PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 6, June 2010, Pages 2113–2123
S 0002-9939(10)10270-6
Article electronically published on February 9, 2010

TOEPLITZ-COMPOSITION C^*-ALGEBRAS FOR CERTAIN FINITE BLASCHKE PRODUCTS

HIROYASU HAMADA AND YASUO WATATANI

(Communicated by Marius Junge)

ABSTRACT. Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. Then there exists a relation between the associated composition operator C_R on the Hardy space and the C^*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system $(R^{\circ n})_n$ on the Julia set J_R. We study the C^*-algebra $\mathcal{TC}_R$ generated by both the composition operator C_R and the Toeplitz operator T_z to show that the quotient algebra by the ideal of the compact operators is isomorphic to the C^*-algebra $\mathcal{O}_R(J_R)$, which is simple and purely infinite.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H^2(\mathbb{D})$ be the Hardy (Hilbert) space of analytic functions whose power series have square-summable coefficients. For an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is defined by $C_\varphi(g) = g \circ \varphi$ for $g \in H^2(\mathbb{D})$ and is known to be a bounded operator by the Littlewood subordination theorem [14]. The study of composition operators on the Hardy space $H^2(\mathbb{D})$ gives a fruitful interplay between complex analysis and operator theory as shown, for example, in the books of Shapiro [29], Cowen and MacCluer [4] and Martínez-Avendaño and Rosenthal [15]. Since the work by Cowen [2], good representations of adjoints of composition operators have been investigated. Consult Cowen and Gallardo-Gutiérrez [3], Martín and Vukotié [17], Hammond, Moorhouse and Robbins [8], and Bourdon and Shapiro [1] to see recent achievements in adjoints of composition operators with rational symbols. In this paper, we only need an old result by McDonald in [18] for finite Blaschke products.

On the other hand, for a branched covering $\pi : M \to M$, Deaconu and Muhly [6] introduced a C^*-algebra $C^*(M, \pi)$ as the C^*-algebra of the r-discrete groupoid constructed by Renault [27]. In particular, they study rational functions R on the Riemann sphere $\hat{\mathbb{C}}$. Iterations $(R^{\circ n})_n$ of R by composition give complex dynamical systems. In [11] Kajiwara and the second-named author introduced slightly

Received by the editors October 23, 2008, and, in revised form, October 8, 2009.
2010 *Mathematics Subject Classification.* Primary 46L55, 47B33; Secondary 37F10, 46L08.
Key words and phrases. Composition operator, Blaschke product, Toeplitz operator, $C*$-algebra, complex dynamical system.

different C^*-algebras $\mathcal{O}_R(\hat{\mathbb{C}})$, $\mathcal{O}_R(J_R)$ and $\mathcal{O}_R(F_R)$, associated with the complex dynamical system $(R^{\circ n})_n$ on the Riemann sphere $\hat{\mathbb{C}}$, the Julia set J_R and the Fatou set F_R of R. The C^*-algebra $\mathcal{O}_R(J_R)$ is defined as a Cuntz-Pimsner algebra [26] of a Hilbert bimodule, called a C^*-correspondence, $C(\operatorname{graph} R|_{J_R})$ over $C(J_R)$. We regard the algebra $\mathcal{O}_R(J_R)$ as a certain analog of the crossed product $C(\Lambda_\Gamma) \rtimes \Gamma$ of $C(\Lambda_\Gamma)$ by a boundary action of a Kleinian group Γ on the limit set Λ_Γ.

The aim of this paper is to show that there exists a relation between composition operators on the Hardy space and the C^*-algebras $\mathcal{O}_R(J_R)$ associated with the complex dynamical systems $(R^{\circ n})_n$ on the Julia sets J_R.

