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THE SZLENK INDEX OF ORLICZ SEQUENCE SPACES

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Abstract. We provide explicit estimates of the Szlenk indices of Orlicz sequence spaces. Applications are given to uniform homeomorphisms between subspaces and quotients of Orlicz spaces.

1. Introduction

Banach spaces are topological spaces with a vector space structure. Whereas the vectorial structure is very rich, the topological one is less informative : for example, Kadec's theorem asserts that any two separable Banach spaces are homeomorphic. The metric structure is in-between. A uniform homeomorphism between two spaces gives some information about their linear structure (see [8] and [1]) but in general it does not imply the existence of an isomorphism between those spaces. However, it does for some particular Banach spaces: spaces uniformly homeomorphic to an ℓ_p-space, for $1 < p < \infty$, are those which are isomorphic to it. We refer to [2] for an authoritative book on nonlinear geometry. It seems natural to study the case of the Orlicz sequence spaces, which are, in a way, a generalization of the ℓ_p-spaces. The aim of this article is to present in Theorem 2.3 a uniformly homeomorphic invariant for Orlicz sequence spaces obtained through the use of a more general invariant, the convex Szlenk index [6]. This result will allow us to improve on a result of [5] and establish that the smallest p such that a given Orlicz space contains ℓ_p is the same for two uniformly homeomorphic Orlicz sequence spaces. We refer to [10] for an updated account of the nonlinear geometry of Banach spaces.

We recall ([13], [3]) that an Orlicz function F is a continuous nondecreasing and convex function defined on $\mathbb{R}_+$ such that $F(0) = 0$. We will consider only nondegenerate Orlicz functions, that is, Orlicz functions which vanish only at zero.

An Orlicz function is said to satisfy the Δ_2-condition at zero if

$$\limsup_{t\to 0} F(2t)/F(t) < +\infty.$$

To any Orlicz function F we associate the Banach space ℓ_F of all sequences of scalars $(x_n)_{n\in\mathbb{N}^*}$ such that

$$\sum_{n=1}^{+\infty} F\left(\frac{|x_n|}{r}\right) < +\infty$$

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for some $r > 0$, equipped with the Luxemburg norm

$$\|x\|_F = \inf\left\{ r > 0, \quad \sum_{n=1}^{+\infty} F\left(\frac{|x_n|}{r}\right) \leq 1 \right\}.$$

We will be particularly interested in the subspace h_F of ℓ_F consisting of those sequences $(x_n) \in \ell_F$ such that $\sum_{n=1}^{+\infty} F(|x_n|/r) < +\infty$ for all $r > 0$. We will exclude the case when F is equivalent to t in the sense that there exist positive constants k, K and t_0 such that for all $0 \leq t \leq t_0$, $K^{-1}F(t/k) \leq t \leq KF(kt)$. It is equivalent to the case when ℓ_F is isomorphic to ℓ_1.

It is well known that F satisfies the Δ_2-condition at zero if and only if $\ell_F = h_F$ if and only if l_F is separable (see [13]).

Associated to an Orlicz function F such that $\lim_{t \to +\infty} F(t)/t = +\infty$ is another Orlicz function F^*, which is its dual Young function, i.e.

$$F^*(u) = \sup\{uv - F(v), 0 < v < +\infty\}.$$

It is a classical result that the dual of h_F is isomorphic to ℓ_{F^*} (see [13]).

Following Kalton [9], we say that a Banach space X has property (M) if whenever u, v are in the unit sphere of X, S_X, and $(x_n) \subset X$ is a w-null sequence, then

$$\limsup_{n \to +\infty} \|u + x_n\| = \limsup_{n \to +\infty} \|v + x_n\|.$$

In [11], the following dual version of property (M) is introduced.

A Banach space X has property (M^*) if whenever u^*, $v^* \in S_{X^*}$ and $(x_n^*) \subset X^*$ is a w^*-null sequence, then

$$\limsup_{n \to +\infty} \|u^* + x_n^*\| = \limsup_{n \to +\infty} \|v^* + x_n^*\|.$$

According to [11], if X is a separable Banach space having property (M^*), then X^* is separable and X has property (M). If X is a separable Banach space not containing ℓ_1 and having property (M), then X has property (M^*).

