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SEMIPRINCIPAL CLOSED IDEALS OF βS

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ABSTRACT. Let S be an infinite discrete semigroup and let βS be the Stone-Čech compactification of S. For every $p \in \beta S$, $\mathrm{cl}((\beta S)p(\beta S))$ is a closed two-sided ideal of βS called the *semiprincipal closed ideal* generated by p. We show that if S can be embedded into a group, then βS contains $2^{2^{|S|}}$ pairwise incomparable semiprincipal closed ideals.

Given a discrete semigroup S, the operation can be naturally extended to the Stone-Čech compactification βS of S making βS a compact right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the right translation

$$\beta S \ni x \mapsto xp \in \beta S$$

is continuous, and for each $a \in S$, the left translation

$$\beta S \ni x \mapsto ax \in \beta S$$

is continuous.

We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$,

$$\overline{A} = \{p \in \beta S : A \in p\},$$

and we write A^* and $U(A)$ for the sets of nonprincipal and uniform ultrafilters from $\overline{A}$, respectively. The family $\{\overline{A} : A \subseteq S\}$ is a base for the topology of βS. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in pq$ if and only if

$$\{x \in S : x^{-1}A \in q\} \in p,$$

where $x^{-1}A = \{y \in S : xy \in A\}$.

The semigroup βS is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. An elementary introduction to βS can be found in [3].

As any compact Hausdorff right topological semigroup, βS has a smallest two-sided ideal $K(\beta S)$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. The intersection of a minimal right ideal and a minimal

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left ideal is a group, and all these groups are isomorphic. The precise description of $K(\beta S)$ is given by the Rees-Suschkewitsch Theorem (see [3, Theorem 1.64]).

For every infinite cancellative semigroup S, βS contains $2^{2^{|S|}}$ minimal left ideals [3, Theorem 6.42], and in the case where S is countable or an Abelian group, βS contains also $2^{2^{|S|}}$ minimal right ideals ([3, Corollary 6.41] and [6], respectively).

The closure of any right (left) ideal of βS is a right (left) ideal [3, Theorems 2.15 and 2.17], so the closure of any two-sided ideal of βS is a two-sided ideal; in particular, $\mathrm{cl}(K(\beta S))$ is a two-sided ideal of βS. In [5], it was shown that for every infinite Abelian group G, βG contains $2^{2^{|G|}}$ closed two-sided ideals. In [2], this result has been extended to an arbitrary countably infinite group (see also the survey [4]). However, the question whether this is true for all infinite groups remained open.

Given a semigroup S and $p \in \beta S$, let

$$I(S,p) = \mathrm{cl}\left((\beta S)p(\beta S)\right).$$

Note that $I(S,p)$ is a closed two-sided ideal of βS and $I(S,p) \cup \{p\}$ is the smallest closed two-sided ideal of βS containing p. We call $I(S,p)$ the *semiprincipal closed ideal* of βS generated by p.

The aim of this paper is to prove the following result.

Theorem 1. *Let S be a semigroup embeddable into a group, let $B \subseteq S$, and let $|B| = |S| = \kappa \geq \omega$. Then there is $A \subseteq B$ with $|A| = \kappa$ such that whenever $p, q \in U(A)$ and $p \neq q$, one has $I(S,p) \setminus I(S,q) \neq \emptyset$ and $I(S,q) \setminus I(S,p) \neq \emptyset$.*

Since $|U(A)| = 2^{2^{|A|}}$, we obtain from Theorem 1 that

Corollary 2. *For every infinite semigroup S embeddable into a group, βS contains $2^{2^{|S|}}$ pairwise incomparable semiprincipal closed ideals.*

In order to prove Theorem 1, we show the following.

Theorem 3. *Let G be a group, let $B \subseteq G$, and let $|B| = |G| = \kappa \geq \omega$. Then there is $A \subseteq B$ with $|A| = \kappa$ such that for every $p \in U(A)$, one has $I(G,p) \cap \overline{A} = \{p\}$.*

Before proving Theorem 3, let us show how it implies Theorem 1.

Proof of Theorem 1. Let G be a group containing S and let $|G| = \kappa$. By Theorem 3, there is $A \subseteq B$ with $|A| = \kappa$ such that for every $p \in U(A)$, $I(G,p) \cap \overline{A} = \{p\}$. Now let $p, q \in U(A)$ and $p \neq q$. Then

$$I(G,p) \cap (GqG) = \emptyset \text{ and } I(G,q) \cap (GpG) = \emptyset.$$

Indeed, if $g, h \in G$ and $gqh \in I(G,p)$, then $q \in g^{-1}I(G,p)h^{-1} \subseteq I(G,p)$. It follows that

$$I(S,p) \cap (SqS) = \emptyset \text{ and } I(S,q) \cap (SpS) = \emptyset,$$

and so

$$I(S,q) \setminus I(S,p) \neq \emptyset \text{ and } I(S,p) \setminus I(S,q). \qquad \square$$

The proof of Theorem 3 is based on the next lemma.