We recall that the C^*-algebra $\mathcal{T}$ generated by the Toeplitz operator T_z contains all continuous symbol Toeplitz operators, and its quotient by the ideal of the compact operators on $H^2(\mathbb{T})$ is isomorphic to the commutative C^*-algebra $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$. For an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, we denote by $\mathcal{TC}_\varphi$ the Toeplitz-compostion C^*-algebra generated by both the composition operator C_φ and the Toeplitz operator T_z. Its quotient algebra by the ideal $\mathcal{K}$ of the compact operators is denoted by $\mathcal{OC}_\varphi$. Recently Kriete, MacCluer and Moorhouse [12, 13] studied the Toeplitz-compostion C^*-algebra $\mathcal{TC}_\varphi$ for a certain linear fractional self-map φ. They describe the quotient C^*-algebra $\mathcal{OC}_\varphi$ concretely as a subalgebra of $C(\Lambda) \otimes M_2(\mathbb{C})$ for a compact space Λ. If $\varphi(z) = e^{-2\pi i\theta} z$ for some irrational number θ, then the Toeplitz-composition C^*-algebra $\mathcal{TC}_\varphi$ is an extension of the irrational rotation algebra A_θ by $\mathcal{K}$ and studied by Park [25]. Jury [9, 10] investigated the C^*-algebra generated by a group of composition operators with the symbols belonging to a non-elementary Fuchsian group Γ to relate it with extensions of the crossed product $C(\mathbb{T}) \rtimes \Gamma$ by $\mathcal{K}$.

In this paper we study the class of finite Blaschke products R of degree $n \geq 2$ with $R(0) = 0$. The boundary $\mathbb{T}$ of the open unit disk $\mathbb{D}$ is the Julia set J_R of the Blaschke product R. We show that the quotient algebra $\mathcal{OC}_R$ of the Toelitz-composion C^*-algebra $\mathcal{TC}_R$ by the ideal $\mathcal{K}$ is isomorphic to the C^*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system $(R^{\circ n})_n$, which is simple and purely infinite. We should remark that the notion of transfer operator considered by Exel in [7] is one of the keys to clarifying the above relation. In fact the corresponding operator of the composition operator in the quotient algebra is the implementing isometry operator.

The Toeplitz-composition C^*-algebra depends on the analytic structure of the Hardy space by construction. The finite Blaschke product R is not conjugate with z^n by any Möbius automorphism unless $R(z) = \lambda z^n$. But we can show that the quotient algebra $\mathcal{OC}_R$ is isomorphic to $\mathcal{OC}_{z^n}$ as a corollary of our main theorem. This enables us to compute $K_0(\mathcal{OC}_R)$ and $K_1(\mathcal{OC}_R)$ easily.

2. Toeplitz-composition C^*-algebras

Let $L^2(\mathbb{T})$ denote the square integrable measurable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure. The Hardy space $H^2(\mathbb{T})$ is the closed subspace of $L^2(\mathbb{T})$ consisting of the functions whose negative Fourier coefficients vanish. We put $H^\infty(\mathbb{T}) := H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$.

The Hardy space $H^2(\mathbb{D})$ is the Hilbert space consisting of all analytic functions $g(z) = \sum_{k=0}^\infty c_k z^k$ on the open unit disk $\mathbb{D}$ such that $\sum_{k=0}^\infty |c_k|^2 < \infty$. The inner product is given by

$$(g|h) = \sum_{k=0}^{\infty} c_k \overline{d_k}$$

for $g(z) = \sum_{k=0}^{\infty} c_k z^k$ and $h(z) = \sum_{k=0}^{\infty} d_k z^k$.

We identify $H^2(\mathbb{D})$ with $H^2(\mathbb{T})$ by a unitary $U : H^2(\mathbb{D}) \to H^2(\mathbb{T})$. We note that $\widetilde{g} = Ug$ is given as

$$\widetilde{g}(e^{i\theta}) := \lim_{r \to 1-} g(re^{i\theta}) \quad \text{a.e. } \theta$$

for $g \in H^2(\mathbb{D})$ by Fatou's theorem. Moreover the inverse $\breve{f} = U^* f$ is given as a Poisson integral

$$\breve{f}(re^{i\theta}) := \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(e^{it}) dt$$

for $f \in H^2(\mathbb{T})$, where P_r is the Poisson kernel defined by

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \quad 0 \le r < 1,\ 0 \le \theta \le 2\pi.$$

Let $P_{H^2} : L^2(\mathbb{T}) \to H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ be the projection. For $a \in L^\infty(\mathbb{T})$, the Toeplitz operator T_a on $H^2(\mathbb{T})$ is defined by $T_a f = P_{H^2} a f$ for $f \in H^2(\mathbb{T})$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map. Then the composition operator C_φ on $H^2(\mathbb{D})$ is defined by $C_\varphi g = g \circ \varphi$ for $g \in H^2(\mathbb{D})$. By the Littlewood subordination theorem, C_φ is always bounded.

