DIAMONDS

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Abstract. If \( \lambda = \chi^+ = 2^\chi > \aleph_1 \), then diamond on \( \lambda \) holds. Moreover, if \( \lambda = \chi^+ = 2^\chi \) and \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) \neq \text{cf}(\chi) \} \) is stationary, then \( \Diamond_S \) holds.

Earlier this was known only under additional assumptions on \( \chi \) and/or \( S \).

1. Introduction

In this paper we prove several results about diamonds. Let us recall the basic definitions and sketch the (pretty long) history of related questions.

The diamond principle was formulated by Jensen, who proved that it holds in \( \mathbb{L} \) for every regular uncountable cardinal \( \kappa \) and stationary \( S \subseteq \kappa \). This is a prediction principle, which asserts the following:

Definition 1.1. \( \Diamond_S \) (the set version).

Assume \( \kappa = \text{cf}(\kappa) > \aleph_0 \) and \( S \subseteq \kappa \) is stationary; \( \Diamond_S \) holds when there is a sequence \( \langle A_\alpha : \alpha \in S \rangle \) such that \( A_\alpha \subseteq \alpha \) for every \( \alpha \in S \) and the set \( \{ \alpha \in S : A \cap \alpha = A_\alpha \} \) is a stationary subset of \( \kappa \) for every \( A \subseteq \kappa \).

The diamond sequence \( \langle A_\alpha : \alpha \in S \rangle \) guesses enough (i.e., stationarily many) initial segments of every \( A \subseteq \kappa \). Several variants of this principle were formulated, for example:

Definition 1.2. \( \Diamond_S^* \).

Assume \( \kappa = \text{cf}(\kappa) > \aleph_0 \) and \( S \) is a stationary subset of \( \kappa \). Now \( \Diamond_S^* \) holds when there is a sequence \( \langle A_\alpha : \alpha \in S \rangle \) such that each \( A_\alpha \) is a subfamily of \( \mathcal{P}(\alpha), |A_\alpha| \leq |\alpha| \) and for every \( A \subseteq \kappa \) there exists a club \( C \subseteq \kappa \) such that \( A \cap \alpha \in A_\alpha \) for every \( \alpha \in C \cap S \).

We know that \( \Diamond_S^* \) holds in \( \mathbb{L} \) for every regular uncountable \( \kappa \) and stationary \( S \subseteq \kappa \). Kunen proved that \( \Diamond_S^* \Rightarrow \Diamond_S \). Moreover, if \( S_1 \subseteq S_2 \) are stationary subsets of \( \kappa \), then \( \Diamond_{S_2} \Rightarrow \Diamond_{S_1} \) (hence \( \Diamond_{S_1} \)). But the assumption \( V = \mathbb{L} \) is heavy. Trying to avoid it, we can walk in several directions. On weaker relatives see [12] and references there. We can also use other methods, aiming to prove the diamond without assuming \( V = \mathbb{L} \).
There is another formulation of the diamond principle, phrased via functions (instead of sets). Since we use this version in our proof, we introduce the following:

**Definition 1.3.** ♦_S (the functional version).

Assume \( \lambda = \text{cf}(\lambda) > \aleph_0, S \subseteq \lambda, S \) is stationary. ♦_S holds if there exists a diamond sequence \( \langle g_\delta : \delta \in S \rangle \) which means that \( g_\delta \in \delta \) for every \( \delta \in S \), and for every \( g \in \lambda \) the set \( \{ \delta \in S : g|\delta = g_\delta \} \) is a stationary subset of \( \lambda \).

By Gregory [4] and Shelah [6] we know that assuming \( \lambda = \chi^+ = 2^\chi \) and \( \kappa = \text{cf}(\kappa) \neq \text{cf}(\chi), \kappa < \lambda \), and that G.C.H. holds (or actually just \( \chi^\kappa = \chi \) or \( (\forall \alpha < \chi)(|\alpha|^\kappa < \chi) \land \text{cf}(\chi) < \kappa) \), then ♦_S holds (recall that \( S_\kappa^\lambda = \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \}).

We also have results which show that the failures of the diamond above a strong limit cardinal are limited. For instance, if \( \lambda = \chi^+ = 2^\chi > \mu \) and \( \mu > \aleph_0 \) is a strong limit, then (by [8]) the set \( \{ \kappa < \mu : \Diamond_{S_\kappa} \text{ fails} \} \) is bounded in \( \mu \) (recall that \( \kappa \) is regular). Note that the result here does not completely subsume the earlier results when \( \lambda = 2^\chi = \chi^+ \), as we get a “diamond on every stationary set \( S \subseteq \lambda^{\text{cf}(\chi)} \)” but not on ♦_S; this is inherent as noted in Observation 3.4. In [12], a similar, stronger result is proved for ♦_S: for every \( \lambda = \chi^+ = 2^\chi > \mu \), a strong limit for some finite \( \delta \subseteq \text{Reg} \cap \mu \), for every regular \( \kappa < \mu \) not from \( \delta \) we have ♦_S, and even ♦_S for “most” stationary \( S \subseteq S_\kappa^\lambda \). In fact, for the relevant good stationary sets \( S \subseteq S_\kappa^\lambda \) we get ♦_S. Also weaker related results are proved there for other regular \( \lambda \) (mainly \( \lambda = \text{cf}(2^\chi) \)).

The present work does not resolve:

**Problem 1.4.** Assume \( \chi \) is singular and that \( \lambda = \chi^+ = 2^\chi \). Do we have ♦_S?

(You may even assume G.C.H.)

However, the full analog result for Problem 1.4 consistently fails; see [7] or [10]. That is, if G.C.H., \( \chi > \text{cf}(\chi) = \kappa \), then we can force a non-reflecting stationary \( S \subseteq S_\kappa^\lambda \) such that the diamond on \( S \) fails and cardinals and cofinalities are preserved; also G.C.H. continue to hold. But if \( \chi \) is a strong limit, \( \lambda = \chi^+ = 2^\chi \), we still know something on guessing equalities for every stationary \( S \subseteq S_\kappa^\lambda \); see [10].

Note that this \( S \) (by [7], [10]) in some circumstances has to be “small”

(*) if \( \chi \) is singular, \( 2^\chi = \chi^+ = \lambda, \kappa = \text{cf}(\chi) \) and we have the square \( \square_\chi \) (i.e. there exists a sequence \( \langle C_\delta : \delta < \lambda, \delta \land \text{cf}(\delta) < \chi \Rightarrow |C_\delta| < \chi, \text{ and if } \gamma \text{ is a limit point of } C_\delta, \text{ then } C_\gamma = C_\delta \cap \gamma \rangle \), then ♦_S holds. Moreover, if \( S \subseteq S_\kappa^\lambda \) reflects in a stationary set of \( \delta < \lambda \), then ♦_S holds; see [7] §3.