Lemma 4. *Let G be a group, let $B \subseteq G$, and let $|G| = |B| = \kappa \geq \omega$. Then there is $A \subseteq B$ with $|A| = \kappa$ such that*

(1) *for every $g \in G \setminus \{1\}$, $|gA \cap A| \leq 3$, and*
(2) *for every $g, h \in G$, $|\{a \in A : a \neq gah \in A\}| < \kappa$.*

Note that Lemma 4 without condition (2) is a well-known fact [1, Proposition 4.1].

Proof. Enumerate G as $\{g_\alpha : \alpha < \kappa\}$. Construct recursively a κ-sequence $(a_\alpha)_{\alpha<\kappa}$ in B such that

$$a_\alpha \in B \setminus (A_\alpha A_\alpha^{-1} A_\alpha \cup C_\alpha A_\alpha C_\alpha),$$

where $A_\alpha = \{a_\beta : \beta < \alpha\}$ and $C_\alpha = \{g_\beta^{\pm 1} : \beta < \alpha\}$. We claim that the set $A = \{a_\alpha : \alpha < \kappa\}$ is as required. Clearly $|A| = \kappa$.

To see (1), suppose instead that one has $g \in G \setminus \{1\}$ and distinct α, β, and γ such that $\{a_\alpha, a_\beta, a_\gamma\} \subseteq gA$. Pick distinct δ, μ, and ν such that $a_\alpha = ga_\delta$, $a_\beta = ga_\mu$, and $a_\gamma = ga_\nu$. Consider two cases.

Case 1. $\max\{\alpha, \beta, \gamma, \delta, \mu, \nu\} \in \{\alpha, \beta, \gamma\}$. Then without loss of generality, this maximum is α. Also $\alpha > \delta$. Since $\mu \neq \nu$, without loss of generality we have $\alpha > \mu$ so that $a_\alpha = ga_\delta = a_\beta a_\mu^{-1} a_\delta \in A_\alpha A_\alpha^{-1} A_\alpha$, a contradiction.

Case 2. $\max\{\alpha, \beta, \gamma, \delta, \mu, \nu\} \notin \{\alpha, \beta, \gamma\}$. Then without loss of generality that maximum is δ and one has that $\delta > \alpha$, $\delta > \beta$, and $\delta > \mu$. Thus $a_\delta = g^{-1} a_\alpha = a_\mu a_\beta^{-1} a_\alpha \in A_\delta A_\delta^{-1} A_\delta$, a contradiction.

To verify (2), suppose instead that we have some β and δ such that

$$|\{a \in A : a \neq g_\beta a g_\delta \in A\}| = \kappa.$$

Pick $\alpha > \max\{\beta, \delta\}$ such that $a_\alpha \neq g_\beta a_\alpha g_\delta \in A$ and pick γ such that $g_\beta a_\alpha g_\delta = a_\gamma$. If $\gamma > \alpha$, then $a_\gamma \in C_\gamma A_\gamma C_\gamma$. If $\gamma < \alpha$, then $a_\alpha = g_\beta^{-1} a_\gamma g_\delta^{-1} \in C_\alpha A_\alpha C_\alpha$. In either case we have a contradiction. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let A be a subset of B guaranteed by Lemma 4. We claim that for every $p \in U(A)$, one has $I(G,p) \cap \overline{A} = \{p\}$. Since $I(G,p) = \mathrm{cl}\,\big(Gp(\beta G)\big)$, it suffices to prove that $\big(Gp(\beta G)\big) \cap \overline{A} = \{p\}$.

We first show that $(G^*G^*) \cap \overline{A} = \emptyset$.

Assume on the contrary that $qr \in \overline{A}$ for some $q, r \in G^*$. Let

$$Q = \{x \in G : x^{-1}A \in r\}.$$

Then $Q \in q$, so pick distinct x and y in Q. Then $(x^{-1}A) \cap (y^{-1}A) \in r$ and is therefore infinite. But then $A \cap (xy^{-1}A)$ is infinite, contradicting condition (1) of Lemma 4.

From $(G^*G^*) \cap \overline{A} = \emptyset$ and the fact that G^* is a left ideal of βG [3, Corollary 4.33] we obtain that $\big(Gp(\beta G)\big) \cap \overline{A} = (GpG) \cap \overline{A}$. Hence, in order to finish the proof, it remains to show that $(GpG) \cap \overline{A} = \{p\}$.

Assume on the contrary that $gph = q$ for some $g, h \in G$ and $q \in \overline{A} \setminus \{p\}$. Then there is $P \in p$ such that $P \subseteq A$, $gPh \subseteq A$ and $P \cap (gPh) = \emptyset$. It follows that $\{a \in A : a \neq gah \in A\} \in p$. Since $p \in U(A)$, this contradicts condition (2) of Lemma 4. □

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