We can regard Toeplitz operators and composition operators as acting on the same Hilbert space by the unitary U above. More precisely, we put $\breve{T_a} = U^* T_a U$ and $\widetilde{C_\varphi} = U C_\varphi U^*$. If φ is an inner function, then we know that $\widetilde{g \circ \varphi} = \widetilde{g} \circ \widetilde{\varphi}$ for $g \in H^2(\mathbb{D})$ by Ryff [28, Theorem 2]. Therefore we may write $\widetilde{C_\varphi} f = f \circ \widetilde{\varphi}$ for $f \in H^2(\mathbb{T})$.

Definition. For an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, we denote by $\mathcal{TC}_\varphi$ the C^*-algebra generated by the Toeplitz operator $\breve{T_z}$ and the composition operator C_φ on $H^2(\mathbb{D})$. The C^*-algebra $\mathcal{TC}_\varphi$ is called the Toeplitz-composition C^*-algebra with symbol φ. Since $\mathcal{TC}_\varphi$ contains the ideal $K(H^2(\mathbb{D}))$ of compact operators, we define a C^*-algebra $\mathcal{OC}_\varphi$ to be the quotient C^*-algebra $\mathcal{TC}_\varphi / K(H^2(\mathbb{D}))$. By the unitary $U : H^2(\mathbb{D}) \to H^2(\mathbb{T})$ above, we usually identify $\mathcal{TC}_\varphi$ with the C^*-algebra generated by T_z and $\widetilde{C_\varphi}$. We also use the same notation $\mathcal{TC}_\varphi$ and $\mathcal{OC}_\varphi$ for the operators on $H^2(\mathbb{T})$. But we sometimes need to treat them carefully. Therefore we often use the notation $\widetilde{g} = Ug$ and $\breve{f} = U^* f$ to avoid confusion in the paper. Wise readers may neglect this troublesome notation.

3. C^*-ALGEBRAS ASSOCIATED WITH COMPLEX DYNAMICAL SYSTEMS

We recall the construction of Cuntz-Pimsner algebras [26]. Let A be a C^*-algebra and X be a Hilbert right A-module. We denote by $L(X)$ the algebra of the adjointable bounded operators on X. For $\xi, \eta \in X$, the operator $\theta_{\xi,\eta}$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)_A$ for $\zeta \in X$. The closure of the linear span of these operators is denoted by $K(X)$. We say that X is a Hilbert C^*-bimodule (or C^*-correspondence) over A if X is a Hilbert right A-module with a *-homomorphism $\phi : A \to L(X)$. We always assume that X is full and ϕ is injective. Let $F(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$ be the full Fock module of X with a convention $X^{\otimes 0} = A$. For $\xi \in X$, the creation operator $T_\xi \in L(F(X))$ is defined by

$$T_\xi(a) = \xi a \quad \text{and} \quad T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

We define $i_{F(X)} : A \to L(F(X))$ by

$$i_{F(X)}(a)(b) = ab \quad \text{and} \quad i_{F(X)}(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a, b \in A$. The Cuntz-Toeplitz algebra $\mathcal{T}_X$ is the C*-algebra acting on $F(X)$ generated by $i_{F(X)}(a)$ with $a \in A$ and T_ξ with $\xi \in X$.