On Orlicz spaces h_F, there is an equivalent norm such that h_F endowed with this norm has property (M). This construction is due to Kalton.

Theorem 1.1 ([9])**.** *Every Orlicz space h_F can be renormed to have property (M).*

The complete proof can be found in [9] and [7]. We recall the definition of this norm. We may and do assume that $F(1) = 1$. Let us define a new Orlicz function, M_F as below :

$$M_F(t) = \begin{cases} F(t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ F\left(\frac{1}{2}\right) + 2t - 1 & \text{if} \quad t > \frac{1}{2}. \end{cases}$$

Let N_2 be the norm on $\mathbb{R}^2$ such that

$$\begin{cases} N_2(s,t) &= & |s|\left[1 + M_F\left(|t/s|\right)\right] & \text{if} \quad s \neq 0 \\ N_2(0,t) &= & 2|t| & \text{otherwise.} \end{cases}$$

Define inductively norms on $\mathbb{R}^d$ by

$$N_d(x_1, \cdots, x_d) = N_2(N_{d-1}(x_1, \cdots, x_{d-1}), x_d)$$

and for $x = (x_n) \in h_F$,

$$\|x\|_N = \sup_{d \geq 2} N_d(x_1, \cdots, x_d).$$

$\|\cdot\|_N$ is a norm on h_F equivalent to the Luxemburg norm. The Orlicz space h_F endowed with $\|\cdot\|_N$ has property (M).

We recall that for a Banach space X the modulus of w^*-asymptotic convexity δ_X^* is defined as follows.

Definition 1.2. For $x^* \in S_{X^*}$, $t > 0$, and $Y^* \subseteq X^*$ a w^*-closed finite codimensional subspace,

$$\delta_X^*(x^*, t, Y^*) = \inf_{y^* \in Y^*, \|y^*\|=t} \|x^* + y^*\| - 1$$

and

$$\delta_X^*(x^*, t) = \sup_{Y^*, \dim(X^*/Y^*)<\infty} \delta_X^*(x^*, t, Y^*).$$

Then define

$$\delta_X^*(t) = \inf_{x^* \in S_{X^*}} \delta_X^*(x^*, t).$$

δ_X^* gives information on X which can be read on its dual.

As in [6], we introduce $\theta_X(t)$ for $0 \le t \le 1$ to be the greatest constant so that

$$\liminf_{n \to +\infty} \|x^* + x_n^*\| \ge 1 + \theta_X(t)$$

whenever x^*, $x_n^* \in X^*$, $\|x^*\| = 1$, (x_n^*) is a w^*-null sequence and $\liminf \|x_n^*\| \ge t$.

We then define ψ_X by

$$\psi_X(t) = \sup\{\theta_Y(t), \quad d(X,Y) \le 2\}$$

for $0 \le t \le 1$, where d is the Banach-Mazur distance. Observe that θ_X and ψ_X are isometric notions, although sometimes the norm does not appear in the notation.

We will need the following lemmas.

Lemma 1.3. *Let X be a separable Banach space. For every $t \in [0,1]$,*

$$\delta_X^*(t) = \theta_X(t).$$

The proof of this lemma is based on classical duality arguments. Lemma 37 of [4] provides a similar result for the modulus of uniform asymptotic smoothness with a very similar proof.

Lemma 1.4 ([5], Proposition 2.3)**.** *Let $(X, |\cdot|)$ be a separable Banach space with property (M^*) and let $\|\cdot\|$ be an equivalent norm on X. Let d be the Banach-Mazur distance between the two norms. Then for any $t > 0$, we have*

$$\delta_{|\cdot|}^*(t) \ge \delta_{\|\cdot\|}^*(t/d).$$

Let f, g be continuous monotone increasing functions on $[0,1]$ which verify $f(0) = g(0) = 0$. We will say, as in [6], that f C-dominates g if $f(t) \ge g(t/C)$ for every $0 \le t \le 1$. The functions f and g are C-equivalent if f C-dominates g and g C-dominates f.

