AN ALEKSANDROV TYPE ESTIMATE
FOR $\alpha$-CONVEX FUNCTIONS

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Abstract. In the context of $\alpha$-convexity, using an operator similar to the Monge-Ampère operator based on the notion of normal mapping, we estimate the difference between a function $u$ and the solution of the homogeneous problem $U$ in terms of the measure of the normal mapping of $u$ and a power of the distance to the boundary.

1. Introduction

In the theory of the Monge-Ampère equation the following estimate due to Aleksandrov is of great importance: if $u$ is convex in $\Omega$, an open bounded convex subset of $\mathbb{R}^n$, and $u \in C(\overline{\Omega})$ with $u = 0$ on $\partial \Omega$, then

$$|u(x)|^n \leq C \operatorname{dist}(x, \partial \Omega) \operatorname{diam}(\Omega)^{n-1} |Du(\Omega)|,$$

for all $x \in \Omega$, where

$$Du(\Omega) = \{ p \in \mathbb{R}^n : \exists \overline{x} \in \Omega \text{ such that } u(x) \geq u(\overline{x}) + p \cdot (x - \overline{x}) \forall x \in \Omega \},$$

with a constant $C$ depending only on $n$ and independent of $u$. The estimate plays a crucial role in the theory of sections of solutions to the Monge-Ampère equation and consequently in regularity theory; see [Caf90], [Gut01], [GH00]. More generally, if $u$ is not necessarily convex but satisfies $u(x_0) \leq 0$ at some $x_0 \in \Omega$, then (1.1) holds at $x = x_0$. Indeed, taking $v$ to be the convex function defining a cone with base in $\partial \Omega$ and vertex at the point $(x_0, u(x_0))$, and following the argument in [Gut01, Lemma 1.4.1], we obtain $Dv(x_0) \subset Du(\Omega)$. Then the proof of [Gut01, Theorem 1.4.2] applies in this case.

The purpose of this paper is to prove this estimate in the context of $\alpha$-convex functions, $\alpha > 1$; see Definition 2.3. In the language of optimal mass transportation these are functions that are convex with respect to the cost function $c(x,y) = |x-y|^\alpha$. In our estimate, the subdifferential $Du(\Omega)$ on the right hand side of (1.1) is replaced by the quantity

$$F_u(\Omega) = \{ y \in \mathbb{R}^n : \exists \overline{x} \in \Omega \text{ such that } u(x) \geq u(\overline{x}) + |\overline{x} - y|^\alpha - |x - y|^\alpha \forall x \in \Omega \},$$

and $|u(x)|$ on left hand side of (1.1) gets replaced by $U(x) - u(x)$, where $U$ is the solution of the homogeneous problem $|F_U(\Omega)| = 0$ with $U = 0$ on $\partial \Omega$. The case $\alpha = 2$ is related to standard convexity since $F_u(x) = x + 2Du(x)$; see the
end of the proof of Lemma 3.4. The structure of the set $F_u$ is related to the condition (A3w) introduced in [TW09] for general cost functions, and it is proven there, in Example 4, that $|x - y|^{2\alpha}$ satisfies this condition only when $\alpha = 2$ or when $-2 \leq \alpha < 1$. Consequently, from the results of Loeper [Loe09] Proposition 2.11 and Theorem 3.1, the set $F_u(\bar{x})$ defined in Definition 2.1 is in general not connected. We refer to the paper [GN07] for results on Monge-Ampère type equations arising in optimal mass transportation for general cost functions and properties of the subdifferential $F_u$. Optimal mass transportation has recently become a very active area of research; we mention, in particular, the fundamental work of Ma, Trudinger and Wang [MTW05], for smooth cost functions. For further references see [Vil07].

The main estimate proved in this paper is the following:

Let $\alpha > 1$ and let $\Omega$ be an open, bounded, convex domain in $\mathbb{R}^n$. If $u \in C(\Omega)$ with $u = 0$ on $\partial \Omega$ and such that $0 \leq u \leq U$ in $\Omega$, then for all $x \in \Omega$ we have

$$((U(x) - u(x))^n \leq C(\text{diam}(\Omega))^{\alpha ^2} |x|^{2n - 2\alpha} |F_u(\Omega)|$$

whenever $n(2\alpha - 3) - 1 \geq 0,$ and

$$((U(x) - u(x))^n \leq C(\text{diam}(\Omega))^{\alpha ^2} |F_u(\Omega)|$$

whenever $n(2\alpha - 3) - 1 \leq 0$. The constant $C$ depends only on $n$ and $\alpha$, and $U$ is the solution of the homogeneous problem as stated above. Depending on the value of $\alpha$, the hypothesis $u \geq 0$ is essential for the validity of the estimates. Indeed, if $\alpha > 2n/(n - 1)$ and $u < 0$, then it is not possible to give an estimate of $|u|$ by any positive power of the distance; see Remark 5.3. However, if $\alpha \leq 2n/(n - 1)$ and $u < 0$, then such an estimate holds; see Theorem 5.2.

The paper is organized as follows. Section 2 contains preliminary results. In Section 3 we solve the homogeneous Dirichlet problem giving an explicit characterization of the solution. In Section 4 we find the solution $u$ to the Dirichlet problem when the right hand side is a multiple $\beta$ of the Dirac delta function at a point $\bar{x} \in \Omega$, and we estimate the size of $\beta$ in terms of $U(\bar{x}) - u(\bar{x})$ and $\text{dist}(\bar{x}, \partial \Omega)$. The whole Section 4 is devoted to proving Theorem 4.1 and Lemma 4.3 and the main estimates are finally proved in Section 4.

2. Definitions and Preliminary Results

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, convex set, and $\alpha > 1$.

**Definition 2.1.** Let $u : \Omega \to \mathbb{R}$ and $\bar{x} \in \Omega$. We define

$$F_u(\bar{x}) := \{y \in \mathbb{R}^n : u(x) \geq u(\bar{x}) + |x - y|^{\alpha} - |x|^{\alpha} \forall x \in \Omega\}.$$ 

If $E \subseteq \Omega$, we define $F_u(E) = \bigcup_{x \in E} F_u(x)$.

**Remark 2.2.** If $u \in C(\overline{\Omega})$ and $\bar{x} \in \partial \Omega$, then we say that $y \in F_u(\bar{x})$ if there exists $\bar{x} \in \Omega$ such that $y \in F_u(\bar{x})$ and $u(x) \geq u(\bar{x}) + |x - y|^{\alpha} - |x|^{\alpha}$ for all $x \in \Omega$.

**Definition 2.3.** We say that $u : \Omega \to R$ is $\alpha$-convex in $\Omega$ if $F_u(x) \neq \emptyset$, $\forall x \in \Omega$.

**Lemma 2.4.** If $u : \Omega \to R$ is $\alpha$-convex in $\Omega$, then $u$ is locally Lipschitz continuous in $\Omega$.

**Proof.** First we check the boundedness of $u$. That $u$ is bounded below is trivial. We show that $u$ is locally bounded above in $\Omega$. Let $K \subseteq \Omega$ be compact and suppose there exist $x_0 \in K$ and $\{x_n\} \subset K$ with $x_n \to x_0$ and $u(x_n) \to +\infty$. If
Indeed, the function $u(x) = u(x_n) + |x_n - y_n|^\alpha - |x - y_n|^\alpha \forall x \in \Omega$. If $|y_n| \leq M$, then clearly $\lim_{n \to \infty} u(x_n) + |x_n - y_n|^\alpha - |x - y_n|^\alpha = +\infty$ for each $x \in \Omega$ (passing through a subsequence if necessary) which yields $u(x) = +\infty$ for $x \in \Omega$. So, we can assume that $|y_n| \to +\infty$. Let $I = \bigcup_{k=1}^{\infty} \cap_{n=k}^{\infty} B(x_n - y_n)(y_n)$. If $x \in I$, then $\lim_{n \to \infty} u(x_n) + |x_n - y_n|^\alpha - |x - y_n|^\alpha = +\infty$ and so $I \cap \Omega = \emptyset$.

Letting $\xi_n = \frac{y_n - x_n}{|y_n - x_n|}$, we may assume that $\xi_n \to e_1$. Consider the cone $C = \{x : e_1 \cdot \left(\frac{x-x_0}{|x-x_0|}\right) \geq \delta\}$. We claim that $C \subset I$. If we let $x \in C$, there exists $N$ such that $\xi_n \cdot \left(\frac{x-x_n}{|x-x_n|}\right) \geq \frac{\delta}{2}$ for all $n \geq N$. Since $|x_n - y_n| \to \infty$, there exists $N'$ such that $\frac{\delta}{2} |x - x_n| \leq \frac{\delta}{2} \text{diam}(\Omega) \leq |x_n - y_n|$ for all $n \geq N'$. Thus, $\frac{\delta}{2} |x_n - y_n| \leq |x - x_n| \leq \frac{\delta}{2} |x_n - y_n| |x - x_n|$ for all $n$ sufficiently large and the claim is proved. Since $x_0 \in \Omega$ there exists $x' \in \Omega \cap C$, and therefore $x' \in \Omega \cap I$, a contradiction. Therefore, $u$ is bounded on compact subsets of $\Omega$.

