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THE REMAINDER IN ASYMPTOTIC INTEGRATION II

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Abstract. Levinson's Theorem in asymptotic integration of linear differential systems is strengthened in a quantitative way. It is shown that any decay in excess of absolute integrability appears with the remainder.

1. Introduction

Levinson's Theorem is arguably one of the strongest results in the theory of asymptotic integration of linear differential systems of the form

$$u' = (\Lambda(x) + R(x))u \quad \text{on } [a, \infty). \tag{1}$$

It states that the solutions of (1) behave almost like the solutions of the unperturbed system $u' = \Lambda u$, if the eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ satisfy a dichotomy condition and if R, the perturbing term, is small. In this case the k-th solution $u_k, k = 1, \dots, n$, is given by

$$u_k(x) = (e_k + r_k(x)) \exp \int_a^x \lambda_k(t)dt \quad \text{with } r_k = o(1) \tag{2}$$

and e_k the k-th unit vector.

In the classical version [3, 4] small means integrable, $R \in \mathcal{L}^1 = \mathcal{L}^1([a, \infty))$. But meanwhile the dichotomy conditions and R have been modfied many times, to give variants of the original result [3] by strengthening the dichotomy condition while weakening the decay or vice versa. In this paper, which can be considered a sequel to [1], the plausible quantitative version of Levinson's Theorem:

Small perturbing terms R imply small remainders r_k,

is made more precise. We show that any decay in excess of integrability is reflected in the decay of r_k. It is obvious that a better control of the remainder terms is needed in a more detailed analysis. Such estimates are also needed in the study of higher order differential operators with unbounded coefficients [4, 3.3, 3.8] where R and the unperturbed solutions differ strongly in their asymptotics. It was in fact this set of problems which initiated these results. Details for this will, however, be pursued in a separate publication with D. Hinton and F.O. Nyamwala.

The notation in this paper is standard and mostly self-explanatory. By $h \nearrow \infty$ we denote a function h which tends to infinity monotonically.

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In a nutshell the arguments we are using are as follows: Let h be continuous with $h \nearrow \infty$ and let r be measurable with $r_1 = rh \in \mathcal{L}^1$; then $\int_x^\infty r dt = O(h^{-1})$. To see this write

$$|\int_x^\infty r \text{ dt}| = h^{-1}(x)|\int_x^\infty h(x)h^{-1}(t)r_1(t)\text{dt}| \leq h^{-1}(x)\int_x^\infty |r_1(t)| \text{ dt}| = O(h^{-1}).$$

This result will also hold for quasimonotonic functions, where $h(x) \leq K\ h(t)$ for $t \geq x$. The example with $r(x) = x^{-\alpha}$ shows that this result cannot be improved substantially.

2. The dichotomy condition

The dichotomy condition guarantees that the eigenfunctions of the unperturbed system are x-uniformly comparable on $[a, \infty)$. We will formulate it as a trichotomy condition, which is more suitable for our proofs.

Definition. The diagonal matrix $\Lambda = diag(\lambda_i)$ satisfies the dichotomy condition if for $i \neq j$, $i, j = 1, \ldots, n$, either of the following conditions holds for constants K_i:

(a) $K_1 \leq \text{Re} \int_t^x (\lambda_i(s) - \lambda_j(s))ds \leq K_2, \quad a \leq t \leq x < \infty$.

(b) $\int_a^x \text{Re}(\lambda_i(s) - \lambda_j(s))ds \to \infty$ as $x \to \infty$ and $\int_t^x \text{Re}(\lambda_i(s) - \lambda_j(s))ds \geq K_3, a \leq t \leq x < \infty$.

(c) $\int_a^x \text{Re}(\lambda_i(s) - \lambda_j(s))ds \to -\infty$ as $x \to \infty$ and $\int_t^x \text{Re}(\lambda_i(s) - \lambda_j(s))ds \leq K_4$.

The standard form in which the dichotomy condition arises in applications is sign $\text{Re}(\lambda_i - \lambda_j) =$ constant modulo integrable terms. In asymptotic integration the proof of Levinson's theorem or the $(1+Q)$-transformation [1, 3] leads to equations of the type

$$q'_{ij} = (\lambda_i - \lambda_j)q_{ij} + r_{ij} \quad \text{on } [a, \infty). \tag{3}$$

For (3) solutions are needed which decay at infinity. Write (3) in short as

$$q' = \lambda q + r. \tag{4}$$

In spectral analysis the occurrence of an absolutely continuous spectrum leads to equations of the form

$$q' = (i\lambda_1 + \eta\lambda_2)q + r \ \text{ with } \lambda_1, \lambda_2 \text{ real valued, } 0 \leq \eta \leq \epsilon > 0. \tag{5}$$

In this case one has to derive η-uniform estimates of the solutions of (5).