Lemma 1.5. *Let X be a separable Banach space with property (M^*). Then ψ_X is 2-equivalent to θ_X.*

Proof. Let $0 \le t \le 1$.

By definition, $\psi_X(t) = \sup\{\theta_Y(t), \quad d(X,Y) \le 2\}$. It is obvious that $\theta_X(t) \le \psi_X(t)$ and, since θ_X is increasing, $\theta_X(t/2) \le \psi_X(t)$.

Let Y be a Banach space such that $d = d(X,Y) \le 2$. According to Lemma 1.4, $\delta_X^*(t) \ge \delta_Y^*(t/d) \ge \delta_Y^*(t/2)$. Thus, using Lemma 1.3, $\psi_X(t/2) \le \theta_X(t)$. □

We now define the Szlenk index and the convex Szlenk index. Suppose X is a separable Banach space and $K \subseteq X^*$ is a w^*-compact set. Let $\varepsilon > 0$ and set $F_0(\varepsilon) = K$. If $\alpha < \omega_1$, given $F_\alpha(\varepsilon)$, we define

$$F_{\alpha+1}(\varepsilon) = \{x^* \in F_\alpha(\varepsilon); \text{for any } w^*\text{-neighborhood } V \text{ of } x^*, \operatorname{diam}(V \cap F_\alpha(\varepsilon)) \geq \varepsilon\}.$$

If α is a limit ordinal, $F_\alpha(\varepsilon) = \cap_{\beta<\alpha} F_\beta(\varepsilon)$.

When $K = B_{X^*}$, we define the Szlenk index of X at ε, denoted $Sz(X, \varepsilon)$, to be the least countable ordinal α so that $F_\alpha(\varepsilon) = \emptyset$, if such an ordinal exists. The convex Szlenk index of X, $Cz(X, \varepsilon)$, is defined the same way (see [6]) except that at each derivation, we take the w^*-closed convex hull of the sets. It is shown in [12] that $Sz(X) = \omega_0$ if and only if $Cz(X) = \omega_0$, where ω_0 denotes the first limit ordinal. The convex Szlenk index has a remarkable property regarding uniform homeomorphisms: when finite, it is an invariant under uniform homeomorphism.

Proposition 1.6 ([6], Theorem 5.5)**.** *Suppose X and Y are uniformly homeomorphic. Then $Sz(X) \leq \omega_0$ if and only if $Sz(Y) \leq \omega_0$. If $Sz(X) \leq \omega_0$, there is a constant C so that if $0 \leq t \leq 1$, then*

$$Cz(X, Ct) \leq Cz(Y, t) \leq Cz(X, t/C).$$

2. Main results

Let us consider h_F as an Orlicz space with a separable dual, that is to say, such that F^* has the property Δ_2 at zero. As above, we can construct an Orlicz function $M^* = M_{F^*}$ and a norm $\|\cdot\|_N$ such that h_{F^*} equipped with this norm has property (M).

Lemma 2.1. *$\|\cdot\|_N$ is a dual norm. The space h_F endowed with the associated norm has property (M^*).*

Proof. By definition, for $x = (x_n)$ an element of h_{F^*}, $\|x\|_N = \sup_{d\in\mathbb{N}^*} N_d(x_1, \cdots, x_d)$. The norm $\|\cdot\|_N$ is the supremum of lower semicontinuous functions for the w^*-topology. Thus it is w^*-lower semicontinuous and so a dual norm.

