetici, 71 (1996), 60–69. MR1371678 (97a:53094)


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