If the dichotomy condition holds for Λ, the solutions of (3), respectively (4), which are small at infinity, are given by

$$q(x) = -\int_x^\infty \exp(\mu(x) - \mu(t))r(t) \text{ dt} \quad \text{in cases (a), (b),} \tag{6}$$

$$q(x) = \int_a^x \exp(\mu(x) - \mu(t))\ r(t) \text{ dt} \ \text{ in case (c), where } \mu(x) = \int_a^x \lambda\ dt. \tag{7}$$

In order to explain the basic idea of the following results, consider the integrand in (6) and (7) and write as in the example above:

$$\exp(\mu(x) - \mu(t))r(t) = \exp(\mu(x) - \mu(t))h(t)^{-1}(h(t)r(t)).$$

This suggests we write $\exp(-\mu(t)) \cdot h(t)^{-1} = \exp(-\nu(t))$ and split off a term $h(x)^{-1}$ from $\exp(\mu(x))$ in a similar fashion.

Lemma 1. *Let h be differentiable $h \nearrow \infty$ on $[a, \infty)$ so that $rh = r_1 \in \mathcal{L}^1$. Then (4) has a solution $q = O(h^{-1})$ if $\lambda + \frac{h'}{h}$ satisfies the dichotomy condition.*

Proof. We may assume $r \geq 0$ and λ, μ are real valued. Now write $\nu(x) = \mu(x) + \ln h(x)$. Then the first integral in (6) can be written as

$$q(x) = -\int_x^\infty \exp(\nu(x) - \ln h(x)) \exp(-\nu(t)) r_1(t)\ dt = O(h^{-1}),$$

because for $t \geq x$ we have $\nu(x) - \nu(t) = \mu(x) + \ln h(x) - \nu(t)$. The same trick on the second integral will only work if $\lambda + \frac{h'}{h}$ satisfies (a) or (b). So we have to go back to (4) and write $y = hq$. Then

$$y' = (\lambda + \frac{h'}{h})y + r_1, \tag{8}$$

and the result $y = O(1)$ for some solution of (8) follows by assumption. □

Remark. Even though we could have used the two-line argument with (8) right away, the seemingly longer first argument has its merits, because it yields directly $q = o(h^{-1})$ and because it works uniformly for any λ, for which Re λ is essentially positive, i.e. Re $\mu(x) -$ Re $\mu(t) \geq K$ for $t \geq x$. By modifying h slightly, it is also possible to obtain $q = o(h^{-1})$ in the second case.

In applications of asymptotic integration to spectral theory one usually encounters linear systems, which still depend on the spectral parameter z or even more often $\eta =$ Im z. In this case Levinson's results have to be extended to cover z or η-uniform results. The following lemma is such a uniform variant of Lemma 1.

Lemma 2. *On* $[a, \infty)$ *consider* (5) *with* Re $\lambda(x) \geq 0$ *or* Re $\lambda(x) < 0$ *and let* $h \nearrow \infty$ *so that* $r_1 = rh \in \mathcal{L}^1$. *In addition assume that there is an* $X(\eta)$ *so that* $(\frac{h'}{h} + \eta \text{ Re } \lambda)(x) \geq 0, x \leq X(\eta)$, *and* $(\frac{h'}{h} + \eta \text{ Re } \lambda)(x) \leq 0$ *for* $x \geq X(\eta)$ *with* $X(\eta) = \infty$ *permitted. Then* (5) *has solutions* $q(x, \eta)$ *with* $q(x, \eta)h$ *bounded uniformly in* $0 \leq \eta \leq \epsilon > 0$ *for some* $\epsilon > 0$.