Let x^* and y^* be two elements of the unit sphere of h_{F^*} and let $(x_n^*) \subset h_{F^*}$ be a w^*-null sequence. We can suppose without loss of generality that x^* and y^* have finite supports disjoint from x_n^* support for all n. We then remark that $\|x^* + x_n^*\|_N = \|y^* + x_n^*\|_N$ for all n. Taking the upper limit provides the condition which defines property (M^*). □

The notation h_F, N will be used below to index the quantities θ and ψ relative to the space h_F endowed with this new equivalent norm. For an Orlicz function F, we will introduce for $0 \leq \varepsilon \leq 1$,

$$\tilde{F}(\varepsilon) = \inf_{0<t\leq 1} \frac{F^*(\varepsilon t)}{F^*(t)}.$$

Lemma 2.2. *Let F be an Orlicz function. There is a constant $C \geq 1$ such that for $0 \leq \varepsilon \leq 1$*

$$\frac{1}{C}\tilde{F}(\varepsilon) \leq \theta_{h_F,N}(\varepsilon) \leq C\tilde{F}(\varepsilon).$$

Proof. First we will prove the lower estimate.

Let $0 \leq \varepsilon \leq 1$. Let $(h_n)_{n\in\mathbb{N}} \subset h_{F^*}$ be a w^*-null sequence such that $\|h_n\|_N = \varepsilon$ for all n. Let us note $h_n = (h_i^n)_{i\in\mathbb{N}^*}$. Since h_F has property (M^*), for all $x^* \in S_{X^*}$,

$\delta^*_{h_F,N}(x^*,t) = \delta^*_{h_F,N}(e_1,t)$, where $\{e_n\}_{n\in\mathbb{N}^*}$ is the natural basis of h_{F^*}. We can suppose without loss of generality that $h_1^n = 0$ for all n. Let $m > 1$ and $n \geq 0$. Then,

$$N_m(1,h_2^n,\cdots,h_m^n) \geq N_{m-1}(1,h_2^n,\cdots,h_{m-1}^n) + M^*\left(\frac{|h_m^n|}{N_{m-1}(1,h_1^n,\cdots,h_{m-1}^n)}\right).$$

Since $N_{m-1}(1,h_2^n,\cdots,h_{m-1}^n) \leq \|e_1+h_n\|_N \leq 2$ and M^* verifies the Δ_2-condition at zero, there is $K \geq 1$ such that

$$N_m(1,h_2^n,\cdots,h_m^n) \geq N_{m-1}(1,h_2^n,\cdots,h_{m-1}^n) + \frac{1}{K}M^*(|h_m^n|).$$

By a direct induction,

$$N_m(1,h_2^n,\cdots,h_m^n) \geq 1 + \frac{1}{K}\sum_{i=2}^{m} M^*(|h_i^n|).$$

Now $h_n = \varepsilon u_n$ with $\|u_n\|_N = 1$. Since $\|\cdot\|_N$ and the Luxemburg norm $\|\cdot\|_{M^*}$ are equivalent, there is a constant C such that $1/C \leq \|u_n\|_{M^*}$. This implies that $\sum_{i=1}^{\infty} M^*(C|u_i^n|) \geq 1$. By noticing that $|u_i^n| \leq \|u_n\|_N = 1$ and assuming that $u_i^n \neq 0$, we get

$$\sum_{i=2}^{m} M^*(|h_i^n|) = \sum_{i=2}^{m} M^*(\varepsilon|u_i^n|) = \sum_{i=2}^{m} \frac{M^*(\varepsilon|u_i^n|)}{M^*(C|u_i^n|)} M^*(C|u_i^n|) \geq \inf_{0<t\leq 1} \frac{M^*(\varepsilon t)}{M^*(Ct)}.$$

Using again the Δ_2-condition at zero, there is a constant $K_C \geq 1$ such that

$$\inf_{0<t\leq 1} \frac{M^*(\varepsilon t)}{M^*(Ct)} \geq \frac{1}{K_C} \inf_{0<t\leq 1} \frac{M^*(\varepsilon t)}{M^*(t)}.$$

The lower estimate follows from the fact that there is a constant K_F such that $F^* \leq M^* \leq K_F F^*$ on $[0,1]$.

Let us now prove the upper estimate.

Let $(s_n) \in]0,1]^{\mathbb{N}}$ be such that $\lim_{n\to+\infty} M^*(\varepsilon s_n)/M^*(s_n) = \inf_{0<t\leq 1} M^*(\varepsilon t)/M^*(t)$.