Let $j_K : K(X) \to \mathcal{T}_X$ be the homomorphism defined by $j_K(\theta_{\xi,\eta}) = T_\xi T_\eta^*$. We consider the ideal $I_X := \phi^{-1}(K(X))$ of A. Let $\mathcal{J}_X$ be the ideal of $\mathcal{T}_X$ generated by $\{i_{F(X)}(a) - (j_K \circ \phi)(a)\,;\, a \in I_X\}$. Then the Cuntz-Pimsner algebra $\mathcal{O}_X$ is defined as the quotient $\mathcal{T}_X/\mathcal{J}_X$. Let $\pi : \mathcal{T}_X \to \mathcal{O}_X$ be the quotient map. We set $S_\xi = \pi(T_\xi)$ and $i(a) = \pi(i_{F(X)}(a))$. Let $i_K : K(X) \to \mathcal{O}_X$ be the homomorphism defined by $i_K(\theta_{\xi,\eta}) = S_\xi S_\eta^*$. Then $\pi((j_K \circ \phi)(a)) = (i_K \circ \phi)(a)$ for $a \in I_X$.

The Cuntz-Pimsner algebra $\mathcal{O}_X$ is the universal C*-algebra generated by $i(a)$ with $a \in A$ and S_ξ with $\xi \in X$, satisfying the fact that $i(a)S_\xi = S_{\phi(a)\xi}$, $S_\xi i(a) = S_{\xi a}$, $S_\xi^* S_\eta = i((\xi|\eta)_A)$ for $a \in A$, $\xi, \eta \in X$ and $i(a) = (i_K \circ \phi)(a)$ for $a \in I_X$. We usually identify $i(a)$ with a in A. If A is unital and X has a finite basis $\{u_i\}_{i=1}^n$ in the sense that $\xi = \sum_{i=1}^n u_i(u_i|\xi)_A$, then the last condition can be replaced by the fact that there exists a finite set $\{v_j\}_{j=1}^m \subset X$ such that $\sum_{j=1}^m S_{v_j} S_{v_j}^* = I$. Then $\{v_j\}_{j=1}^m$ becomes another finite basis of X automatically.

Next we introduce the C^*-algebras associated with complex dynamical systems as in [11]. Let R be a rational function of degree at least two. The sequence $(R^{\circ n})_n$ of iterations of composition by R gives a complex dynamical system on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Fatou set F_R of R is the maximal open subset of $\hat{\mathbb{C}}$ on which $(R^{\circ n})_n$ is equicontinuous (or a normal family) and the Julia set J_R of R is the complement of the Fatou set in $\hat{\mathbb{C}}$. We denote by $e(z_0)$ the branch index of R at z_0. Let $A = C(\hat{\mathbb{C}})$ and $X = C(\operatorname{graph} R)$ be the set of continuous functions on $\hat{\mathbb{C}}$ and $\operatorname{graph} R$ respectively, where $\operatorname{graph} R = \{(x, y) \in \hat{\mathbb{C}}^2\,;\, y = R(x)\}$ is the graph of R. Then X is an A-A bimodule by

$$(a \cdot \xi \cdot b)(x, y) = a(x)\xi(x, y)b(y), \quad a, b \in A,\ \xi \in X.$$

We define an A-valued inner product $(\ |\ )_A$ on X by

$$(\xi|\eta)_A(y) = \sum_{x \in R^{-1}(y)} e(x)\overline{\xi(x, y)}\eta(x, y), \quad \xi, \eta \in X,\ y \in \hat{\mathbb{C}}.$$

Thanks to the branch index $e(x)$, the inner product above gives a continuous function and X is a full Hilbert bimodule over A without completion. The left action of A is unital and faithful.

Since the Julia set J_R is completely invariant under R, i.e., $R(J_R) = J_R = R^{-1}(J_R)$, we can consider the restriction $R|_{J_R} : J_R \to J_R$, which will often be denoted by the same letter R. Let $\operatorname{graph} R|_{J_R} = \{(x, y) \in J_R \times J_R\,;\, y = R(x)\}$ be the graph of the restriction map $R|_{J_R}$ and $X_R = C(\operatorname{graph} R|_{J_R})$. In the same way as above, X_R is a full Hilbert bimodule over $C(J_R)$.

Definition. The C^*-algebra $\mathcal{O}_R(\hat{\mathbb{C}})$ on $\hat{\mathbb{C}}$ is defined as the Cuntz-Pimsner algebra of the Hilbert bimodule $X = C(\operatorname{graph} R)$ over $A = C(\hat{\mathbb{C}})$. We also define the C^*-algebra $\mathcal{O}_R(J_R)$ on the Julia set J_R as the Cuntz-Pimsner algebra of the Hilbert bimodule $X_R = C(\operatorname{graph} R|_{J_R})$ over $A = C(J_R)$.