Proof. As mentioned above, the proof of Lemma 1 applies if Re $\lambda(x) \geq 0$. This proof will also apply if $\mu(x) \searrow -\infty$, but $\eta_0 \mu + \ln\ h \nearrow \infty$ for some $\eta_0 > 0$. Thus assume Re $\lambda \leq 0$. With the same factorization as above it suffices to show that with $\nu_\eta = \eta\mu + \ln\ h$,

$$-\int_x^{X(\eta)} (\exp \nu_\eta(x) - \nu_\eta(t)) r_1(t) dt = -\int_x^{X(\eta)} (\exp \int_t^x (\eta\lambda + \frac{h'}{h}) ds) r_1(t) dt$$

is η-uniformly bounded. For $x \leq X(\eta)$ we have $x \leq t \leq X(\eta)$ so that the integral in the exponential is negative. For $x \geq X(\eta)$ we have $X(\eta) \leq t \leq x$, and this integral is negative again. □

Remark. The trick with the variation of the initial point $X(\eta)$ was used earlier in [2]. Lemma 2 can be generalized in various ways. For example Re $\lambda \gtrless 0$ need only hold modulo integrable terms. Also $X(\eta)$ need not be unique, as long as the integrals can be controlled in the domain where the change in sign takes place.

In the cases above, Re λ played no role as long as the dichotomy condition was valid. If Re λ can be controlled, better results are possible.

Lemma 3. *Consider* (4) *with* Re $\lambda(x) = \gamma > 0 (< 0)$. *Let* h *be differentiable with* $h \nearrow \infty, |\gamma| \geq rh$ *and* $\frac{h'}{h} = o(\gamma)$. *Then* (4) *has a solution* $q = o(h^{-1})$.

Proof. The solutions of (4) vanishing at infinity are again given by (6), respectively (7). Let us begin with (6). Take absolute values. Then as before

$$\begin{aligned}|hq(x)| &\leq -\int_x^\infty |\exp[(\mu(x)+\ln h)-(\mu(t)+\ln h(t))]||h(t)r(t)|\mathrm{dt}\\ &\leq -\int_x^\infty \exp(-\int_x^t(\gamma(s)+\frac{h'}{h}(s))ds)\gamma(t)\mathrm{dt}\end{aligned}$$

because $|h\cdot r|\leq\gamma$. The latter integral can be estimated by

$$-\int_x^\infty \exp-\int_x^t\left|\gamma+\frac{h'}{h}\right|ds\left|\frac{\gamma}{(\gamma+\frac{h'}{h})}\right|\left|\gamma+\frac{h'}{h}\right|\ \mathrm{dt}.$$

Now change variables to $u=\int_a^x|\gamma+\frac{h'}{h}|ds$. The other integral is transformed in the same way by adjoining h to r and h^{-1} to the exponentials. □

In order to apply these lemmata to the asymptotic integraton of (1), we follow Eastham's proof [4, ch. 1.4]. For easier comparison we use his notation. In order to eliminate the exponential factor with the k-th solution of u_k of (1), only the "neutral" solution $Z=u_k\cdot\exp-\int_a^x\lambda_k\ \mathrm{dt}$ is analyzed. Z satisfies

$$Z'=(\Lambda_k+R)Z \text{ with } \quad \Lambda_k=\Lambda-\lambda_k\cdot 1. \tag{9}$$

So only the dichotomy conditions for λ_j versus λ_k are relevant. The fundamental matrix Φ for the unperturbed system is then $\Phi(x)=\mathrm{diag}(\exp\mu_i(x))$ with $\mu_i(x)=\int_a^x(\lambda_i-\lambda_k)\mathrm{dt}$. In addition the indices are ordered so that condition (c) applies to the first l-components of Φ, thus determining the matrix Φ_1. With this Z, Z will satisfy the integral equation

$$Z(x)=e_k+\Phi_1(x)\int_a^x\Phi^{-1}(t)RZ(t)\mathrm{dt}-\Phi_2(x)\int_x^\infty\Phi^{-1}(t)R\ Z(t)\mathrm{dt}, \tag{10}$$

where $\Phi_2=\Phi-\Phi_1$. This equation is then solved by iteration with recourse to Banach's fix point theorem. For our purposes it is more convenient to write the iteration scheme as

$$Z_i^{(m+1)}(x)=e_k+\sum_j\int_j\exp(\mu_i(x)-\mu_i(t))R_{ij}Z_j^{(m)}(t)\mathrm{dt}, \tag{11}$$

where $\int_j$ stands for $\int_a^x$, respectively $\int_\infty^x$, if $(\lambda_i-\lambda_k)$ satisfies (c), respectively (a), (b), of the dichotomy conditions. Now (11) can be solved iteratively if one chooses a sufficiently large and lets $Z^{(0)}=e_k$. Since we will assume R to decay faster than $\mathcal{L}^1$, we will also adjoin the diagonal of R to Λ and thus achieve $R_{ii}=0$. In order to prove a quantitative version of Levinson's Theorem we assume that there exist differentiable functions $h_{ij}\nearrow\infty$ so that