We next show that $u$ is locally Lipschitz. Let $B$ be a ball with $2B \Subset \Omega$. Then we show first that the set $F_u(B)$ is bounded. Otherwise, there exist $y_n \in F_u(x_n)$, with $x_n \in B$ and $|y_n| \to \infty$. Since $u$ is bounded above in $2B$ and bounded below in $\Omega$, we get that $|x_n - y_n|^\alpha - |x - y_n|^\alpha \leq M$ for all $x \in 2B \Subset \Omega$. Let $x = x_n + \beta \xi_n$, with $\xi_n = \frac{y_n - x_n}{|y_n - x_n|}$. We have $x \in 2B$ for $\beta$ small, and therefore $|x_n - y_n|^\alpha - |x - y_n|^\alpha = |x_n - y_n|^\alpha - |x_n + \beta \xi_n - y_n|^\alpha = \alpha \beta |x_n + \beta \xi_n - y_n|^\alpha - |x_n - y_n|^\alpha$ for some $0 < \beta < \beta$ from the mean value theorem. But the last expression tends to $+\infty$ as $n \to \infty$, which yields a contradiction. Finally, let $B \Subset \Omega$ and let $x_0, x_1 \in B$ and $y_0 \in F_u(x_0), y_1 \in F_u(x_1)$. Then $u(x) \geq u(x_0) + |x_0 - y_0|^\alpha - |x - y_0|^\alpha$ and $u(x) \geq u(x_1) + |x_1 - y_1|^\alpha - |x - y_1|^\alpha \forall x \in \Omega$. Thus, $|x_0 - y_0|^\alpha - |x_0 - y_0|^\alpha \leq u(x_1) - u(x_0) \leq |x_0 - y_0|^\alpha - |x_1 - y_1|^\alpha$. Again from the mean value theorem it follows that $\alpha |x - y_0|^\alpha - \beta(x - x_0) \leq u(x_1) - u(x_0) \leq |x_0 - y_0|^\alpha - |x_1 - y_1|^\alpha$. Consequently, $-\alpha |x - y_0|^\alpha - |x_0 - x_1| \leq u(x_1) - u(x_0) \leq \alpha |x - y_1|^\alpha |x_0 - x_1|$ and the Lipschitz continuity of $u$ in $B$ follows.

Remark 2.5. If $u$ is $\alpha$-convex in $\Omega$, then

$$\{(y \in \mathbb{R}^n : y \in F_u(x_1) \cap F_u(x_2), x_1 \neq x_2 \in \Omega)\} = 0.\)$$

Indeed, the function $u^*(z) = \inf_{x \in \Omega} u(x) + |x - z|^\alpha$ is locally Lipschitz in $\mathbb{R}^n$. Suppose $y_1 \in F_u(x_1)$, where $x_1 \in \Omega$. Then $u^*(z) \leq u(x_1) + |x_1 - y_1|^\alpha = u(x_1) + |x_1 - y_1|^\alpha - |x_1 - y_1|^\alpha + |x_1 - z|^\alpha = u^*(y_1) - |x_1 - y_1|^\alpha + |x_1 - z|^\alpha, \forall z \in \mathbb{R}^n$. Therefore, we see that if $y \in F_u(x_1) \cap F_u(x_2)$ for some $x_1 \neq x_2 \in \Omega$, then $u^*$ cannot be differentiable at $y$. This proves the remark.

Remark 2.6. The conclusion in the previous remark also holds if $u \in C(\overline{\Omega})$ is $\alpha$-convex and the $y$'s are taken so that $y \in F_u(x_1) \cap F_u(x_2)$, where $x_1 \in \Omega$ and $x_2 \in \partial\Omega$.

For each $y_0 \in \Omega$ consider the set

$$A_{y_0} = \{x \in \partial\Omega : \text{dist}(y_0, \partial\Omega) = |y_0 - x|\}.\)$$

We have the following lemma, which will be used in the proof of Theorem 3.1.
Lemma 2.7. For each $y_0 \in \Omega$ and for each $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, there exists $\bar{x} \in A_{y_0}$, a sequence $y_k = y_0 + \delta_k \xi$ with $\delta_k > 0$ and $\delta_k \to 0$, and $x_k \in A_{y_k}$ such that $x_k \to \bar{x}$.

Proof. For each $k$, let $x_k \in A_{y_k} + \frac{1}{k} \xi$ and since $\{x_k\}$ is a bounded sequence, passing through a subsequence, we can assume $x_k \to \bar{x} \in \partial \Omega$. Then $\text{dist}(y_0, \partial \Omega) = \lim_{k \to \infty} \text{dist}(y_0 + \frac{1}{k} \xi, \partial \Omega) = \lim_{k \to \infty} |y_0 + \frac{1}{k} \xi - x_k| = |y_0 - \bar{x}|$, i.e. $\bar{x} \in A_{y_0}$.\[\square\]

3. Homogeneous Dirichlet problem

In this section, we solve the Dirichlet problem with zero boundary data and give a characterization of the solution. This problem was considered in [GN07] for general cost functions, but in our case we need to have a more precise characterization of the solution; see [GN07], Theorem 6.7.

Theorem 3.1. Let $\Omega$ be a bounded, open, convex domain in $\mathbb{R}^n$. Let $\alpha > 1$. Given $y \in \mathbb{R}^n$, let $v_{\lambda, y}(x) = \lambda - |x - y|^{\alpha}$ ($v_{\lambda, y}$ is $\alpha$-convex in $\mathbb{R}^n$) and let

$$U(x) = \sup \{v_{\lambda, y}(x) : v_{\lambda, y} \leq 0 \text{ on } \partial \Omega\}.$$ 

We then have:

1. $U$ is $\alpha$-convex in $\Omega$;
2. $U \in C(\overline{\Omega})$ and $U = 0$ on $\partial \Omega$;
3. $|F_U| = 0$;
4. $U(x) = \sup \{\text{dist}^\alpha(y, \partial \Omega) - |x - y|^{\alpha} : y \in \Omega\}$;
5. if $U(x_0) = \text{dist}^\alpha(y_0, \partial \Omega) - |x_0 - y_0|^{\alpha}$, then $x_0 \in A_{y_0}$ with

$$\Lambda_{y_0} = \{x \in \Omega : \text{for each } \xi \text{ with } |\xi| = 1 \text{ there exists } \bar{x} \in A_{y_0} \text{ such that}$$

$$(|\bar{x} - y_0|^{\alpha-2}(y_0 - \bar{x}) - |x - y_0|^{\alpha-2}(y_0 - x)) \cdot \xi \leq 0\},$$

where the set $A_{y_0}$ is defined in (2.3).

Proof. 1. To define $U$ it is enough to consider the set of functions $v_{\lambda, y}$ with $y \in \Omega$. Because if $y \notin \Omega$, and $v_{\lambda, y}(x) = \lambda - |x - y|^{\alpha}$ satisfies $v_{\lambda, y} \leq 0$ on $\partial \Omega$, then $v_{\lambda, y} \leq 0$ in $\Omega$. While if $y \in \Omega$, then $v(x) = \text{dist}^\alpha(y, \partial \Omega) - |x - y|^{\alpha}$ satisfies $v \leq 0$ on $\partial \Omega$, and hence $U(x) \geq v(x) \forall x \in \Omega$; in particular,

$$(3.1) \quad U(y) \geq \text{dist}^\alpha(y, \partial \Omega) \forall y \in \Omega.$$

Let $x_0 \in \Omega$. From the definition of $U$, $U(x_0) = \lim_{n \to \infty} \lambda_n - |x_0 - y_n|^{\alpha} = \lim_{n \to \infty} v_n(x_0)$ with $v_n(x) = \lambda_n - |x - y_n|^{\alpha}$. Since $y_n \to y_0$, we may assume $y_n \to y_0 \in \overline{\Omega}$, and so also $\lambda_n \to \lambda_0$. Thus, $U(x_0) = \lambda_0 - |x_0 - y_0|^{\alpha}$ and $v_n \to v_0$ uniformly in $\overline{\Omega}$ where $v_0(x) = \lambda_0 - |x - y_0|^{\alpha}$. It follows that $v_0 \leq 0$ on $\partial \Omega$ and since $v_0(x_0) = U(x_0) \geq \text{dist}^\alpha(x_0, \partial \Omega) > 0$, we have $y_0 \in \Omega$. Since $U \geq v_n$, it follows that $U \geq v_0$ in $\Omega$ and therefore $y_0 \in F_U(x_0)$, so $U$ is $\alpha$-convex.