Also note that our results are of the form “♦_S for every stationary \( S \subseteq S^* \)” for suitable \( S^* \subseteq \lambda \). Usually this was deduced from the stronger statement ♦_S. However, the results on ♦_S cannot be improved; see Observation 3.4.

Also, if \( \chi \) is regular we cannot improve the result to ♦_S (see [7] or [9]), even assuming G.C.H. Furthermore, the question on ♦_S when \( 2^{\aleph_1} = \aleph_2 = 2^{\aleph_0} \) was raised. Concerning this we show in Claim 3.2 that ♦_S may fail (this works in other cases, too).

**Question 1.5.** Can we deduce any ZFC result on \( \lambda \) strongly inaccessible?

By Džamonja-Shelah [2] we know that failure of SNR helps (SNR stands for strong non-reflection); a parallel here is Claim 2.3.2).
For $\lambda = \lambda^{<\lambda} = 2^\mu$ weakly inaccessible, we do not know if the diamond holds (in ZFC). Nevertheless, we have proved (in [11] and [5]) that the weaker version of the diamond (as formulated in Definition 3.5(2)) holds in this case. Again failure of SNR helps.

Regarding consistency results on SNR see Cummings-Džamonja-Shelah [11] and Džamonja-Shelah [3].

**Notation** 1.6. 1) If $\kappa = \text{cf}(\kappa) < \lambda = \text{cf}(\lambda)$, then we let $S^\lambda_\kappa := \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$. 2) $D_\lambda$ is the club filter on $\lambda$ for $\lambda$ a regular uncountable cardinal.

### 2. Diamond on successor cardinals

Recall (needed only for part (2) of Claim 2.3):

**Definition 2.1.** We say $\lambda$ on $S$ has $\kappa$-SNR or SNR($\lambda, S, \kappa$) or $\lambda$ has strong non-reflection for $S$ in $\kappa$ when $S \subseteq S^\lambda_\kappa := \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$, so $\lambda = \text{cf}(\lambda) > \kappa = \text{cf}(\kappa)$. Also, there are $h : \lambda \to \kappa$ and a club $E$ of $\lambda$ such that for every $\delta \in S \cap E$ for some club $C$ of $\delta$, the function $h \upharpoonright C$ is one-to-one and even increasing (note that without loss of generality $\alpha \in \text{nacc}(E) \Rightarrow \alpha$ is a successor and without loss of generality $E = \lambda$, so $\mu \in \text{Reg} \cap \lambda \text{|}^{\text{cf}} \Rightarrow \text{SNR}(\mu, S \cap \mu, \kappa)$). If $S = S^\lambda_\kappa$ we may omit it.

**Remark 2.2.** Note by Fodor’s lemma that if $\text{cf}(\delta) = \kappa > \aleph_0$ and $h$ is a function from some set $\supseteq \delta$ and the range of $h$ is $\subseteq \kappa$, then the following conditions are equivalent:

- (a) $h$ is one-to-one on some club of $\delta$,
- (b) $h$ is increasing on some club of $\delta$,
- (c) $\text{Rang}(h|S)$ is unbounded in $\kappa$ for every stationary subset $S$ of $\delta$.

Our main theorem is:

**Claim 2.3.** Assume $\lambda = 2^\chi = \chi^+$.

1) If $S \subseteq \lambda$ is stationary and $\delta \in S \Rightarrow \text{cf}(\delta) \neq \text{cf}(\chi)$, then $\Diamond_S$ holds.

2) If $\aleph_0 < \kappa = \text{cf}(\chi) < \chi$ and $\Diamond_S$ fails where $S = S^\lambda_\kappa$ (or just $S \subseteq S^\lambda_\kappa$ is a stationary subset of $\lambda$), then SNR($\lambda, S, \kappa$) or just $\lambda$ has a strong non-reflection for $S \subseteq S^\lambda_\kappa$ in $\kappa$.

**Definition 2.4.** 1) For a filter $D$ on a set $I$ let $\text{Dom}(D) := I$, and $S$ is called $D$-positive when $S \subseteq I \land (I \setminus S) \notin D$ and $D^+ = \{S \subseteq \text{Dom}(D) : S$ is $D$-positive$\}$. Also, we let $D + A = \{B \subseteq I : B \cup (I \setminus A) \in D\}$ (so if $D = D_\lambda$, the club filter on the regular uncountable $\lambda$, then $D^+$ is the family of stationary subsets of $X$).

2) For $D$ a filter on a regular uncountable cardinal $\lambda$ which extends the club filter, let $\Diamond_D$ mean: there is $f = \langle f_\alpha : \alpha \in S \rangle$ which is a diamond sequence for $D$ (or a $D$-diamond sequence), which means that $S \in D^+$ and for every $g \in \lambda \lambda$ the set $\{\alpha < \lambda : g(\alpha) = f_\alpha\}$ belongs to $D^+$, so $f$ is also a diamond sequence for the filter $D + S$ (clearly $\Diamond_S$ means $\Diamond_{D_\lambda + S}$ for $S$ a stationary subset of the regular uncountable $\lambda$).

A somewhat more general version of the theorem is

**Claim 2.5.** 1) Assume $\lambda = \chi^+ = 2^\chi$ and $D$ is a $\lambda$-complete filter on $\lambda$ which extends the club filter. If $S \in D^+$ and $\delta \in S \Rightarrow \text{cf}(\delta) \neq \text{cf}(\chi)$, then we have $\Diamond_{D + S}$.

2) We have $\Diamond_D$ when:

- (a) $\lambda = \lambda^{<\lambda}$,
(b) $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ lists $\cup \{ \alpha \lambda : \alpha < \lambda \}$,
(c) $S \in D^+$,
(d) $\bar{u} = \langle u_\alpha : \alpha \in S \rangle$ and $u_\alpha \subseteq \alpha$ for every $\alpha \in S$,
(e) $\chi = \sup \{ |u_\alpha|^+ : \alpha < \lambda \} < \lambda$,
(f) $D$ is a $\chi^+$-complete filter on $\lambda$ extending the club filter,
(g) $(\forall \bar{g} \in \lambda)(\exists D^+ \delta \in S)[\delta = \sup \{ \alpha \in u_\delta : g[\alpha] \in \{ f_\beta : \beta \in u_\delta \} \}].$

3) Assume $\lambda = \chi^+ = 2^\chi$ and $\kappa_0 < \kappa = \text{cf}(\chi) < \chi$, $S \subseteq S^\lambda_\chi$ is stationary, and $D$ is a $\lambda$-complete filter extending the club filter on $\lambda$ to which $S$ belongs. If $\diamond_D$ fails, then $\text{SNR}(\lambda, S, \kappa)$.