4. Transfer operators and adjoints of composition operators

In the rest of this paper, we assume that R is a finite Blaschke product of degree at least two with $R(0) = 0$; that is,

$$R(z) = \lambda z \prod_{k=1}^{n-1} \frac{z - z_k}{1 - \overline{z_k} z} = \lambda \prod_{k=0}^{n-1} \frac{z - z_k}{1 - \overline{z_k} z}, \quad z \in \hat{\mathbb{C}},$$

where $n \geq 2,\ z_1, \ldots, z_{n-1} \in \mathbb{D},\ |\lambda| = 1$ and $z_0 = 0$. Thus R is a rational function with degree $\deg R = n$. Since R is an inner function, R is an analytic self-map on $\mathbb{D}$. We consider the composition operator C_R with symbol R. Since R is inner and $R(0) = 0$, C_R is an isometry by Nordgren [24]. We note the following fact:

$$\text{(1)} \qquad \frac{zR'(z)}{R(z)} = 1 + \sum_{k=1}^{n-1} \frac{1 - |z_k|^2}{|z - z_k|^2} > 0, \quad z \in \mathbb{T}.$$

Thus R has no branched points on $\mathbb{T}$ and the branch index $e(z) = 1$ for any $z \in \mathbb{T}$. Furthermore the Julia set J_R of R is $\mathbb{T}$ (see, for example, [19, pages 70-71]), and it coincides with the boundary of the disk $\mathbb{D}$.

Let $A = C(\mathbb{T})$ and $h \in A$ be a positive invertible element. Set $\operatorname{graph} R|_{\mathbb{T}} = \{(z, w) \in \mathbb{T}^2\,;\, w = R(z)\}$ and $X_{R,h} = C(\operatorname{graph} R|_{\mathbb{T}})$. We define

$$(a \cdot \xi \cdot b)(z, w) = a(z)\xi(z, w)b(w),$$
$$(\xi|\eta)_{A,h}(w) = \sum_{z \in R^{-1}(w)} h(z)\overline{\xi(z, w)}\eta(z, w)$$

for $a, b \in A$ and $\xi, \eta \in X_{R,h}$. We see that $X_{R,h}$ is a pre-Hilbert A-A bimodule whose left action is faithful. Since the Julia set $J_R = \mathbb{T}$ has no branched point, for the constant function $h = 1$, $X_{R,1}$ has a finite basis and coincides with the Hilbert A-A bimodule $X_R = C(\operatorname{graph} R|_{\mathbb{T}})$. Let $\{u_k\}_{k=1}^N$ be a basis of X_R. For a positive invertible element $h \in A$, put $v_k = h^{-1/2} u_k$. Then $\{v_k\}_{k=1}^N$ is a basis of $X_{R,h}$, and $X_{R,h}$ is also a Hilbert module without completion.

Definition. The C^*-algebra $\mathcal{O}_{R,h}(\mathbb{T})$ is defined as the Cuntz-Pimsner algebra of the Hilbert bimodule $X_{R,h} = C(\operatorname{graph} R|_{\mathbb{T}})$ over $A = C(\mathbb{T})$. The C^*-algebra $\mathcal{O}_{R,h}(\mathbb{T})$ is the universal C^*-algebra generated by $\{\hat{S}_\xi\ ;\ \xi \in X_{R,h}\}$ and A satisfying the following relations:

$$a\hat{S}_\xi = \hat{S}_{a \cdot \xi},\ \hat{S}_\xi b = \hat{S}_{\xi \cdot b},\ \hat{S}_\xi^* \hat{S}_\eta = (\xi|\eta)_{A,h},\ \sum_{k=1}^N \hat{S}_{v_k} \hat{S}_{v_k}^* = I$$

for $a, b \in A$ and $\xi, \eta \in X_{R,h}$. The C^*-algebra $\mathcal{O}_{R,h}(\mathbb{T})$ is in fact a topological quiver algebra in the sense of Muhly and Tomforde [21]. See also Muhly and Solel [20, Example 5.4].