2. Since $U$ is $\alpha$-convex in $\Omega$, from Lemma 2.4, $U$ is continuous in $\Omega$. We show that $U$ is continuous up to the boundary. Fix $\bar{x} \in \partial \Omega$ and let $\eta(\bar{x})$ be the unit inner normal to some supporting plane to $\partial \Omega$ at $\bar{x}$, and let $W(x) = 2\text{diam}(\Omega)^{\alpha-1}(|x - \bar{x}, \eta(\bar{x}))$. Let $v_{\lambda, y}(x) = \lambda - |x - y|^{\alpha}$ with $v_{\lambda, y} \leq 0$ on $\partial \Omega$. Since $|D(W - v_{\lambda, y})(x)| > 0$ for $x \in \Omega$ and $W - v_{\lambda, y} \geq 0$ on $\partial \Omega$, it follows that $W - v_{\lambda, y} \geq 0$ in $\Omega$. Therefore $V(x) \geq U(x)$ for $x \in \Omega$. This together with (3.1) yields that $U \in C(\overline{\Omega})$ and $U = 0$ on $\partial \Omega$.

3. Let $y \in F_U(\bar{x})$ for some $\bar{x} \in \Omega$, so $U(\bar{x}) \geq U(\bar{x}) + |\bar{x} - y|^{\alpha} - |x - y|^{\alpha}$ for all $x \in \Omega$ and we claim that there exists $\bar{x} \in \partial \Omega$ such that $U(\bar{x}) + |\bar{x} - y|^{\alpha} - |x - y|^{\alpha} = 0$.\[\square\]
Otherwise, since $U = 0$ on $\partial \Omega$, we must have that $U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha < 0$ for all $x \in \partial \Omega$, and hence, there exists $\epsilon > 0$ such that $U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha + \epsilon < 0$ for all $x \in \partial \Omega$. Then by the definition of $U$, we must have $U(x) \geq U(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha + \epsilon$ for all $x \in \Omega$, and in particular, $U(\bar{x}) \geq U(\bar{x}) + \epsilon$, a contradiction, and the claim is proved. Therefore, there exists $\hat{x} \in \partial \Omega$ such that $U(\hat{x}) + |\hat{x} - y|^\alpha - |x - y|^\alpha = 0$ and since $U(\hat{x}) = 0$ this clearly implies that $y \in F_U(\hat{x})$. We have then proved that $F_U(\hat{\Omega}) \subseteq \{y : \exists x_1 \in \Omega, x_2 \in \partial \Omega \text{ such that } y \in F_U(x_1) \cap F_U(x_2)\}$, which implies that $|F_U(\Omega)| = 0$.

We remark that if $\alpha = 2$, we have that $U(x) = |\bar{x} - y|^2 - |x - y|^2$ for all $x \in [\bar{x}, \tilde{x}]$, and hence $y \in F_U(x)$ for all $x \in [\tilde{x}, \bar{x}]$, where $\tilde{x} \in \partial \Omega$ is as above.

(4) Let $\bar{x} \in \Omega$ be arbitrary and let $\tilde{y} \in F_U(\bar{x})$. From the claim in (3), we conclude that $U(x) \geq |\bar{x} - y|^\alpha - |x - \tilde{y}|^\alpha$ for all $x \in \Omega$ with equality at $\bar{x}$ and at $\tilde{x} \in \partial \Omega$. Hence, $|x - \tilde{y}|^\alpha \geq |\bar{x} - \tilde{y}|^\alpha$, for all $x \in \partial \Omega$. This implies that $|\bar{x} - \tilde{y}|^\alpha = \text{dist}^\alpha(\tilde{y}, \partial \Omega)$ and so $U(x) \geq \text{dist}^\alpha(\tilde{y}, \partial \Omega) - |x - \tilde{y}|^\alpha$ for all $x \in \Omega$ with equality at $\bar{x}$. Therefore, we get $U(\bar{x}) = \text{dist}^\alpha(\tilde{y}, \partial \Omega) - |\bar{x} - \tilde{y}|^\alpha \leq \text{dist}^\alpha(y, \partial \Omega) - |\bar{x} - y|^\alpha : y \in \Omega$.

The reverse inequality follows by noting that for any $y$ fixed in $\Omega$, the function $v(x) = \text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha$ satisfies $v \leq 0$ on $\partial \Omega$, $v$ $\alpha$-convex, and so $U \geq v$.

(5) Suppose $U(x_0) = \text{dist}^\alpha(y_0, \partial \Omega) - |x_0 - y_0|^\alpha$, and we will show $x_0 \in \Lambda_{y_0}$. Otherwise, there exists $\tilde{\xi}$ with $|\tilde{\xi}| = 1$ such that for all $x \in A_{y_0}$,

$$
(3.2) \quad \langle |x - y_0|^{\alpha - 2}(y_0 - x) - |x_0 - y_0|^{\alpha - 2}(y_0 - x_0), \tilde{\xi} \rangle > 0.
$$

From Lemma 2.7 applied to $\tilde{\xi}$, we know that there exists $\bar{x} \in A_{y_0}$ and $\bar{\delta}_k \to 0$, $\bar{\delta}_k > 0$, and $x_k \in A_{y_0 + \bar{\delta}_k \tilde{\xi}}$ such that $x_k \to \bar{x}$. Using $x = \bar{x}$ in equation (3.2), by definition of $U$ and the fact that $x_k \in A_{y_0 + \bar{\delta}_k \tilde{\xi}}$, we have

$$
(3.3) \quad 0 \geq |x_k - (y_0 + \bar{\delta}_k \tilde{\xi})|^\alpha - |x_0 - (y_0 + \bar{\delta}_k \tilde{\xi})|^\alpha - U(x_0)
\quad = |\bar{\delta}_k \tilde{\xi} - (x_k - y_0)|^\alpha - |\bar{x} - y_0|^\alpha - (|\bar{\delta}_k \tilde{\xi} - (x_k - y_0)|^\alpha - |x_0 - y_0|^\alpha)
\quad = |x_k - y_0|^\alpha - |\bar{x} - y_0|^\alpha - \alpha \bar{\delta}_k \left( |\bar{\delta}_k \tilde{\xi} - (x_k - y_0)|^{\alpha - 2} (\bar{\delta}_k \tilde{\xi} + y_0 - x_k, \tilde{\xi}) - |\bar{\delta}_k \tilde{\xi} - (x_0 - y_0)|^{\alpha - 2} (\bar{\delta}_k \tilde{\xi} + y_0 - x_0, \tilde{\xi}) \right),
$$

by the mean value theorem for some $0 < \bar{\delta}_k, \bar{\delta}_k < \delta_k$. Notice that $\lim_{\delta_k \to 0} \{\ldots\} = \langle |\bar{x} - y_0|^{\alpha - 2}(y_0 - x) - |x_0 - y_0|^{\alpha - 2}(y_0 - x_0), \tilde{\xi} \rangle > 0$ and since $x_k \in \partial \Omega$ and $\bar{x} \in A_{y_0}$ we also have $|x_k - y_0|^\alpha - |\bar{x} - y_0|^\alpha \geq 0$ and hence, we conclude that for $\delta_k$ small enough, 3.3 is positive, which is a contradiction, thus proving the claim. This completes the proof. \hfill \Box

**Remark 3.2.** We analyze in passing the case $\alpha = 2$. In this case, $\Lambda_{y_0} = \{x : \forall \xi, |\xi| = 1, \exists \bar{x} \in A_{y_0} \text{ such that } (x - \bar{x}, \xi) \leq 0\} = \text{convex hull}(A_{y_0})$ and we have the following conclusions:

If $x_0 \in \Omega$ and $U(x_0) = \text{dist}^2(y_0, \partial \Omega) - |x_0 - y_0|^2$, then $\Lambda_{y_0} = \{x \in \partial \Omega : \text{dist}(y_0, \partial \Omega) = |y_0 - x|\}$ is not a singleton. Moreover, if $U(x_0) = \text{dist}^2(y_0, \partial \Omega) - |x_0 - y_0|^2 = |x - y_0|^2 - |x_0 - y_0|^2$, where $\bar{x} \in A_{y_0}$, then $U(x) = |x - y_0|^2 - |x - y_0|^2$ for all $x \in \text{convex hull}(A_{y_0})$. Also, $x_0 \in B_{|y_0 - \bar{x}|/2}$ (3.3) and $|x_0 - \bar{x}|^2 \leq U(x_0)$. This can be realized by taking $\xi = x_0 - y_0$ in the definition of $\Lambda_{y_0}$ and $\bar{x}$ to be the corresponding point in $A_{y_0}$.