**Proof of Claim 2.3** Part (1) follows from Claim 2.5(1) for $D$ the filter $D_\lambda + S$. Part (2) follows from Claim 2.5(3) for $D$ the filter $D_\lambda + S$. □

**Proof of Claim 2.5** Proof of part (1).

Clearly we can assume
\[ \diamond_0 \chi > \kappa_0, \] as for $\chi = \kappa_0$ the statement is empty.

Let
\[ \diamond_1 \langle f_\alpha : \alpha < \lambda \rangle \text{ list the set } \{ f : f \text{ is a function from } \beta \text{ to } \lambda \text{ for some } \beta < \lambda \}. \]

For each $\alpha < \lambda$ clearly $|\alpha| \leq \chi$, so let
\[ \diamond_2, \alpha \ (u_{\alpha, \varepsilon} : \varepsilon < \chi) \text{ be } \subseteq\text{-increasing continuous with union } \alpha \text{ such that } \varepsilon < \chi \Rightarrow |u_{\alpha, \varepsilon}| \leq \kappa_0 + |\varepsilon| < \chi. \]

For $g \in ^\lambda \lambda$ let $h_g \in ^\lambda \lambda$ be defined by
\[ \diamond_3, g \ h_g(\alpha) = \text{Min}\{ \beta < \lambda : g \upharpoonright \alpha = f_\beta \}. \]

Let $cd_\varepsilon (cd_\varepsilon : \varepsilon < \chi)$ be such that
\[ \diamond_4 \ (a) \ cd_\varepsilon \text{ is a one-to-one function from } ^\chi \lambda \text{ onto } \lambda \text{ such that } cd_\varepsilon(\bar{\alpha}) \geq \sup(\alpha_{\varepsilon} : \varepsilon < \chi) \text{ (when } \bar{\alpha} = \langle \alpha_{\varepsilon} : \varepsilon < \chi \rangle), \]
\[ \diamond_5 \ (b) \text{ for } \varepsilon < \chi, \ cd_\varepsilon \text{ is a function from } \lambda \text{ to } \lambda \text{ such that } \bar{\alpha} = \langle \alpha_{\varepsilon} : \varepsilon < \chi \rangle \in ^\chi \lambda \Rightarrow cd_\varepsilon(cd_\varepsilon(\bar{\alpha})) = \alpha_{\varepsilon} \]

(taking that $\lambda = \lambda^\chi$; in the present case this holds as $2^\chi = \chi^+ = \lambda$).

Now we let (for $\beta < \lambda, \varepsilon < \chi$)
\[ \diamond_5, f_{\beta, \varepsilon}^1, \text{ be the function from Dom}(f_\beta) \text{ into } \lambda \text{ defined by } f_{\beta, \varepsilon}^1(\alpha) = cd_\varepsilon(f_{\beta}(\alpha)), \]
so Dom($f_{\beta, \varepsilon}^1$) = Dom($f_\beta$).

Without loss of generality
\[ \diamond_6 \ \alpha \in S \Rightarrow \alpha \text{ is a limit ordinal}. \]

For $g \in ^\lambda \lambda$ and $\varepsilon < \chi$ we let
\[ S^\varepsilon_g = \{ \delta \in S : \delta = \sup\{ \alpha \in u_{\delta, \varepsilon} : \text{ for some } \beta \in u_{\delta, \varepsilon} \text{ we have } g \upharpoonright \alpha = f_{\beta, \varepsilon}^1 \}. \]

Next we shall show
\[ \diamond_7 \text{ for some } \varepsilon(*) < \chi \text{ and for every } g \in ^\lambda \lambda \text{ the set } S^\varepsilon_g \text{ is a } D\text{-positive subset of } \lambda. \]

**Proof of \( \diamond_7 \).** Assume this fails, so for every $\varepsilon < \chi$ there is $g_\varepsilon \in ^\lambda \lambda$ such that $S^\varepsilon_{g_\varepsilon}$ is not $D$-positive and let $E_{\varepsilon}$ be a member of $D$ disjoint to $S^\varepsilon_{g_\varepsilon}$. Define $g \in ^\lambda \lambda$ by
$g(\alpha) := \text{cd}(\{g_\varepsilon(\alpha) : \varepsilon < \chi\})$ and let $h_\beta \in \lambda^{\lambda}$ be as in $@3,g$; i.e. $h_\beta(\alpha) = \text{Min}\{\beta : g \upharpoonright \alpha = f_\beta\}$.

Let $E_\beta = \{\delta < \lambda : \delta$ is a limit ordinal such that $\alpha < \delta \Rightarrow h_\beta(\alpha) < \delta\}$. Clearly it is a club of $\lambda$; hence it belongs to $D$, and so $E = \cap\{E_\varepsilon : \varepsilon < \chi\} \cap E_\varepsilon$ belongs to $D$ as $D$ is $\lambda$-complete and $\chi + 1 < \lambda$.

As $S$ is a $D$-positive subset of $\lambda$ there is $\delta_4 \in E \cap S$. For each $\alpha < \delta_4$, as $\delta_4 \in E \subseteq E_\varepsilon$, clearly $h_\beta(\alpha) < \delta_4$, and $\alpha$ as well as $h_\beta(\alpha)$ belong to $\cup\{u_{\delta_4,\varepsilon} : \varepsilon < \chi\} = \delta_4$.

However, $\langle u_{\delta_4,\varepsilon} : \varepsilon < \chi\rangle$ is $\subseteq$-increasing; hence $\varepsilon_{\delta_4,\alpha} = \text{Min}\{\varepsilon : \alpha \in u_{\delta_4,\varepsilon}$ and $h_\beta(\alpha) \in u_{\delta_4,\varepsilon}\}$ is not just well defined but also $\varepsilon \in [\varepsilon_{\delta_4,\alpha}, \chi) \Rightarrow \{\alpha, h_\beta(\alpha)\} \subseteq u_{\delta_4,\varepsilon}$. As $\text{cf}(\delta_4) \neq \text{cf}(\chi)$, by an assumption on $S$, it follows that for some $\varepsilon(*) < \chi$ the set $B := \{\alpha < \delta_4 : \varepsilon_{\delta_4,\alpha} < \varepsilon(*)\}$ is unbounded below $\delta_4$.