We will use the symbol a and S_ξ to denote the generator of $\mathcal{O}_R(J_R)$ for $a \in A$ and $\xi \in X_R$.

In the rest of this paper, we choose and fix $h \in A$ defined by

$$h(z) = \frac{nR(z)}{zR'(z)} = \frac{n}{|R'(z)|}, \quad z \in \mathbb{T}.$$

Then h is positive and invertible by (1).

We need and collect several facts as lemmas to prove our main theorem. Some of them might be considered folklore.

Lemma 1. *Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. Then the C^*-algebra $\mathcal{O}_{R,h}(\mathbb{T})$ is isomorphic to the C^*-algebra $\mathcal{O}_R(J_R)$ by an isomorphism Φ such that $\Phi(a) = a$ and $\Phi(\hat{S}_\xi) = h^{1/2}S_\xi$ for $a \in A$, $\xi \in X_{R,h}$.*

Proof. Let $V_\xi = h^{1/2}S_\xi$ for $\xi \in X_{R,h}$. Then we have that

$$aV_\xi = V_{a\cdot\xi},\ V_\xi b = V_{\xi\cdot b},\quad V_\xi^* V_\eta = (h^{1/2}\cdot\xi|h^{1/2}\cdot\eta)_A = (\xi|\eta)_{A,h},\quad \sum_{k=1}^N V_{v_k}V_{v_k}^* = I$$

for $a, b \in A$ and $\xi, \eta \in X_{R,h}$. By universality, we have the desired isomorphism. □

Definition. For a function f on $\mathbb{T}$, we define a function $\mathcal{L}_R(f)$ on $\mathbb{T}$ by

$$\mathcal{L}_R(f)(w) = \frac{1}{n}\sum_{z\in R^{-1}(w)} h(z)f(z) = \sum_{z\in R^{-1}(w)} \frac{R(z)}{zR'(z)}f(z), \qquad w \in \mathbb{T}.$$

We mainly consider the restrictions of $\mathcal{L}_R$ to $A = C(\mathbb{T})$ and $H^2(\mathbb{T})$ and use the same notation if no confusion can arise.

The notion of transfer operator by Exel in [7] is one of the keys to clarifying our situation.

Lemma 2. *Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. Let $A = C(\mathbb{T})$ and $\alpha : A \to A$ be the unital *-endomorphism defined by $(\alpha(a))(z) = a(R(z))$ for $a \in A$ and $z \in \mathbb{T}$. Then the restriction of $\mathcal{L}_R$ to $A = C(\mathbb{T})$ is a transfer operator for the pair (A, α) in the sense of Exel; that is, $\mathcal{L}_R$ is a positive linear map such that*

$$\mathcal{L}_R(\alpha(a)b) = a\mathcal{L}_R(b) \text{ for } a, b \in A.$$

Moreover $\mathcal{L}_R$ satisfies the fact that $\mathcal{L}_R(1) = 1$.

Proof. The only non-trivial statement is to show that $\mathcal{L}_R(1) = 1$. But this is an easy calculation as follows: We fix $w \in \mathbb{T}$ and let $\alpha_1, \dots, \alpha_n$ be exactly different solutions of $R(z) = w$. By the partial fraction decomposition, there exists $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that

$$\frac{R(z)/z}{R(z) - w} = \sum_{k=1}^n \frac{\lambda_k}{z - \alpha_k}.$$

Multiplying by z and letting $z \to \infty$, we have $\sum_{k=1}^n \lambda_k = 1$. Moreover multiplying by $z - \alpha_l$ and letting $z \to \alpha_l$, we get $\lambda_l = \frac{R(\alpha_l)}{\alpha_l R'(\alpha_l)}$. This shows that $\mathcal{L}_R(1) = 1$. See, for example, [5]. □

Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. J. N. McDonald [18] calculated the adjoint of C_R and gave a formula. We follow some of his argument, but we also need a different formula to prove our main theorem. We claim that there exist $\theta_0 \in [0, 2\pi]$ and a strictly increasing continuously differentiable function $\psi : [\theta_0 - 2\pi\,,\,\theta_0] \to \mathbb{R}$ such that $\psi(\theta_0 - 2\pi) = 0$, $\psi(\theta_0) = 2n\pi$ and $R(e^{i\theta}) = e^{i\psi(\theta)}$.