Now, let us look at the case $\alpha > 1$ and consider the set $\Lambda_{y_0}$. Set $p_{y_0} = (x - y_0)|x - y_0|^{-\alpha - 2}$. One can check that $\Lambda_{y_0} = p_{y_0}^{-1}(\text{convex hull}(p_{y_0}(A_{y_0})))$. 


Let $\xi = x_0 - y_0$. Then there exists $\bar{x} \in A_{y_0}$ such that $\langle |\bar{x} - y_0|^{\alpha - 2}(y_0 - \bar{x}) - |x_0 - y_0|^{\alpha - 2}(y_0 - x_0), x_0 - y_0 \rangle \leq 0$, which gives $|x_0 - y_0|^\alpha \leq |\bar{x} - y_0|^{\alpha - 2}(\bar{x} - y_0, x_0 - y_0)$. Taking for instance $y_0 = 0$ and $\bar{x}$ along $e_1$, we see that if $x_0 \in A_{y_0}$, then $x_0$ is on the set obtained by rotating the polar curve $r = R(\cos \theta)^{\alpha-2}$ around the $e_1$-axis, where $R = |\bar{x}|$.

3.1. Regularity of $U$. We prove the following theorem.

Theorem 3.3. The function $U$ in Theorem 3.1 is $C^1(\Omega)$.

We first prove a lemma.

Lemma 3.4. Let $u$ be $\alpha$-convex in $\Omega$. Then $u \in C^1(\Omega)$ if and only if $F_u(x)$ is a singleton for each $x \in \Omega$.

Proof. It is clear that if for some $x \in \Omega$, $F_u(x)$ has more than one point, then $u$ is not differentiable at $x$.

To prove the other implication, fix $\bar{x} \in \Omega$, and let $\{\tilde{y}\} = F_u(\bar{x})$. We claim first that if $x_n \to \bar{x}$, and $\{y_n\} = F_u(x_n)$, then $y_n \to \tilde{y}$. If not, then there exists $\epsilon > 0$ and infinitely many points $y_n \not\in B_{\epsilon}(\tilde{y})$. Since the sequence $\{y_n\}$ is bounded, extracting a subsequence we may assume $y_{n_k} \to \tilde{y}$. But then it follows that $\tilde{y} \in F_u(\bar{x})$ and hence by assumption that $\tilde{y} = y$, and this is a contradiction, proving the claim.

Using this claim we show that $u$ has first-order partial derivatives. Without loss of generality, we can assume $u(0) = 0$ and $\{0\} = F_u(0)$, so $u(x) \geq -|x|^\alpha$ for all $x \in \Omega$. Suppose $\left.\frac{\partial u}{\partial y_1}\right|_{y_1 = 0}$ does not exist. For $t > 0$ we have $\frac{u(te_1)}{t} \geq -t^{\alpha-1}$ and hence $\liminf_{t \to 0^+} \frac{u(te_1)}{t} \geq 0$. Suppose $\limsup_{t \to 0^+} \frac{u(te_1)}{t} = a > 0$. Let $t_n \to 0^+$ such that $\frac{u(te_n)}{t_n} \geq a$ and let $\{y_n\} = F_u(te_n)$. It follows by the claim that $y_n \to 0$, and we have $u(x) \geq u(t_n e_1) + |t_n e_1 - y_n|^\alpha - |x - y_n|^\alpha$ for all $n$ and for all $x \in \Omega$. In particular, $0 = u(0) \geq u(t_n e_1) + |t_n e_1 - y_n|^\alpha - |y_n|^\alpha = u(t_n e_1) + |\xi - y_n|^\alpha - |\xi - y_n, t_n e_1|^\alpha$ for some $\xi \in [0, t_n e_1]$. Dividing by $t_n$ we get $0 \geq \frac{u(te_n)}{t_n} + |\xi - y_n|^\alpha - |\xi - y_n, e_1|^\alpha \geq a + |\xi - y_n|^\alpha - |\xi - y_n, e_1|^\alpha \geq a$ for $n$ large enough, a contradiction. Exactly the same argument works for $t < 0$, and hence we conclude that $\frac{\partial u}{\partial y_1}(0)$ exists. By the claim once again we can also conclude that the partial derivatives are continuous because if $y \in F_u(x)$, then $y = x + a(1/\alpha - 1)|Du(x)|^{1/(\alpha - 1)}Du(x)$.

Proof of Theorem 3.3. Let us recall that $\Omega$ is convex and $U(x) = \sup\{\text{dist}^\alpha(y, \partial \Omega) + |x - y|^\alpha : y \in \Omega\}$. Fix $x_0 \in \Omega$. We show that $F_{U}(x_0)$ is a singleton.

Set $t = U(x_0) > 0$ and suppose by contradiction that $y_1, y_2 \in F_U(x_0)$ with $y_1 \neq y_2$. It follows that $\text{dist}^\alpha(y_i, \partial \Omega) - |x_0 - y_i|^\alpha = t$ for $i = 1, 2$. We also have that $B_{\frac{1}{\alpha}(|x_0 - y_i|^\alpha + t)}(y_i) \subseteq \Omega$ for $i = 1, 2$ and $\partial B_1 \cap \partial B_2 \neq \emptyset$.

Let $\Lambda$ be the convex hull of $B_1 \cup B_2$ and let $T$ be a supporting hyperplane to $\Lambda$ that touches $\Lambda$ at more than one point. Set $\Phi(y) = \text{dist}^\alpha(y, T) - |x_0 - y|^\alpha$. We will prove in Lemma 4.2 that the set $S = \{y : \Phi(y) \geq t\}$ is strictly convex. Since $\Phi(y_i) = t$ for $i = 1, 2$ it follows that $[y_1, y_2] \subseteq S$. Then, for $y \in [y_1, y_2]$ we have $\text{dist}^\alpha(y, \partial \Omega) - |x_0 - y|^\alpha \geq \text{dist}^\alpha(y, \Lambda) - |x_0 - y|^\alpha = \text{dist}^\alpha(y, T) - |x_0 - y|^\alpha \geq \Phi(y) > t$, and this is a contradiction with the definition of $U$ since $U(x_0) = t$. ☐
4. Nonhomogeneous Dirichlet problem

Let \( \alpha > 1 \) and let \( U \) be the solution of \( |F_U(\Omega)| = 0, \) \( U = 0 \) on \( \partial \Omega \) from the previous section. We shall prove the following theorem; see [GN07, Lemma 6.19].

**Theorem 4.1.** Let \( x_0 \in \Omega \) and \( t < U(x_0) \) and define

\[
(4.1) \quad u(x) = \sup \{v_{\lambda,y}(x) = \lambda - |x-y|^\alpha : v_{\lambda,y} \leq 0 \text{ on } \partial \Omega \text{ and } v_{\lambda,y}(x_0) \leq t\}.
\]

Then \( u \in C(\Omega), \) \( u = 0 \) on \( \partial \Omega, \) \( u(x_0) = t, \) \( u \) is \( \alpha \)-convex and satisfies the equation \( F_u = \beta \delta_{x_0}, \) for some \( \beta \geq 0. \) Moreover, when \( t \geq 0 \) we have the following estimates for \( \beta: \)

(a) Suppose \( 1 < \alpha \leq 2. \) If \( n(3-2\alpha) + 1 \geq 0, \) then

\[
(4.2) \quad \beta = |F_u(x_0)| \geq C \frac{(U(x_0) - t)^n}{\text{dist}(x_0, \partial \Omega)^n(\alpha-1)};
\]

and if \( n(3-2\alpha) + 1 < 0, \) then

\[
(4.3) \quad \beta = |F_u(x_0)| \geq C \frac{(U(x_0) - t)^n}{\text{dist}(x_0, \partial \Omega)^n(\alpha-1)} \frac{\alpha n + 1}{\alpha n - 2} \frac{\text{diam}(\Omega)^{\alpha n - 1}}{2}.
\]

(b) If on the other hand \( \alpha \geq 2, \) then we have

\[
(4.4) \quad \beta = |F_u(x_0)| \geq C \frac{(U(x_0) - t)^n}{\text{dist}(x_0, \partial \Omega)^n(\alpha-1)} \frac{\alpha n + 1}{\alpha n - 2} \frac{\text{diam}(\Omega)^{\alpha n - 1}}{2}.
\]

Here \( C \) is a positive constant depending only on \( \alpha \) and \( n. \)

Finally, for \( \alpha > 1 \) and \( t \geq 0, \) the set \( F_u(x_0) \) is convex.