So

(a) $\alpha \in B \Rightarrow \{\alpha, h_\beta(\alpha)\} \subseteq u_{\delta_4,\varepsilon(*)}$ and
(b) $\alpha \in B \Rightarrow g(\alpha) = f_{h_\beta(\alpha)} \Rightarrow \bigwedge\varepsilon_{\varepsilon(*)} g(\alpha) = f_{h_\beta(\alpha),\varepsilon(*)} \Rightarrow g(\alpha) = f_{h_\beta(\alpha),\varepsilon(*)}.$

But $\delta_4 \in E \subseteq E_\varepsilon(*)$; hence $\delta_4 \notin S_{\delta_4}^{\varepsilon(*)}$ by the choice of $E_\varepsilon(*)$. But by (a) + (b) and the definition of $S_{\delta_4}^{\varepsilon(*)}$ recalling $\delta_4 \in S$, we have sup($B$) $= \delta_4 \Rightarrow \delta_4 \in S_{\delta_4}^{\varepsilon(*)}$ (where $h_\beta(\alpha)$ plays the role of $\beta$ in the definition of $S_{\delta_4}^{\varepsilon(*)}$ above), a contradiction. So the proof of $\otimes$ is finished.

Let $\chi_\varepsilon = (\varepsilon(*) + \varepsilon_0)$; hence $\delta \in S \Rightarrow |u_{\delta,\varepsilon(*)}| \leq \chi_\varepsilon$ and $\chi_\varepsilon + < \chi_\varepsilon$ as $\chi_\varepsilon < \chi_\varepsilon < \chi < \lambda$ because $\varepsilon_0, \varepsilon(*) < \chi < \lambda$. Now we apply Claim 2.5, which is proved below with $\lambda, S, D, \chi_\varepsilon := f_{\beta,\varepsilon(*)}.(\beta < \lambda), \{u_{\delta,\varepsilon(*)} : \delta \in S\}$ here standing for $\lambda, S, D, \chi, f, \bar{u}$ there.

The conditions there are satisfied, hence also the conclusion which says that $\Diamond_D$ holds.

**Proof of Claim 2.5.** Let

$\mathbb{E}_1 \langle \text{cd}_{\varepsilon} : \varepsilon < \chi\rangle$ and $\text{cdD}$ be as in $\otimes$ in the proof of part (1), possible as we are assuming $\chi < \lambda = \lambda^{\lambda}$;

$\mathbb{E}_2$ for $\beta < \lambda$ and $\zeta < \chi$ let $f_{\beta,\zeta}$ be the function with domain $\text{Dom}(f_{\beta})$ such that $f_{\beta,\zeta}(\alpha) = \text{cd}(f_{\beta}(\alpha))$;

$\mathbb{E}_3$ for $g \in \lambda^{\lambda}$ define $h_\beta \in \lambda^{\lambda}$ as in $\otimes$ in the proof of part (1), i.e. $h_\beta(\alpha) = \text{Min}\{\beta : g \upharpoonright \alpha = f_\beta\}$.

If $2^{<\chi} < \lambda$ our life would be easier, but we do not assume this. For $\delta \in S$ let $\xi_\delta^\varepsilon$ be a cardinal, and let $\langle(\alpha_1^\varepsilon, \xi_\delta^\varepsilon, \alpha_2^\varepsilon) : \xi < \xi_\delta^\varepsilon\rangle$ list the set $\{\alpha_1^\varepsilon, \alpha_2^\varepsilon \in u_{\delta_4,\varepsilon} : \text{Dom}(f_{\alpha_2^\varepsilon}) = \alpha_1^\varepsilon\}$. Note that $\xi_\delta^\varepsilon < \chi$, recalling $|u_{\delta_4,\varepsilon}| < \chi$ by clause (e) of the assumption. We now try to choose $(\bar{v}_\varepsilon, g_\varepsilon, E_\varepsilon)$ by induction on $\varepsilon < \chi$ (note that $\bar{v}_\varepsilon$ is defined from $\langle g_\varepsilon : \varepsilon < \chi\rangle$ (see clause (e) of $\mathbb{E}_4$ below), so we choose just $(g_\varepsilon, E_\varepsilon)$) such that

$\mathbb{E}_1$ (a) $E_\zeta$ is a member of $D$ and $\langle E_\zeta : \zeta \leq \varepsilon\rangle$ is $\subseteq$-decreasing with $\zeta$;
(b) $\bar{v}_\varepsilon = \langle v_\zeta : \zeta \in \delta \cap E_\varepsilon\rangle$ when $E_\varepsilon = \cap\{E_\zeta : \zeta < \varepsilon\} \cap \lambda$, so is $\lambda$ if $\varepsilon = 0$;
(c) $\langle v_\zeta : \zeta \leq \varepsilon\rangle$ is $\subseteq$-decreasing with $\zeta$ for each $\delta \in \delta' \cap E_\varepsilon$;
(d) $g_\varepsilon \in \lambda^{\lambda};$
(e) $v_\zeta = \langle \xi < \xi_\delta^\varepsilon : \xi < \varepsilon, \text{then } g_\zeta \upharpoonright \alpha_1^\varepsilon = f_{\zeta}^\varepsilon \upharpoonright \alpha_1^\varepsilon \rangle$ (so if $\varepsilon$ is a limit ordinal then $v_0 = \bigcap\varepsilon_\zeta$ and $\varepsilon = 0 \Rightarrow v_\zeta = \xi^\varepsilon_\delta$);
(f) if $\delta \in E_\varepsilon' \cap S$, then $v^\bar{v}_\varepsilon \subseteq v^\varepsilon_\delta$ or $\delta \geq \sup\{\alpha^\varepsilon_\delta : \xi \in v^\varepsilon_\delta\}.$

Next $\otimes$ we cannot carry the induction, that is for all $\varepsilon < \chi$.  