Lemma 3. *Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. Then $\mathcal{L}_R(H^2(\mathbb{T})) \subset H^2(\mathbb{T})$ and the restriction $\mathcal{L}_R : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a bounded operator such that $\widetilde{C_R}^* = \mathcal{L}_R$.*

Proof. Let ψ be the map defined above. Set $t = \psi(\theta)$ and $\sigma_k(t) = \psi^{-1}(t+2(k-1)\pi)$ for $1 \le k \le n$ and $0 \le t \le 2\pi$. If we differentiate $R(e^{i\sigma_k(t)}) = e^{it}$ with respect to t, then

$$\sigma_k'(t) = \frac{R(e^{i\sigma_k(t)})}{R'(e^{i\sigma_k(t)})e^{i\sigma_k(t)}}.$$

Therefore for $\xi_l(z) := z^l$ we have that

$$\begin{aligned}(\widetilde{C_R}\xi_l|f) &= \frac{1}{2\pi}\int_0^{2\pi} R(e^{i\theta})^l\overline{f(e^{i\theta})}d\theta = \frac{1}{2\pi}\int_{\theta_0-2\pi}^{\theta_0} R(e^{i\theta})^l\overline{f(e^{i\theta})}d\theta \\ &= \frac{1}{2\pi}\int_0^{2n\pi} e^{ilt}\overline{f(e^{i\psi^{-1}(t)})(\psi^{-1}(t))'}dt = \frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi} e^{ilt}\overline{f(e^{i\sigma_k(t)})\sigma_k'(t)}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{ilt}\overline{\mathcal{L}_R(f)(e^{it})}dt = (\xi_l|\mathcal{L}_R(f))\end{aligned}$$

for $f \in H^2(\mathbb{T})$ and $l \ge 0$. Thus $\widetilde{C_R}^* = \mathcal{L}_R$ and $\mathcal{L}_R$ is bounded. □

We need an appropriate basis of $H^2(\mathbb{T})$ for R. Let

$$e_l(z) = \begin{cases} \dfrac{\sqrt{1-|\beta_0|^2}}{1-\overline{\beta_0}z}, & l = 0, \\ \dfrac{\alpha_l\sqrt{1-|\beta_l|^2}}{1-\overline{\beta_l}z}\displaystyle\prod_{k=0}^{l-1}\frac{z-\beta_k}{1-\overline{\beta_k}z}, & l \ge 1, \end{cases}$$

where $\alpha_{kn+l} = \lambda^k$, $\beta_{kn+l} = z_l$ for $k \ge 0$ and $0 \le l \le n-1$. Since $\sum_{k=0}^{\infty}(1-|\beta_k|) = \infty$, $\{e_k\}_{k=0}^{\infty}$ is an orthonormal basis of $H^2(\mathbb{T})$ as in Ninness, Hjalmarsson and Gustafsson [23, Theorem 2.1], and Ninness and Gustafsson [22, Theorem 1]. We write

$$Q_l(z) = \frac{\sqrt{1-|z_l|^2}}{1-\overline{z_l}z} \quad \text{and} \quad R_l(z) = \begin{cases} 1, & l = 0, \\ \displaystyle\prod_{k=0}^{l-1}\frac{z-z_k}{1-\overline{z_k}z}, & l \ge 1. \end{cases}$$

Thus $e_{kn+l} = Q_lR_lR^k$ for $k \ge 0$ and $0 \le l \le n-1$, where R^k is the k-th power of R with respect to pointwise multiplication.