**Proof.** The set of functions \( v_{\lambda,y} \) is clearly nonempty, so \( u \) is well defined and also \( u(x_0) \leq t. \) We will prove that \( u(x_0) \leq t. \) Let’s assume first that \( t \geq 0. \) Since \( t < U(x_0), \) there exists \( y_0 \in \Omega \) such that \( t < \text{dist}(y_0, \partial \Omega) - |x_0 - y_0|^\alpha. \) Since the function \( \Psi(z) = \text{dist}^\alpha(y_0, \partial \Omega) - |x_0 - z|^\alpha \) is continuous and \( \Psi(y_0) > t \) and \( \Psi \leq 0 \) on \( \partial \Omega, \) this implies that there exists \( z_0 \in \Omega \) such that \( \Psi(z_0) = t. \) Letting \( v(x) = \text{dist}^\alpha(z_0, \partial \Omega) - |x - z_0|^\alpha, \) then \( v \leq 0 \) on \( \partial \Omega \) and \( v(x_0) = t; \) this implies \( u(x_0) \geq t. \) If \( t < 0, \) then we can take \( u_{x_0} \) in the definition of \( u. \) Therefore \( u(x_0) = t. \)

We prove that \( u \) is \( \alpha \)-convex in \( \Omega. \) Let \( \bar{z} \in \Omega. \) We first claim the supremum is attained; i.e., there exists \( \bar{\lambda} \in \mathbb{R} \) and \( \bar{y} \in \mathbb{R}^n \) such that \( u(\bar{z}) = \bar{\lambda} - |\bar{z} - \bar{y}|^\alpha, \) where \( v_{\bar{\lambda},\bar{y}} \) satisfies \( v_{\bar{\lambda},\bar{y}} \leq 0 \) on \( \partial \Omega \) and \( v_{\bar{\lambda},\bar{y}}(x_0) \leq t. \) Assuming the claim we get that \( u(x) \geq \bar{\lambda} - |x - \bar{y}|^\alpha = u(\bar{z}) + |\bar{z} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha, \) for all \( x \in \Omega, \) which implies that \( \bar{y} \in F_u(\bar{z}); \) that is, \( u \) is \( \alpha \)-convex.

To prove the claim, we have from the definition of \( u \) that \( u(\bar{z}) = \lim_{n \to \infty} \lambda_n - |\bar{z} - y_n|^\alpha. \) If \( t \geq 0, \) then we may assume that \( y_n \in \Omega \) otherwise \( v_{\lambda_n, y_n} \leq 0 \) in \( \partial \Omega \) while \( u \geq 0 \) in \( \Omega. \) We may also assume that \( y_n \to \bar{y} \in \Omega \) and hence also \( \lambda_n \to \bar{\lambda} \) and the claim is then proved. If \( t < 0, \) suppose by contradiction that \( |\bar{z} - y_n| \to +\infty, \) in which case also \( \lambda_n \to +\infty. \) This implies that for \( n \) large, \( y_n \notin \Omega. \) Let \( x_n \in \partial \Omega \) be such that \( \text{dist}(y_n, \partial \Omega) = |x_n - y_n|. \) Set \( v_n(x) = \lambda_n - |x - y_n|^\alpha. \)
Then \( v_n \leq 0 \) on \( \partial \Omega \) and hence \( \lambda_n \leq \text{dist}^\alpha(y_n, \partial \Omega) = |x_n - y_n|^\alpha \), which implies that \( v_n(x) \leq |x_n - y_n|^\alpha - |x - y|^\alpha \), so

\[
v_n(\bar{x}) \leq |x_n - y_n|^\alpha - |\bar{x} - y|^\alpha = \alpha|\bar{x}_n - y_n|^\alpha - (\bar{x}_n - y_n, x_n - \bar{x}) \quad \text{for some } \bar{x}_n \in [\bar{x}, x_n],
\]

\[
= \alpha|\bar{x}_n - y_n|^\alpha - |x_n - \bar{x}| \cos \theta_n,
\]

where \( \theta_n = \angle(y_n - \bar{x}_n, x_n - \bar{x}_n) = \angle(y_n - \bar{x}_n, \bar{x} - \bar{x}_n) = \angle(y_n - x_n, \bar{x} - \bar{x}_n) \geq \frac{\pi}{2} + \delta(\epsilon) \) for some \( \delta(\epsilon) > 0 \), where \( B_{\epsilon}(\bar{x}) \subseteq \Omega \) (here \( \angle(x, y) \) denotes the angle between the vectors \( x \) and \( y \)). Hence \( \cos \theta_n \leq -C_{\epsilon} \). Then \( |x_n - y_n|^\alpha - |\bar{x} - y|^\alpha \leq -\alpha|\bar{x}_n - y_n|^\alpha - 1 \epsilon C_{\epsilon} \leq -\alpha|\bar{x}_n - y_n|^\alpha - 1 \epsilon C_{\epsilon} \rightarrow -\infty \) as \( n \rightarrow \infty \). This is a contradiction, since \( v_n(\bar{x}) \rightarrow u(\bar{x}) \).

Next we show that \( u \in \mathcal{C}(\bar{\Omega}) \). First, notice that by definition, \( u \leq U \) in \( \Omega \). If \( t \geq 0 \), then \( u \geq 0 \) and we are done. Suppose \( t < 0 \). Fix \( \bar{x} \in \partial \Omega \), and let \( y \) be the unit outer normal to \( \partial \Omega \) at \( \bar{x} \). For \( s > 0 \), let \( v_s(x) = |\bar{x} - (\bar{x} + \eta s)|^\alpha - |x - (\bar{x} + \eta s)|^\alpha \).

We have \( v_s \leq 0 \) on \( \partial \Omega \), and exactly as above we see that \( v_s(x_0) \rightarrow -\infty \) as \( s \rightarrow +\infty \). Hence, for \( s \) large enough, \( v_s \) is an admissible function and hence \( u \geq v_s \) in \( \Omega \). Since \( v(\bar{x}) = 0 \), we get that \( u \in \mathcal{C}(\bar{\Omega}) \) and \( u = 0 \) on \( \partial \Omega \).

We now show that \( F_u = \beta \delta_{x_0} \) for some \( \beta = \beta(x_0, t) \). Indeed, let \( \bar{x} \in \Omega \), \( \bar{x} \neq x_0 \), and let \( y \in F_u(\bar{x}) \); we claim that \( y \notin F_u(\bar{x}) \) for some \( \bar{x} = \bar{x} \). Assuming the claim, we get from (2.2) that \( |F_u(E)| = 0 \) for each \( E \) with \( x_0 \notin \Omega \), and so \( F_u \) is concentrated at \( x_0 \).

To prove the claim, since \( y \in F_u(\bar{x}) \), \( u(x) \geq u(\bar{x}) + |\bar{x} - y|^\alpha - |x - y|^\alpha \), for all \( x \in \Omega \). Let \( v(x) = u(x) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha = \lambda - |x - \bar{y}|^\alpha \). Since \( u \leq 0 \) on \( \partial \Omega \), \( v \leq 0 \) on \( \partial \Omega \) and \( v(x_0) \leq u(x_0) = t \). If \( v(x_0) = t \), then \( y \in F_u(x_0) \); and if \( v(x_0) < t \), then, as before, there exists \( \bar{x} \in \partial \Omega \) such that \( v(\bar{x}) = 0 \) and hence \( y \in F_u(\bar{x}) \).