Why? Assume by contradiction that \( \langle (\vec{v}_\gamma, g_\varepsilon, E_\varepsilon) : \varepsilon < \chi \rangle \) is well defined. Let 
\[ E := \bigcap \{ E_\varepsilon : \varepsilon < \chi \} \text{; it is a member of } D \text{ as } D \text{ is } \chi^+\text{-complete.} \]
Define \( g \in \lambda^\delta \) by \( g(\alpha) := \text{cd}(g_\varepsilon(\alpha) : \varepsilon < \chi) \). Let \( E_\delta = \{ \delta < \lambda : \delta \text{ a limit ordinal such that } h_\varepsilon(\alpha) < \delta \text{ and } \delta > \sup(\text{Dom}(f_\alpha) \cup \text{Rang}(f_\alpha)) \text{ for every } \alpha < \delta \} \), so \( E_\delta \) is a club of \( \lambda \); hence it belongs to \( D \). By assumption \( (g) \) of the claim the set
\[ S_g := \{ \delta \in S : \delta = \sup(\{ \alpha \in u_\delta : (\exists \beta \in u_\delta)(f_\beta = g \upharpoonright \alpha)\} \} \]
is \( D \)-positive, so we can choose \( \delta \in E \cap E_\delta \cap S_g \). Hence \( B := \{ \alpha \in u_\delta : (\exists \beta \in u_\delta)(f_\beta = g(\alpha)) \} \) is an unbounded subset of \( u_\delta \) and let \( h : B \to u_\delta \) be \( h(\alpha) = \min(\beta \in u_\delta : f_\beta = g(\alpha)) \); clearly \( h \) is a function from \( B \) into \( u_\delta \). Now \( \alpha \in B \land \xi < \chi \Rightarrow f_\delta(h(\alpha)) = g(\alpha) \upharpoonright \lambda \wedge \zeta < \chi \Rightarrow f_\delta^2(h(\alpha), \zeta) = g(\alpha) \), so for \( \alpha \in B \) the pair \( (h(\alpha)) \) belongs to \( \{ (\delta_{\alpha, \xi}^2, \delta_{\alpha, \zeta}^2) : \xi \in \nu_\delta^g \} \) for every \( \varepsilon < \chi \). Hence for any \( \varepsilon < \chi \) we have
\[ B \subseteq \{ \delta_{\alpha, \xi}^2, \delta_{\alpha, \zeta}^2 : \xi \in \nu_\delta^g \} , \text{ so } \delta = \sup(\{ \alpha_{\delta, \xi}^2, \delta_{\alpha, \zeta}^2 : \xi \in \nu_\delta^g \} \].

So for the present \( \delta \), in clause (f) of \( \mathbb{H}_4 \) the second possibility never occurs.

So clearly \( \langle \nu_\delta^g, \varepsilon < \chi \rangle \) is strictly \( \leq \)-decreasing, i.e. is \( \leq \)-decreasing, which is impossible as \( |\nu_\delta^g| = \xi_{\lambda, \delta}^g < \chi \). So we have proved \( \oplus_1 \); hence we can assume
\[ \oplus_2 \text{ there is } \varepsilon < \chi \text{ such that we have defined our triple for every } \zeta < \varepsilon \text{, but we cannot define it for } \varepsilon \text{. So we have } \langle (\vec{v}_\zeta, g_\xi, E_\xi) : \zeta < \varepsilon \rangle \].

As in \( \mathbb{H}_4(e) \), let
\[ \circ_1 E'_\varepsilon \text{ be } \lambda \text{ if } \varepsilon = 0 \text{ and } \bigcap \{ E_\varepsilon : \zeta < \varepsilon \} \text{ if } \varepsilon > 0 \text{, and let } S_* := S \cap E'_\varepsilon. \]
Clearly \( \vec{v}_\varepsilon \) is well defined (see clauses (b) and (e) of \( \mathbb{H}_4 \)), and for \( \delta \in S_* \) let \( F_\delta = \{ f_{\alpha, \xi, \varepsilon}^2 : \xi \in \nu_\delta^g \} \), so each member is a function from some \( \alpha \in u_\delta \subseteq \delta \) into some ordinal \( < \delta \).

Let
\[ \circ_2 S_1 := \{ \delta \in S_* : \text{there are } f', f'' \in F_\delta \text{ which are incompatible as functions} \}, \]
\[ \circ_3 S_2 := \{ \delta \in S_* : \delta \notin S_1 \}, \text{ but the function } \bigcup \{ f : f \in F_\delta \}, \text{ has domain } \neq \delta, \]
\[ \circ_4 S_3 = S_1 \setminus (S_1 \cup S_2). \]

For \( \delta \in S_3 \) let \( g_\delta = \bigcup \{ f : f \in F_\delta \} \), so by the definition of \( S_\ell : \ell = 1, 2, 3 \), clearly \( g_\delta \in \delta. \) Now if \( \langle g_\delta : \delta \in S_3 \rangle \) is a diamond sequence for \( D \), we are done.

Assume that this fails. So for some \( g \in \lambda^\delta \) and member \( E \) of \( D \) we have \( \delta \in S_3 \cap E \Rightarrow g_\delta \neq g \upharpoonright \delta \). Without loss of generality \( E \) is included in \( E'_\varepsilon \). But then we could have chosen \( (g, E) \) as \( (g_\varepsilon, E_\varepsilon) \), recalling that \( \vec{v}_\varepsilon \) was already chosen. Then easily the triple \( \langle g_\varepsilon, E_\varepsilon, \vec{v}_\varepsilon \rangle \) is as required in \( \oplus_1 \), contradicting the choice of \( \varepsilon \) in \( \oplus_2 \), so we are done proving part (2) of Claim 2.5 and hence also part (1). \( \square \)

**Proof of Claim 2.5(3).** We use \( \text{cd} \) \( \text{cd}_\varepsilon \) (for \( \varepsilon < \chi \)), \( \langle (u_{\alpha, \varepsilon} : \varepsilon < \chi) : \alpha < \lambda \rangle, \langle f_\alpha : \alpha < \lambda \rangle, \langle f_{\alpha, \varepsilon}^2 : \alpha < \lambda, \varepsilon < \chi \rangle \) and \( S_\delta^g \) for \( \varepsilon < \chi \) as in the proof of part (1).

Recall that \( \kappa \), a regular uncountable cardinal, is the cofinality of the singular cardinal \( \chi \) and let \( (\chi_\gamma : \gamma < \kappa) \) be increasing with limit \( \chi \). For every \( \gamma < \kappa \) we ask:

**The \( \gamma \)-Question.** For every \( g \in \lambda^\delta \), do we have that the following is a \( D \)-positive subset of \( \lambda^\delta \)?
\[ \{ \delta \in S_\delta^g : \delta \cap \delta \text{ is a stationary subset of } \delta, \text{ where } S_\delta^g \} := \{ \zeta < \lambda : \text{cf}(\zeta) \in [\kappa_0, \kappa), \sup(u_{\zeta, \chi_\gamma}) = \zeta, \text{ and for arbitrarily large } \alpha \in u_{\zeta, \chi_\gamma} \text{, for some } \beta \in u_{\zeta, \chi_\gamma} \text{ and } \varepsilon < \chi_\gamma, \text{ we have } \text{Dom}(f_\beta) = \alpha \text{ and } g \upharpoonright \alpha = f_\beta^\varepsilon \}. \]