For all n, we construct u^n as follows: $u_i^n = s_n$ if $i \in \{n+1,\cdots,n+m_n\}$ and $u_i^n = 0$ otherwise, where m_n is such that $1/2 < \|u^n\|_{F^*} \leq 1$, with $\|\cdot\|_{F^*}$ the Luxemburg norm relative to F^*.

Let $h_n = \varepsilon u^n/\|u^n\|_N$. By construction, $\|h_n\|_N = \varepsilon$ and (h_n) is a w^*-null sequence.

The convexity of M^* implies that

$$||e_1+h_n||_N \leq 1 + \sum_{i=2}^{+\infty} M^*(h_i^n) = 1 + m_n M^*(\varepsilon s_n/||u^n||_N).$$

Since the Luxemburg norm and the norm $\|\cdot\|_N$ are equivalent, there is a constant $C \geq 1$ such that $1/2C \leq \|u^n\|_N$. Moreover, $m_n \leq 1/F^*(s_n) \leq K_F/M^*(s_n)$ by construction. Then

$$\|e_1+h_n\|_N \leq 1 + K_F \frac{M^*(2C\varepsilon s_n)}{M^*(s_n)}.$$

The function M^* verifies the Δ_2-condition at zero, so there is a constant K_{2C} such that

$$\|e_1+h_n\|_N \leq 1 + K_F K_{2C} \frac{M^*(\varepsilon s_n)}{M^*(s_n)}.$$

Taking the lower limit of the above expression gives the upper estimate. □

We now state and prove our main result.

Theorem 2.3. *Let F be an Orlicz function such that h_F has a separable dual. Then there are a universal constant $\tilde{C}$ and a constant C, depending on F, such that for all $0 \le \varepsilon \le 1$*

$$\frac{1}{C}\tilde{F}(\varepsilon/2d\tilde{C}) \le (Cz(h_F,\varepsilon)-1)^{-1} \quad \textit{and} \quad \frac{1}{C}(Cz(h_F,\varepsilon/2d\tilde{C})-1)^{-1} \le \tilde{F}(\varepsilon),$$

where d is the distance between the Luxemburg norm and the norm whose dual norm is $\|\cdot\|_N$, and

$$\frac{1}{C}\tilde{F}(\varepsilon/2) \le \psi_{h_F,N}(\varepsilon) \quad \textit{and} \quad \frac{1}{C}\psi_{h_F,N}(\varepsilon/2) \le \tilde{F}(\varepsilon).$$

Proof. We equip the Orlicz space h_F with the norm whose dual norm is $\|\cdot\|_N$. We will still denote this space h_F. Let $H_F(\varepsilon) = (Cz(h_F,\varepsilon)-1)^{-1}$ for $0 \le \varepsilon \le 1$. As in Theorem 4.4 of [6], there is a universal constant $\tilde{C}$ such that H_F is $\tilde{C}$-equivalent to ψ_{h_F} since the space h_F contains no copy of ℓ_1. By Lemma 1.5, ψ_{h_F} is 2-equivalent to θ_{h_F}, and so H_F and θ_{h_F} are $2\tilde{C}$-equivalent.

Using Lemma 2.3 of [6] and Lemma 2.2, the theorem is proved. □

Corollary 2.4. *Let F and G be two Orlicz functions such that h_F and h_G have separable duals. If h_F is uniformly homeomorphic to Y, a subspace of a quotient of h_G, then there are constants K and C such that for all $0 \le \varepsilon \le 1$*

$$K\tilde{G}(C\varepsilon) \le \tilde{F}(\varepsilon).$$

Proof. The spaces h_F and h_G can be renormed to have property (M^*), which implies that $Sz(h_F) = Sz(h_G) = \omega_0$. According to Proposition 1.6, there is a constant C such that $Cz(h_F, C\varepsilon) \le Cz(Y,\varepsilon) \le Cz(h_F,\varepsilon/C)$. Since $Cz(Y,\varepsilon) \le Cz(h_G,\varepsilon)$, we conclude with Theorem 2.3. □