Before estimating \( \beta \), we need the following characterization of \( u \) (it will be easier to work with \( \bar{u} \) rather than with \( u \)). If \( \bar{u}(x) = \sup \{ \text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha \} \), then \( u = \bar{u} \). Notice that \( \bar{u} \leq u \). To show that \( \bar{u} \geq u \), let \( \bar{x} \in \Omega \) and \( \bar{y} \in F_u(\bar{x}) \), so \( u(\bar{x}) \geq u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha \) for all \( x \in \Omega \), and we claim that there exists \( \bar{x} \in \partial \Omega \) such that \( u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha = 0 \). Let’s assume this claim holds. We have \( \geq u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha \) for all \( x \in \partial \Omega \) and \( |\bar{x} - \bar{y}|^\alpha = u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha \), which implies that \( |x - \bar{y}| \geq |\bar{x} - \bar{y}| \) for all \( x \in \partial \Omega \), and hence \( \text{dist}(\bar{y}, \partial \Omega) = |\bar{x} - \bar{y}| \). This implies that \( u(\bar{x}) = \text{dist}^\alpha(\bar{y}, \partial \Omega) - |\bar{x} - \bar{y}|^\alpha \leq \bar{u}(\bar{x}) \) and we are done. We now prove the claim. If the claim is not true, there exists \( \epsilon > 0 \) such that \( u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + \epsilon < 0 \) for all \( x \in \partial \Omega \). By continuity, there exists \( \delta > 0 \) such that \( \text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha - \epsilon < 0 \) for all \( x \in \partial \Omega \). If \( \text{dist}^\alpha(y, \partial \Omega) - |x - y|^\alpha < t \), then \( y \notin F_u(\bar{x}) \), and let \( v(x) = u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha + \epsilon \). Then \( M > 1 \) is large enough such that \( 0 < \frac{M - (u(\bar{x}) + |\bar{x} - \bar{y}|^\alpha - |x - \bar{y}|^\alpha)}{M} < \frac{1}{2} \). It follows that \( v \leq 0 \) on \( \partial \Omega \) and \( v(x_0) \leq t \), and this implies that \( v \leq u \in \Omega \), but \( v(\bar{x}) > u(\bar{x}) \), and this contradiction proves the claim.

We next prove that

\[
F_u(x_0) = \{ y \in \mathbb{R}^\alpha : \text{dist}^\alpha(y, \partial \Omega) - |x_0 - y|^\alpha \geq t \} := F_t(x_0).
\]
It will be easier to work with $F_u(x_0)$ written in this way. For, if $y \in F_u(x_0)$, then $u(x) \geq u(x_0) + |x_0 - y|^{\alpha} - |x - y|^{\alpha}$ for all $x \in \Omega$, which implies that $|x - y|^{\alpha} \geq u(x_0) + |x_0 - y|^{\alpha}$ for all $x \in \partial \Omega$, and hence, $\text{dist}^{\alpha}(y, \partial \Omega) \geq u(x_0) + |x_0 - y|^{\alpha} = t + |x_0 - y|^{\alpha}$.

On the other hand, if $\text{dist}^{\alpha}(y, \partial \Omega) - |x_0 - y|^{\alpha} \geq t$, then $|x - y|^{\alpha} \geq |x_0 - y|^{\alpha} + t$ for all $x \in \partial \Omega$, and hence $0 \geq |x_0 - y|^{\alpha} + t - |x - y|^{\alpha} = |x_0 - y|^{\alpha} + u(x_0) - |x - y|^{\alpha}$ for all $x \in \partial \Omega$. Set $v(x) = |x_0 - y|^{\alpha} + u(x_0) - |x - y|^{\alpha}$. Then $v \leq 0$ on $\partial \Omega$, $v(x_0) = t$, and hence $u \geq v$ in $\Omega$, which implies $y \in F_u(x_0)$.

The remainder of the proof is devoted to establishing (1.2)-(1.4).

Recall that $u(x_0) = t < U(x_0)$, and say $U(x_0) = \text{dist}^{\alpha}(y_0, \partial \Omega) - |x_0 - y_0|^{\alpha}$ for some $y_0 \in \Omega$. Without loss of generality we will assume from now on that $y_0 = 0$, $\text{dist}(y_0, \partial \Omega) = R$ and $x_0 = |x_0|e_1$ with $|x_0| < R$.

Since $B_R(0) \subseteq \Omega$ we have that $|F_l(x_0)| \geq |\{y \in B_R(0) : \text{dist}^{\alpha}(y, \partial \Omega) - |y - x_0|^{\alpha} \geq t\}| \geq |\{y \in B_R(0) : \text{dist}^{\alpha}(y, \partial B_R(0)) - |y - x_0|^{\alpha} \geq t\}| = |\{y \in B_R(0) : (R - |y|)^{\alpha} - |y - x_0|^{\alpha} \geq t\}|$. We shall estimate the measure of the last set.

**Estimation of $\beta$ when $\alpha = 2$.**

First notice that the set $\{y \in B_R(0) : (R - |y|)^2 - |y - x_0|^2 \geq t\}$ is the ellipsoid

$$\left\{ y : \frac{(|y_1| - \delta)^2}{a^2} + \frac{\sum_{i=2}^{n}|y_i|^2}{b^2} \leq 1 \right\}$$

with

$$\delta = \frac{(R^2 - |x_0|^2 - t)|x_0|}{2(R^2 - |x_0|^2)}, \quad a = \frac{(R^2 - |x_0|^2 - t)R}{2(R^2 - |x_0|^2)}, \quad b = \frac{(R^2 - |x_0|^2 - t)}{2(R^2 - |x_0|^2)^{\frac{n-1}{n}}}.$$

The volume of this ellipsoid equals

$$C_n \frac{(R^2 - |x_0|^2 - t)^n R}{(R^2 - |x_0|^2)^{\frac{n-1}{n}}}.$$

We also notice that the set $\{y \in B_R(0) : (R - |y|)^2 - |y - x_0|^2 \geq t\}$ equals the set

$$\left\{ \rho \xi : \xi \in S^{n-1}, 0 \leq \rho \leq \frac{R^2 - |x_0|^2 - t}{2(R - \langle x_0, \xi \rangle)} \right\},$$

and using polar coordinates we get that its volume is equal to $C_n(R^2 - |x_0|^2)^{\frac{n}{2}} \int_{S^{n-1}} \frac{1}{(R - \langle x_0, \xi \rangle)^{\frac{n}{2}}} d\xi$. This implies that

$$\int_{S^{n-1}} \frac{1}{(R - \langle x_0, \xi \rangle)^{\frac{n}{2}}} d\xi = C_n \frac{R^n}{(R^2 - |x_0|^2)^{\frac{n}{2}}}$$

which also implies for $\alpha > 1$ that

$$\int_{S^{n-1}} \frac{1}{(|x_0|^{-\alpha} - \langle |x_0|^{-\alpha} - x_0, \xi \rangle)^{\frac{n}{2}}} d\xi = \frac{R^{\alpha - 1}}{(R^{2(\alpha - 1)} - |x_0|^{2(\alpha - 1)})^{\frac{n}{2}}},$$

for $|x_0| < R$.