**Case 1.** For some \( \gamma < \kappa \), the answer is yes.
Choose \( \{C_\delta : \delta \in S\} \) such that \( C_\delta \) is a club of \( \delta \) of order type \( \text{cf}(\delta) = \kappa \).
For \( \delta \in S \subseteq S^3_\kappa \) let \( u_\delta := \bigcup \{ u_{\alpha, \chi_\gamma} : \alpha \in C_\delta \} \).
Clearly,
\[
\Pi_2 \ | u_\delta | \leq \kappa + \chi_\gamma < \chi ;
\]
\( \Pi_3 \) for every \( g \in \lambda \lambda \) for \( D \)-positively many \( \delta \in S \), we have \( \delta = \sup \{ \alpha \in u_\delta : g|\alpha \in \{ f'_\beta, \varepsilon : \varepsilon < \chi_\gamma \text{ and } \beta < \delta \} \} \).

Why does \( \Pi_3 \) hold? Given \( g \in \lambda \lambda \), let \( h_g \in \lambda \lambda \) be defined by \( h_g(\alpha) = \min \{ \beta < \lambda : g|\alpha = f'_\beta \} \), so \( h_g(\alpha) \geq \alpha \) (but is less than \( \lambda \)). Let \( E_g = \{ \delta < \lambda : \delta \) is a limit ordinal such that \( (\forall \alpha < \delta) h_g(\alpha) < \delta \} \), so \( E_g \) is a club of \( \lambda \), and let \( E'_g \) be the set of accumulation points of \( E_g \), so that \( E'_g \), too, is a club of \( \lambda \). By the assumption of this case, the set \( S' := \{ \delta \in S : \delta \cap S_\gamma[g] \) is a stationary subset of \( \lambda \} \) is \( D \)-positive; hence \( S'' := S' \cap E'_g \) is a \( D \)-positive subset of \( \lambda \). Let \( \delta \in S'' \). By \( E'_g \)'s definition, we can find that \( B^0_\delta \subseteq E_g \cap \delta \) unbounded in \( \delta \), so without loss of generality \( B^0_\delta \) is closed.
But \( S_\gamma[g] \cap \delta \) is a stationary subset of \( \delta \), recalling \( \delta \in S'' \), so \( B^1_\delta = B^0_\delta \cap S_\gamma[g] \cap C_\delta \) is a stationary subset of \( \delta \) as \( B^0_\delta, C_\delta \) are closed unbounded subsets of \( \delta \).

Clearly \( \zeta \in B^1_\delta \Rightarrow \zeta \in C_\delta \Rightarrow u_{\chi_\gamma, \chi_\gamma} \subseteq u_\delta \) by the definitions of \( B^1_\delta \) and \( u_\delta \). Also, \( \zeta \in B^1_\delta \Rightarrow \zeta \in S_\gamma[g] \Rightarrow (\zeta \text{ is a limit ordinal}) \land \zeta = \sup(u_{\chi_\gamma, \chi_\gamma}) = \sup \{ \alpha \in u_{\chi_\gamma, \chi_\gamma} : \langle \exists \beta \in u_{\chi_\gamma, \chi_\gamma} \rangle \} \Rightarrow (\zeta \text{ is a limit ordinal}) \land \zeta = \sup \{ \alpha \in u_\delta \cap \zeta : \langle \exists \beta \in u_\delta \rangle \} \).

As \( B^1_\delta \) is unbounded in \( \delta \) being stationary, we are done proving \( \Pi_3 \).

Now without loss of generality every \( \delta \in S \) is divisible by \( \chi \); hence \( \delta = \chi_\gamma \delta \) and let \( u'_\delta = u_\delta \cup \{ \alpha, \chi_\gamma + \varepsilon : \alpha \in u_\delta, \varepsilon < \chi_\gamma \} \), so \( u_\delta \) is an unbounded subset of \( \delta \), and let \( f'_\delta = f'_\delta \) when \( \delta = \chi_\gamma \alpha \varepsilon, \varepsilon < \chi_\gamma \). So translating what we have:

\( \Pi_4 \) (a) \( \{ f'_\alpha : \alpha < \lambda \} \) is a sequence of members of \( \bigcup \{ \beta : \beta < \lambda \} \),
(b) for \( \delta \in S, u'_\delta \) is an unbounded subset of \( \delta \) of cardinality \( \leq \chi_\gamma \times \chi_\gamma = \chi_\gamma(\leq \chi) \),
(c) for every \( g \in \lambda \lambda \) for \( D \)-positively many \( \delta \in S \), we have \( \delta = \sup \{ \alpha \in u'_\delta : \langle \exists \beta \in u'_\delta \rangle \} \).

Now we can apply part (2) with \( \{ f'_\alpha : \alpha < \lambda \}, \{ u'_\delta : \delta \in S \} \) replacing \( f, \{ u_\delta : \delta \in S \} \).

Therefore we can prove \( \diamondsuit_S \) and we are done.

**Case 2.** For every \( \gamma < \kappa \) the answer is no.

Let \( \langle g_\gamma, E_\gamma \rangle \) exemplify that the answer for \( \gamma \) is no, so \( g_\gamma \in \lambda \lambda \) and \( E_\gamma \in D \). Let \( E = \bigcap \limits_{\gamma < \kappa} E_\gamma \), so \( E \) is a member of \( D \). Let \( g \in \lambda \lambda \) be defined by \( g(\alpha) = \text{cd}(g_\gamma(\alpha)) ; \gamma < \kappa \text{ if } \gamma < \kappa \text{ and is 0 if } \varepsilon \in [\kappa, \chi] \).

Let \( E_g := \{ \delta < \lambda : \delta \) is a limit ordinal such that if \( \alpha < \lambda \), then \( h_g(\alpha) < \delta \) and \( \delta > \sup(\text{Dom}(f_\alpha) \cup \text{Rang}(f_\alpha)) \} \).