$$R_{ij}h_{ij}\in\mathcal{L}^1. \tag{12}$$

Theorem 1. *Consider* (1) *and assume that* Λ *as well as* $\lambda_i-\lambda_k+\frac{h'_{ik}}{h_{ik}}$ *satisfy the dichotomy conditions. Then*

$$u_k(x)=(e_k+r_{k,i})\exp\int_a^x\lambda_k(t)dt \text{ with } r_{k,i}=O(h_{i,k}^{-1}), i\neq k \text{ and } r_{k,k}=o(1), \tag{13}$$

provided the compatibility conditions

$$h_{ik} = o(h_{ij}h_{jk}), \qquad j = 1, \ldots, n, j \neq i, k, \tag{14}$$

hold.

Proof. Let $\mathcal{F}$ be the space of all continuous $\mathbb{C}^n$-valued functions f on $[a, \infty)$ which vanish at infinity so that for $f = (f_i), ||f|| = max||f_i h_{ik}||_\infty < \infty$ with $h_{kk} = 1$. Then $\mathcal{F}$ is complete. Let $f \in \mathcal{F}$; then $(Tf)_i = \delta_{ki} + \sum_j \int_j \exp(\mu - \mu_i(t)) R_{ij} f_j(t) \mathrm{d}t$. By assumption $R_{ij}h_{ij} \in \mathcal{L}^1$ and $f_j h_{jk} = O(1)$ so that $R_{ij} f_j h_{ij} h_{ik} \in \mathcal{L}^1$. Thus $R_{ij} f_j h_{ik} \in \mathcal{L}^1$ by assumption. Now apply Lemma 1 to conclude $Tf \in \mathcal{F}$. If a is sufficiently large, the operator derived from the iteration procedure is a contraction. Now apply Banach's fixed point theorem. For the k-th solution u_k this implies (13). □

Remark. Instead of (9) we could define $y_i = h_{ik} Z_i$ and study

$$y_i' = ((\lambda_k)_i + \frac{h_{ik}'}{h_i k}) y_i + \sum (R_{ij} h_{ij})(h_{ij}^{-1} h_{jk}^{-1}) h_{ik} y_j \tag{15}$$

directly to obtain the conclusion of the theorem. In this way the choice of the limits in the integral would be clear from the outset. However, it should be clear that the basis of Levinson's results are really the scalar equations (4) or (5) and their solutions (6), (7) or Lemma 2.

Corollary. *Assume that* $(\lambda_i - \lambda_k)$ *and* h_{ij} *satisfy the strengthened dichotomy conditions of Lemma* 2. *Moreover assume that the eigenvalues* λ_j *admit a representation* $\lambda_j(\eta, x) = i\lambda_{j1}(x) + \eta\lambda_{j2}(x), \lambda_{j1}, \lambda_{j2}$ *real valued and satisfy the dichotomy conditions for all* $j = 1, \ldots, n$ *and fixed* $0 \leq \eta \leq \epsilon > 0$. *Then the* k*-th solution of* (1) *has the form*

$$u_k(x, \eta) = (e_k + r_{k,i}(\eta)) \exp \int_a^x \lambda_k(\eta, t)\, dt \tag{16}$$

with $r_{ki} \cdot h_{ik}$ η*-uniformly bounded and* $r_{k,k}(\eta, x) = o(1)$ η*-uniformly bounded.*

In order to extend the proof of the theorem, we just have to define $\int_j$ as $\int_x^{X(\eta)}$ whenever necessary.

Remark. There will also be a variant of the theorem based on Lemma 3. Such a lemma can be applied in the study of spectral problems. There one is led to equations of the form

$$\begin{aligned} q' &= (\lambda_1(x) + \lambda_2(x, z))q + r(x, z) \text{ with } \lambda_2 = o(\lambda_1) \text{ and Re } \lambda_1 = \gamma > 0 \\ &\quad \text{or Re } \lambda_1 = -\gamma < 0. \end{aligned} \tag{17}$$

Moreover λ_2 and r will be analytic in z for $a \leq \text{Re } z \leq b$ and $|\text{Im } z| \leq \epsilon > 0$. In this case the solutions will be analytic in z likewise. Thus only discrete spectrum can be generated in this way.

The lemmata and the theorem can also be used to obtain better estimates for the $(1 + Q)$-transformations. Finally it should be noted that with little effort parameter uniform estimates can be derived if the coefficients depend continuously on the parameter [2].

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