**Estimation of $\beta$ for $\alpha > 1$.**

Let $\xi \in S^{n-1}$ and consider $\phi(s) = (R - s)^{\alpha} - |s\xi - x_0|^{\alpha}$ for $0 \leq s \leq \frac{R^2 - |x_0|^2}{2(R - \langle x_0, \xi \rangle)}$. We claim that $\phi$ is decreasing, concave for $1 < \alpha \leq 2$ and convex for $\alpha \geq 2$. First notice that for $0 \leq s \leq \frac{R^2 - |x_0|^2}{2(R - \langle x_0, \xi \rangle)}$ we have $|s\xi - x_0| \leq (R - s)$. Next, we compute $\phi'(s) = \alpha(R - s)^{\alpha - 1} - |s\xi - x_0|^{\alpha - 2}(s\xi - x_0) \leq \alpha(R - s)^{\alpha - 1} + \alpha |s\xi - x_0|^{\alpha - 1} < 0$ and $\phi''(s) = (\alpha - 1)(R - s)^{\alpha - 2} - |s\xi - x_0|^{\alpha - 2} \left(1 - (2 - \alpha) \frac{|s\xi - x_0|}{|s\xi - x_0|} \right)$. Hence if $1 < \alpha < 2$, then $\phi''(s) < 0$; and if $\alpha > 2$, then $\phi''(s) > 0$. Therefore we have, for
Proof. From the calculation when $\alpha \leq 2$, that $\phi(s) \geq R^\alpha - |x_0|^\alpha - s\frac{2(R - (x_0, \xi))(R^\alpha - |x_0|^\alpha)}{R^2 - |x_0|^2}$, and for $\alpha \geq 2$ that $\phi(s) \geq R^\alpha - |x_0|^\alpha - s(\alpha R^{\alpha - 1} - \alpha|x_0|^{\alpha - 2}(x_0, \xi))$. Given $0 \leq t < R^\alpha - |x_0|^\alpha$, we set in case $1 < \alpha \leq 2$, $R^\alpha - |x_0|^\alpha - s\frac{2(R - (x_0, \xi))(R^\alpha - |x_0|^\alpha)}{R^2 - |x_0|^2} = t$ and solve for $s$ to get $s(\xi) = \frac{(R^\alpha - |x_0|^\alpha - t)(R^2 - |x_0|^2)}{2(R^\alpha - |x_0|^\alpha)(R - (x_0, \xi))}$. For $\alpha \geq 2$, we set $R^\alpha - |x_0|^\alpha - s(\alpha R^{\alpha - 1} - \alpha|x_0|^{\alpha - 2}(x_0, \xi)) = t$ and we solve for $s$ to get $s(\xi) = \frac{R^\alpha - |x_0|^\alpha - t}{\alpha R^{\alpha - 1} - \alpha|x_0|^{\alpha - 2}(x_0, \xi)}$. Hence in any case we have that $\{y \in B_R(0) : (R - |y|)^\alpha - |y - x_0|^\alpha \geq t\}$ contains the set $\{\rho \xi : \xi \in S^{n-1}, 0 \leq \rho \leq s(\xi)\}$. Using polar coordinates we find that the volume of the last set is $\frac{1}{n} \int_{S^{n-1}} s(\xi)^n d\xi$. If $1 < \alpha \leq 2$, then
\[
|\{\rho \xi : \xi \in S^{n-1}, 0 \leq \rho \leq s(\xi)\}| = C_n (R^\alpha - |x_0|^\alpha - t)n \int_{S^{n-1}} \frac{1}{(R - (x_0, \xi))^{n+1}} d\xi \leq \frac{(R^\alpha - |x_0|^\alpha - t)^n R^{\alpha - 1}}{(R^{2(\alpha - 1)} - |x_0|^{2(\alpha - 1)})^{\frac{n+1}{n}}} , \text{ from (4.6)}. \]
From the mean value theorem, $R^\alpha - |x_0|^\alpha \leq \alpha R^{\alpha - 1}(R - |x_0|)$, for $1 < \alpha \leq 2$, and for $\alpha \geq 2$ we have $R^{2(\alpha - 1)} - |x_0|^{2(\alpha - 1)} \leq 2(\alpha - 1)R^{2\alpha - 3}(R - |x_0|)$, and using also that $R \leq R + |x_0| \leq 2R$, we get the following estimates: if $1 < \alpha \leq 2$, then $|F_t(x_0)| \geq C \frac{R^{2(\alpha - 2(\alpha - 1))} (R^\alpha - |x_0|^\alpha - t)^n}{(R - |x_0|)^{\frac{4n+1}{2n}}}$. Finally, noticing that $R - |x_0| \leq \text{dist}(x_0, \partial \Omega), R \leq \text{diam}(\Omega)$ and $U(x_0) = R^\alpha - |x_0|^\alpha$, the estimates in the theorem follow.

To complete the proof it remains to show that for $t \geq 0$ and $\alpha > 1$ the set $F_t(x_0)$ is convex. We remark that this fact is not used in the estimation of $|F_t(x_0)|$. To show that $F_t(x_0)$ is convex we need the following lemma.

Lemma 4.2. Let $C = \{(x', z) \in \mathbb{R}^n : z \in \mathbb{R}; z^\alpha - (x^2 + (z - z_0)^2)^{\frac{\alpha}{2}} \geq t\}$, where $r = |x'|, z_0 > 0$. Then, the set $C$ is convex for $t \geq 0$.

Proof. We will show that the function $r(z) = \left((z^\alpha - t)^{\frac{2}{\alpha}} - (z - z_0)^2\right)^{\frac{1}{2}}$ is a concave function on the set $z^\alpha - |z - z_0|^\alpha \geq t$. We have $rr' = (z^\alpha - t)^{\frac{2(\alpha - 1)}{\alpha} z^\alpha - 1} - (z - z_0)$ and
\[
(r')^2 + rr'' = \frac{z^\alpha - (\alpha - 1)t - (z^\alpha - t)^{\frac{2(\alpha - 1)}{\alpha} z^\alpha - 1} - (z - z_0)}{(z^\alpha - t)^{\frac{2(\alpha - 1)}{\alpha} z^\alpha - 1} z^2}. 
\]
*If $t > 0$, then $C$ is strictly convex.*
Hence, to show that \( r'' < 0 \) for \( r > 0 \), we must show that
\[
\frac{z^\alpha - (\alpha - 1)t - (z^\alpha - t)\frac{2(\alpha - 1)}{\alpha} z^{\alpha - 1}}{(z^\alpha - t)\frac{2(\alpha - 1)}{\alpha} z^{\alpha - 2}} \leq (r')^2 = \left(\frac{(z^\alpha - t)\frac{2(\alpha - 1)}{\alpha} z^{\alpha - 1} - (z - z_0)^2}{(z^\alpha - t)\frac{2(\alpha - 1)}{\alpha} z^{\alpha - 2} - (z - z_0)^2}\right),
\]
which holds if and only if
\[
0 \leq (\alpha - 1)t((z^\alpha - t)\frac{2(\alpha - 1)}{\alpha} - (z - z_0)^2) + z^\alpha((z - z_0) - (z^\alpha - t)z^{1-\alpha})^2.
\]
This inequality is obviously true for \( t \geq 0 \). □

For \( \bar{x} \in \partial \Omega \), let \( T_{\bar{x}} \) be a supporting hyperplane to \( \Omega \) at \( \bar{x} \) and let \( P_{\bar{x},t} = \{ y \in \Omega : \text{dist}^\alpha(y, T_{\bar{x}}) - |y - x_0|^\alpha \geq t \} \), which by the previous lemma is a convex set. Notice that \( F_t(x_0) = \{ y \in \Omega : \text{dist}^\alpha(y, \partial \Omega) - |y - x_0|^\alpha \geq t \} = \bigcap_{x \in \partial \Omega} P_{x,t} \), and hence it is a convex set. This completes the proof of Theorem 4.1 □

We now consider the case when \( t < 0 \) in Theorem 4.1

**Lemma 4.3.** For \( 1 < \alpha \leq \frac{2n}{n-1} \) there exists \( \delta > 1 \) depending only on \( \alpha \) such that if \( -t \geq \delta \text{dist}(x_0, \partial \Omega)^\alpha \) then
\[
|F_t(x_0) \cap \Omega^c| \geq C \frac{(-t)^{\frac{n-1}{2(n-1)}}}{\text{dist}(x_0, \partial \Omega)^{\frac{n(2-\alpha)+\alpha}{2(n-1)}}},
\]
where \( C \) depends only on \( \alpha \) and \( n \).

**Proof.** Write \( x = (x', x_n) \) and assume \( 0 \in \partial \Omega, \Omega \subseteq \{ x : x_n \leq 0 \} \) and \( x_0 = (0, -\epsilon) \in \Omega \) with \( \text{dist}(x_0, \partial \Omega) = \epsilon \). Assume that \( -t \geq \delta \epsilon^\alpha \), where \( \delta > 1 \) will be chosen momentarily. For \( y \in \mathbb{R}^n \) with \( y_n \geq 0 \), we have \( \text{dist}^\alpha(y, \partial \Omega) - |y - x_0|^\alpha \geq y_n^\alpha - (|y'|^2 + (y_n + \epsilon)^2)^\frac{\alpha}{2} \). Hence \( H := \{ y : y_n^\alpha - (|y'|^2 + (y_n + \epsilon)^2)^\frac{\alpha}{2} \geq t \} \) and \( y_n \geq 0 \} \subseteq F_t(x_0) \). Let \( \tilde{y} > 0 \) satisfy the equation \( y_n^\alpha - t = (y + \epsilon)^\alpha \). Then by slicing, the volume of \( H \) equals
\[
V = C_n \int_0^{\tilde{y}} ((y_n^\alpha - t)^{\frac{n-1}{2}} - (y + \epsilon)^2)^{\frac{n-1}{2}} dy.
\]
Let \( \phi(y) = (y_n^\alpha - t)^{\frac{n}{2}} - (y + \epsilon)^\alpha \) and \( \phi_1 \) be convex, where \( \phi_1(y) = (y_n^\alpha - t)^{\frac{n}{2}} - (y + \epsilon) \) and \( \phi_2(y) = (y_n^\alpha - t)^{\frac{n}{2}} + (y + \epsilon) \). Since \( \phi_1 \) and \( \phi_2 \) are convex, we have \( \phi_1(y) \geq \phi'_2(\tilde{y})(y - \tilde{y}) \) and \( \phi_2(y) \geq \phi_1(0) + \phi'_1(0)y \). Hence \( \phi_1(y) \phi_2(y) \geq p(y) \), with \( p \) a concave parabola.