We now define \( h : \lambda \rightarrow \kappa \) as follows:

\( \Pi_5 \) For \( \beta < \lambda \)
(a) if \( \text{cf}(\beta) \notin [\aleph_0, \kappa) \text{ or } \beta \notin E_g, \) then \( h(\beta) = 0 \);
(b) otherwise
\( h(\beta) = \min \{ \gamma < \kappa : \beta = \sup \{ \alpha_1 \in u_{\beta, \chi_\gamma} : \text{ for some } \alpha_2 \in u_{\beta, \chi_\gamma} \text{ and } \varepsilon < \chi_\gamma \text{ we have } g \upharpoonright \alpha_1 = f''_{\alpha_2, \varepsilon} \} \} \).
Now
\[ \mathbb{E}_6 : h : \lambda \rightarrow \kappa \text{ is well defined.} \]

Why does \( \mathbb{E}_6 \) hold? Let \( \beta < \lambda \). If \( \text{cf}(\beta) \notin [\kappa_0, \kappa] \) or \( \beta \notin E_\gamma \), then \( h(\alpha) = 0 < \kappa \)
by clause (a) of \( \mathbb{E}_5 \). So assume \( \text{cf}(\beta) \in [\kappa_0, \kappa] \) and \( \beta \in E_\gamma \). Let \( (\gamma_1, \epsilon) : \epsilon < \text{cf}(\beta) \)
be increasing with limit \( \beta \) and let \( \gamma_2, \epsilon = \min \{ \gamma : g[\gamma_1, \epsilon] = f_\gamma \} \),
so \( \epsilon < \text{cf}(\beta) \Rightarrow \gamma_2^2 < \beta \) as \( \beta \in E_\gamma \). But \( (u, \chi, \zeta : \zeta < \text{cf}(\chi)) \) is \( \subseteq \)-increasing with union \( \beta \), so
for each \( \epsilon < \text{cf}(\beta) \) there is \( \zeta = \zeta_{\beta, \epsilon} < \text{cf}(\chi) \) such that \( \{ \gamma_{\beta, \epsilon}, \gamma_2^2 \} \subseteq u_{\beta, \chi, \zeta} \).
As \( \text{cf}(\beta) < \kappa = \text{cf}(\chi) \) for some \( \zeta < \kappa \), the set \( \{ \epsilon < \text{cf}(\beta) : \zeta_{\beta, \epsilon} < \zeta \} \)
is unbounded in \( \text{cf}(\beta) \). So \( \zeta \) can serve as \( \gamma \) in clause (b) of \( \mathbb{E}_5 \), so \( h(\beta) \) is well defined. In particular,
it is less than \( \kappa \), so we have proved \( \mathbb{E}_6 \).

\[ \mathbb{E}_7 \text{ If } \delta \in S \cap E_\gamma \text{, then for some club } C \text{ of } \delta \text{ the function } h \upharpoonright C \text{ is increasing.} \]

Why does \( \mathbb{E}_7 \) hold? If not, then by Fodor’s lemma for some \( \gamma < \kappa \) the set \( \{ \delta' \in \delta \cap S : h(\delta') \leq \gamma \} \) is a stationary subset of \( \delta \), and we get a contradiction to the choice of \( E_\gamma \),
so \( \mathbb{E}_7 \) holds indeed.

So \( h \) is as promised in the claim. \( \Box \)

Note that

**Observation 2.6.** If \( \kappa_* < \lambda \) are regular, \( S^\lambda_{\kappa_*} \) strongly does not reflect in \( \lambda \) for every \( \kappa \in \text{Reg} \cap \kappa_* \) and \( \Pi(\text{Reg} \cap \kappa_*) < \lambda \), then:

(a) \( S^\lambda_{\kappa_*} \) can be divided to \( \leq \Pi(\text{Reg} \cap \kappa_*) \) sets, each not reflecting any \( \delta \in S^\lambda_{\kappa_*} \); in particular,

(b) \( S^\lambda_{\kappa_*} \) can be divided to \( \leq \Pi(\text{Reg} \cap \kappa_*) \) sets each not reflecting any \( \delta \in S^\lambda_{\kappa_*} \).

**Remark 2.7.** 1) Of course if \( \lambda \) has \( \kappa \)-SNR, then this holds for every regular \( \lambda' \in (\kappa, \lambda) \).

2) We may state the results using \( \lambda^* \) (see below).

**Definition 2.8.** For each regular \( \kappa \) let \( \lambda^* \kappa = \text{Min} \{ \lambda : \lambda \text{ regular fails to have} \ k\text{-SNR} \} \), and let \( \lambda^* \) be infinite (or not defined) if there is no such \( \lambda \).

### 3. CONSISTENT FAILURE ON \( S^2_1 \)

A known question was:

**Question 3.1.** For \( \theta \in \{ \kappa_0, \kappa_1 \} \) do we have \( 2^{\kappa_0} = 2^{\kappa_1} = \kappa_2 \Rightarrow \langle \otimes \rangle_{\kappa^*} \) ?

So for \( \theta = \kappa_0 \) the answer is yes (by Claim 2.3(1)), but what about \( \theta = \kappa_1 \)? We noted some years ago the following:

**Claim 3.2.** Assume \( V \models \text{GCH} \) or even just \( 2^{\kappa_\ell} = \kappa_{\ell+1} \) for \( \ell = 0, 1, 2 \). Then some forcing notion \( P \) satisfies

(a) \( P \) is of cardinality \( \kappa_3 \);

(b) forcing with \( P \) preserves cardinals and cofinalities;

(c) in \( V^P, 2^{\kappa_0} = 2^{\kappa_1} = \kappa_2, 2^{\kappa_2} = \kappa_3 \);

(d) in \( V^P, \langle \otimes \rangle_S \) fails where \( S = \{ \delta < \kappa_2 : \text{cf}(\delta) = \kappa_1 \} \). Moreover, \( *\) there is a sequence \( A = \langle A_\delta : \delta \in S \rangle \) where \( A_\delta \) an unbounded subset
of \( \delta \) of order type \( \omega_1 \) satisfying \( \langle \ast \rangle \) if \( f = \langle f_\delta : \delta \in S \rangle, f_\delta \in (A_\delta)(\omega_1) \), then there is \( f \in (\omega_2)(\omega_1) \) such that \( \delta \in S \Rightarrow \delta > \text{sup}(\{ \alpha \in A_\delta : f(\alpha) < f_\delta(\alpha) \}) \).
Remark 3.3. Similarly for other cardinals.