Corollary 2.5. *Let F and G be two Orlicz functions such that h_F and h_G are uniformly homeomorphic and have separable duals. Then there are constants K and C such that for all $0 \le \varepsilon \le 1$*

$$K\tilde{F}(C\varepsilon) \le \tilde{G}(\varepsilon) \quad \textit{and} \quad K\tilde{G}(C\varepsilon) \le \tilde{F}(\varepsilon).$$

In a very particular case, we can conclude on the isomorphic character of two uniformly homeomorphic Orlicz spaces.

Corollary 2.6. *Let F and G be two submultiplicative Orlicz functions such that h_F and h_G are uniformly homeomorphic and have separable duals. Then h_F is isomorphic to h_G.*

Proof. We first notice that when F is submultiplicative, F^* is supermultiplicative and $\tilde{F} = F^*$. The inequalities obtained in Corollary 2.5 can be rewritten: there are two constants K and C such that for all $0 \le \varepsilon \le 1$

$$KF^*(C\varepsilon) \le G^*(\varepsilon) \quad \text{and} \quad KG^*(C\varepsilon) \le F^*(\varepsilon),$$

and so h_F is isomorphic to h_G (see [13], Proposition 4.a.5). □

Theorem 2.3 is an improvement of the results of [5]. In Theorem 2.9 of [5] it is shown that if h_F and h_G are two Lipschitz isomorphic Orlicz spaces, they contain the same ℓ_p-spaces. In a way, the exponent part of F^* is invariant under Lipschitz homeomorphism. Theorem 2.3 provides better information and makes it possible to partially generalize Theorem 2.9 of [5] to the uniformly homeomorphic case as done below.

Orlicz functions F lead to the following quantities:

$$\alpha_F = \sup\left\{q; \quad \sup_{0<u,v\le 1} \frac{F(uv)}{u^q F(v)} < \infty\right\}$$

and

$$\beta_F = \inf\left\{q; \quad \inf_{0<u,v\le 1} \frac{F(uv)}{u^q F(v)} > 0\right\}.$$

We always have $1 \le \alpha_F \le \beta_F \le \infty$. It is well known ([13], Theorem 4.a.9) that the space ℓ_p or c_0 if $p=\infty$ is isomorphic to a subspace of an Orlicz sequence space h_F if and only if $p \in [\alpha_F, \beta_F]$.

Corollary 2.7. *Let F and G be two Orlicz functions such that h_F and h_G are uniformly homeomorphic. Then $\alpha_F = \alpha_G$.*

Proof. We consider the following two cases:

Case 1. Suppose $\alpha_F = 1$. In this case, the dual of h_F is not separable, and so $Sz(h_F) = \omega_1$. Suppose $\alpha_G > 1$. Then h_G contains no copy of ℓ_1, and so it can be renormed to verify property (M^*). According to [5], $Sz(h_G) = \omega_0$ and Theorem 1.6 implies that since h_G is uniformly homeomorphic to h_F, $Sz(h_F) = \omega_0$. This contradiction concludes this case.

Case 2. Suppose $\alpha_F > 1$. By the case above, $\alpha_G > 1$. Notice first (see [13]) that $\alpha_F^{-1} + \beta_{F^*}^{-1} = 1$. By definition, for all $\beta > \beta_{F^*}$, there is a constant C_β such that for all $0 \le \varepsilon \le 1$

$$\inf_{0<t\le 1} \frac{F^*(\varepsilon t)}{F^*(t)} \ge C_\beta \varepsilon^\beta.$$

By Corollary 2.5, this implies that for $\beta > \beta_{F^*}$ there is another constant K_β such that for all $0 \le \varepsilon \le 1$

$$\inf_{0<t\le 1} \frac{G^*(\varepsilon t)}{G^*(t)} \ge K_\beta \varepsilon^\beta.$$

Thus, $\beta_{F^*} \ge \beta_{G^*}$, and we conclude by symmetry. □

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