Set \( p(y) = \max\{p(y) : y \in [0, \tilde{y}]\} = h \). Then, \( p(y) \geq \frac{h}{y} \), for \( y \in [0, \tilde{y}] \) and
\[
p(y) \geq \frac{h}{y} - (y - \tilde{y}), \text{ for } y \in [\tilde{y}, \tilde{y}] \). Therefore, we get
\[
V \geq \int_0^{\tilde{y}} (p(y))^{\frac{n-1}{2}} dy \geq \frac{h}{^\frac{n-1}{2}} \tilde{y}.
\]
We estimate \( h \) and \( \tilde{y} \) from below. Notice that \( -t = (\tilde{y} + \epsilon)^\alpha - \tilde{y}^\alpha = \alpha \xi^{\alpha - 1} \epsilon \) for some \( y < \xi < y + \epsilon \), and hence \( -t \leq \alpha(y + \epsilon)^{\alpha - 1} \epsilon \), which implies that \( \tilde{y} \geq (\frac{t}{\alpha \epsilon})^{\frac{1}{\alpha-1}} - \epsilon \).

Choosing \( \delta = \alpha^{2\alpha-1} \), we obtain \( \tilde{y} \geq \frac{1}{2} \left(\frac{t}{\alpha \epsilon}\right)^{\frac{1}{\alpha-1}} \). It follows that \( \frac{\tilde{y}}{y + \epsilon} \geq \frac{1}{2} \) and
U is the hyperplane passing through $\bar{y}$ whenever $\alpha > 1$ and $\Omega$ be an open, bounded, convex domain in $\mathbb{R}^n$. Assume $u \in C(\Omega)$, $u = 0$ on $\partial \Omega$, and $0 \leq u(x_0) \leq U(x_0)$ for some $x_0 \in \Omega$. Then we have

\begin{equation}
(U(x_0) - u(x_0))^n \leq C(\text{dist}(x_0, \partial \Omega))^\frac{n+1}{n} \text{diam}(\Omega)^\frac{n(2\alpha-3)-1}{n} |F_u(\Omega)|
\end{equation}

whenever $n(2\alpha-3) - 1 \geq 0$ and

\begin{equation}
(U(x_0) - u(x_0))^n \leq C(\text{dist}(x_0, \partial \Omega))^{n(\alpha-1)} |F_u(\Omega)|
\end{equation}

whenever $n(2\alpha-3) - 1 \leq 0$. The constant $C$ depends only on $n$ and $\alpha$.

Proof. Suppose $u(x_0) < U(x_0)$. Let $v(x) = \sup \{\lambda - |x - y|^\alpha : v_{\lambda,y}(x_0) \leq u(x_0)\}$ and $v_{\lambda,y} \leq 0$ on $\partial \Omega$. We claim $F_v(x_0) \subseteq F_u(\Omega)$. Let $y \in F_v(x_0)$, so

\begin{equation}
\text{dist}(y, \partial \Omega) = \frac{U(x_0) - u(x_0)}{\alpha |x_0 - y|^{1/\alpha}}\text{diam}(\Omega)^{\frac{1}{\alpha}} |F_v(\Omega)|.
\end{equation}

Since $v_{\lambda,y}(x_0) \leq u(x_0)$, we have $\alpha |x_0 - y|^{1/\alpha} \geq 1$ and

\begin{equation}
\text{dist}(y, \partial \Omega) \leq \text{diam}(\Omega)^{-1} |F_v(\Omega)|,
\end{equation}

which proves the lemma. \hfill \square
\[ v(x) \geq v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha \] for all \( x \in \Omega \). Consider
\[
\sup_{\Omega} (v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha - u(x)),
\]
and let \( \hat{x} \in \hat{\Omega} \) be the point where the supremum is attained. From Theorem 4.1 \( u(x_0) = v(x_0) \). We have \( v(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha \leq 0 \) for all \( x \in \partial \Omega \). If \( \hat{x} \in \partial \Omega \), then \( u(x) \geq u(x_0) + |x_0 - y|^\alpha - |x - y|^\alpha \) for all \( x \in \hat{\Omega} \) and so \( y \in F_u(x_0) \). If on the other hand \( \hat{x} \in \Omega \), then \( u(x) \geq u(\hat{x}) + |\hat{x} - y|^\alpha - |x - y|^\alpha \) for all \( x \in \hat{\Omega} \), and hence \( y \in F_u(\hat{x}) \). Consequently the claim is proved. From Theorem 4.1 applied to \( v \) we then obtain the result. \( \square \)

**Theorem 5.2.** Let \( 1 < \alpha \leq \frac{2n}{n-1} \). Let \( \Omega \) be an open, bounded, convex domain in \( \mathbb{R}^n \). Assume \( u \in C(\Omega) \) with \( u = 0 \) on \( \partial \Omega \). Let \( x \in \Omega \) such that \( u(x) < 0 \). If
\[
|u(x)| \leq \alpha 2^{\alpha-1} \text{dist}^\alpha(x, \partial \Omega),
\]
then
\[
|u(x)| \leq C_{\alpha,n} \text{dist}^\alpha(x, \partial \Omega) |F_u(\Omega)|.
\]
If on the other hand, \( |u(x)| \) does not hold, then
\[
|u(x)| \leq C_{\alpha,n} \text{dist}^{\alpha\frac{n+1}{2(n-1)}}(x, \partial \Omega) |F_u(\Omega)|.
\]

**Proof.** If (5.3) holds and since we always have \( B_{\text{dist}(x, \partial \Omega)}(x) \subseteq F_u(\Omega) \), then (5.4) follows. If on the other hand, \( |u(x)| > \alpha 2^{\alpha-1} \text{dist}^\alpha(x, \partial \Omega) \), then we are under the hypothesis of Lemma 4.3 and the proof of (5.5) follows in the same way as in the previous theorem. \( \square \)

**Remark 5.3.** If \( \alpha > \frac{2n}{n-1} \) and \( u(x) < 0 \), then \( u(x) \) cannot be estimated by any positive power of \( \text{dist}(x, \partial \Omega) \) and consequently neither can \( U(x) - u(x) \).

Considering the cylinder \( \Omega = \{ (x', x_n) : |x'| < 2, |x_n| < 1 \} \), we set \( x = (x', x_n) \). Let \( x_k = (0, 1 - \frac{1}{k}) \). Notice that if \( |x'| \geq 2 \), then \( \text{dist}^\alpha(x, \partial \Omega) - |x - x_k|^\alpha \leq -2^{\alpha} \); this follows since for \( \alpha > 2 \) the function \( h(y) = y^{\alpha} - (a^2 + (y + b)^2)^{\frac{\alpha}{2}} \) is decreasing for \( y > 0 \), for any fixed positive \( a \) and \( b \). Hence, if \( x \notin \Omega \) and \( \text{dist}^\alpha(x, \partial \Omega) - |x - x_k|^\alpha \geq -1 \), then \( |x'| < 2 \), which implies that
\[
F^{-1}(x_k) \cap \Omega^c \subseteq \left\{ x : |x'| < 2, x_n > 1, (x_n - 1)^\alpha - \left( |x'|^2 + \left( x_n - 1 + \frac{1}{k} \right)^2 \right)^{\alpha/2} > -1 \right\},
\]
where \( F^{-1}(x_k) \) is defined in (15), and the set on the right hand side is basically the set \( H \) defined in the proof of Lemma 4.3. We can now estimate from above the measure of this set in the same way as in Lemma 4.3. If \( y > 0 \), \( t < 0 \), then we have
\[
(y^\alpha - t)^{\frac{\alpha}{2}} - y^2 = \frac{\alpha}{2} t^{\frac{\alpha-2}{2}} \text{ for some } \xi \text{ with } y^\alpha < \xi < y^\alpha - t. \] We let \( t = -1 \) and...
\( \epsilon = 1/k \), and with the notation in the proof of Lemma 4.3 we then get that
\[
|F_{-1}(x_k) \cap \Omega^c| \leq \int_0^{\hat{y}_k} ((y^\alpha + 1)^{\frac{1}{\alpha}} - (y + 1/k)^{2}) \, dy
\]
\[
\leq \int_0^{\infty} ((y^\alpha + 1)^{\frac{1}{\alpha}} - y^2) \, dy = \int_0^{1} + \int_0^{\infty}
\]
\[
\leq C_1 + \int_1^{\infty} y^{\left(\frac{2}{(2-\alpha)(n-1)}\right)} \, dy \leq C_{n,\alpha}
\]
for \( \alpha > 2n/(n-1) \). Let \( u_k \) be \( \alpha \)-convex such that \( u_k = 0 \) on \( \partial \Omega \), \( u_k(x_k) = -1 \), whose existence follows from the first part of Theorem 4.1. Then \( |F_{u_k}(\Omega)| \leq C \) for all \( k \) while \( |u_k(x_k)| = 1 \) and \( \text{dist}(x_k, \partial \Omega) = \frac{1}{k} \).

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