Proof. There is an \( N_1 \)-complete \( N_3 \)-c.c. forcing notion \( P \) not collapsing cardinals and not changing cofinalities, preserving \( 2^{N_2} = N_4 \) for \( \ell = 0, 1, 2 \) and \( |P| = N_3 \) such that in \( V^P \), we have \((*)\). In fact more\footnote{\( A = \langle A_\delta : \delta \in S \rangle \), where \( A_\delta \) is an unbounded subset of \( \delta \) of order type \( \omega_1 \) satisfying: \( (*\dagger) \) if \( f = \langle f_\delta : \delta \in S \rangle, f_\delta \in (A_\delta)^{\omega_1} \), then there is \( f \in (\omega_2)^{\omega_1} \) such that for every \( \delta \in S_{N_1}^{N_2} \) for every \( \alpha \in A_\delta \) large enough, we have \( f(\alpha) = f_\delta(\alpha) \).} than \((*)\) holds; see \[\footnote{\( A = \langle A_\delta : \delta \in S \rangle \), where \( A_\delta \) is an unbounded subset of \( \delta \) of order type \( \omega_1 \) satisfying: \( (*\dagger) \) if \( f = \langle f_\delta : \delta \in S \rangle, f_\delta \in (A_\delta)^{\omega_1} \), then there is \( f \in (\omega_2)^{\omega_1} \) such that for every \( \delta \in S_{N_1}^{N_2} \) for every \( \alpha \in A_\delta \) large enough, we have \( f(\alpha) = f_\delta(\alpha) \).}^\]. Let \( Q \) be the forcing of adding an \( N_2 \) Cohen or just any c.c.c. forcing notion of cardinality \( N_2 \) adding \( N_2 \) reals (this can be \( Q \), a \( P \)-name). Now we shall show that \( P \ast Q \), equivalently \( P \times Q \), is as required:

Clause (a):
\[ |P \ast Q| = N_3 \text{, trivial.} \]

Clause (b):
Preserving cardinals and cofinalities; obvious as both \( P \) and \( Q \) do this.

Clause (c):
Easy.

Clause (d):
In \( V^P \) we have \((*)\) as exemplified by, say, \( A = \langle A_\delta : \delta \in S \rangle \). We shall show that \( V^{P \ast Q} \models "A \text{ satisfies } (**\ast)". Otherwise in \( V^{P \ast Q} \) we have \( \bar{f} = \langle f_\delta : \delta \in S \rangle \), say in \( V[G_P, G_Q] \), a counterexample. Then in \( V[G_P] \) for some \( q \in Q \) and \( \bar{f} \) we have
\[ V[G_P] \models (q \Vdash_{P \ast Q} \bar{f} = \langle f_\delta : \delta \in S \rangle \text{, where } f_\delta : A_\delta \rightarrow \omega_1 \text{ for each } \delta \in S \text{ form a counterexample to } (**\ast). \]

Now in \( V[G_P] \) we can define \( \bar{g} = \langle g^1_\delta : \delta \in S \rangle \in V[G_P] \), where \( g^1_\delta \) is a function with domain \( A_\delta \), by
\[ g^1_\delta(\alpha) = \{i : q \Vdash_{P \ast Q} f_\delta(\alpha) \neq i\}. \]
So in \( V[G_P] \) we have \( q \Vdash_{P \ast Q} \bigwedge_{\delta \in S} (\forall \alpha \in A_\delta) f_\delta(\alpha) \in (g^1_\delta(\alpha))^{\omega_1} \). Also \( g^1_\delta(\alpha) \) is a countable subset of \( \omega_1 \) as \( Q \) satisfies the c.c.c.

For \( \delta \in S \) we define a function \( g_\delta : A_\delta \rightarrow \omega_1 \) by letting \( g_\delta(\alpha) = \sup(g^1_\delta(\alpha)) + 1 \); hence \( g_\delta(\alpha) \in \omega_1 \), so \( \langle g_\delta : \delta \in S \rangle \) is as required on \( \bar{f} \) in \((**\ast)\) in \( V[G_P] \), of course.

Apply clause \((**\ast)\) in \( V[G_P] \) to \( \langle g_\delta : \delta \in S \rangle \) so we can find \( g : \omega_2 \rightarrow \omega_1 \) such that \( \bigwedge_{\delta \in S} (\exists \delta \in S) g_\delta(\alpha) > g(\alpha) \). Now \( g \) is as also required in \( V[G_P][G_Q] \). \( \square \)

We may wonder whether we can strengthen the conclusion of Claim \( 2.3 \) to \( \diamondsuit_\delta \) (of course the demand in clauses (e) and (f) in Observation \( 2.4 \) below are necessary; i.e. otherwise \( \diamondsuit_\delta \) holds). The answer is yes as: (the restriction in (e) and in (f) are best possible).

Observation 3.4. Assume \( \lambda = \lambda^{< \lambda}, S \subseteq S_\kappa^{\lambda} \).

Then for some \( P \)

(a) \( P \) is a forcing notion,

(b) \( P \) is of cardinality \( \lambda^+ \) satisfying the \( \lambda^+ \)-c.c.,

(c) forcing with \( P \) does not collapse cardinals and does not change cofinality,

(d) forcing with \( P \) adds no new \( \eta \in \lambda^{> \text{Ord}} \),

(e) \( \diamondsuit_\delta \) fails for every stationary subset \( S \) of \( \lambda \) such that

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\[(\alpha)\] \(S \subseteq S^\lambda\) when \((\exists \mu < \lambda)[\mu^{<\kappa^{\leq \mu}} = \lambda]\) or just
\[(\beta)\] \(\alpha \in S \Rightarrow |\alpha|^{cf(\alpha)}_{tr} > |\alpha|,
\[(f)\] \((D\ell)_S\) (see below) fails for every \(S \subseteq S^\lambda\) when \(\alpha \in S \Rightarrow |\alpha|^{cf(\alpha)}_{tr} = \lambda\).

Recalling

**Definition 3.5.** 1) For \(\mu \geq \kappa = \text{cf}(\kappa)\) let \(\mu^{(\kappa)}_{tr} = \{|T| : T \subseteq \kappa^{\leq \mu} \text{ is closed under initial segments (i.e. a subtree)}\}.|\mu| \leq \mu\}.

2) For \(\lambda\) regular uncountable and stationary \(S \subseteq \lambda\) let \((D\ell)_S\) mean that there is a sequence \(\bar{P} = \langle P_\delta : \delta \in S \rangle\) witnessing it, which means:
\[
(\ast)_P \quad (a)\ T_\delta \subseteq |\delta| \text{ has cardinality } < \lambda,
(b) \text{ for every } f \in \lambda\ the \text{ set } \{\delta \in S : f \upharpoonright \delta \in P_\delta\} \text{ is stationary}
\text{(for } \lambda \text{ a successor it is equivalent to } \diamond_S; \text{ for } \lambda \text{ strong inaccessible it is trivial}).
\]

**Proof of Observation 3.4.** Use \(P\) = adding \(\lambda^+, \lambda\)-Cohen subsets.

The proof is straightforward. \(\square\)

**Remark 3.6.** The consistency results in Observation 3.4 are best possible; see \([12]\).


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