Proposition 4. *Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. Then for any $a \in C(\mathbb{T})$, we have*

$$\widetilde{C_R}^* T_a \widetilde{C_R} = T_{\mathcal{L}_R(a)}.$$

Proof. We first examine the case that $a(z) = z^j$ for $j \ge 0$. Consider the L^2-expansion of a by the basis $\{e_l\}_{l=0}^{\infty}$:

$$a = \sum_{l=0}^{\infty} c_lR^l + g,$$

where $g \in (\operatorname{Im} \widetilde{C_R})^\perp \cap H^2(\mathbb{T})$. For $\xi_m(z) = z^m$ with $m \geq 0$, we have

$$(T_a \widetilde{C_R} \xi_m)(z) = a(z)R(z)^m = \sum_{l=0}^{\infty} c_l R(z)^{l+m} + g(z)R(z)^m.$$

It is clear that

$$\operatorname{Im} \widetilde{C_R} = \overline{\operatorname{span}}\{e_{kn}\,;\, k \geq 0\}$$

and

$$(\operatorname{Im} \widetilde{C_R})^\perp \cap H^2(\mathbb{T}) = \overline{\operatorname{span}}\{e_{kn+l}\,;\, k \geq 0,\, 1 \leq l \leq n-1\},$$

where $\overline{\operatorname{span}}$ means the closure of a linear span. Therefore gR^m is also in $(\operatorname{Im} \widetilde{C_R})^\perp \cap H^2(\mathbb{T})$. Since $\widetilde{C_R}$ is an isometry, we have that

$$(\widetilde{C_R}^* T_a \widetilde{C_R} \xi_m)(z) = \sum_{l=0}^{\infty} c_l z^{l+m}.$$

On the other hand, $\widetilde{C_R}^* = \mathcal{L}_R$ by Lemma 3; hence

$$\mathcal{L}_R(a) = \sum_{l=0}^{\infty} c_l \mathcal{L}_R(R^l).$$

By Lemma 2,

$$\mathcal{L}_R(R^l)(w) = \frac{1}{n} \sum_{z \in R^{-1}(w)} h(z)R(z)^l = (\mathcal{L}_R(1)(w))w^l = w^l.$$

Thus

$$\mathcal{L}_R(a)(w) = \sum_{l=0}^{\infty} c_l w^l \quad \text{as an } L^2\text{-convergence.}$$

Since $\mathcal{L}_R(a) \in H^\infty(\mathbb{T})$, we have

$$(T_{\mathcal{L}_R(a)} \xi_m)(w) = (\mathcal{L}_R(a)\xi_m)(w) = (M_{\xi_m} \mathcal{L}_R(a))(w) = \sum_{l=0}^{\infty} c_l w^{l+m}$$

for $m \geq 0$, where M_{ξ_m} is a multiplication operator by ξ_m. Therefore we obtain that $\widetilde{C_R}^* T_a \widetilde{C_R} = T_{\mathcal{L}_R(a)}$.

For the remaining case where $j \leq 0$, the formula $\widetilde{C_R}^* T_a \widetilde{C_R} = T_{\mathcal{L}_R(a)}$ also holds because $\mathcal{L}_R$ is positive and $T_a^* = T_{\overline{a}}$. □

Lemma 5. *Let the notation be as above. Then*

$$\sum_{k=1}^{n} T_{Q_{k-1}R_{k-1}} \widetilde{C_R} \widetilde{C_R}^* (T_{Q_{k-1}R_{k-1}})^* = I.$$

Proof. We have

$$T_{Q_{k-1}R_{k-1}} \widetilde{C_R} \xi_l = Q_{k-1}R_{k-1}R^l = e_{ln+(k-1)}$$

for $1 \leq k \leq n$ and $l \geq 0$. Thus $T_{Q_{k-1}R_{k-1}} \widetilde{C_R}$ is an isometry, and the desired equality follows. □

We can now state our main theorem.

Theorem 6. *Let R be a finite Blaschke product of degree at least two and let $R(0) = 0$; that is,*

$$R(z) = \lambda z \prod_{k=1}^{n-1} \frac{z - z_k}{1 - \overline{z_k} z},$$

where $n \geq 2$, $|\lambda| = 1$ and $|z_k| < 1$. Then the quotient C^*-algebra $\mathcal{OC}_R$ of the Toeplitz-composition C^*-algebra $\mathcal{TC}_R$ by the ideal of compact operators is isomorphic to the C^*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system on the Julia set J_R by an isomorphism Ψ such that $\Psi(\pi(T_a)) = a$ for $a \in C(\mathbb{T})$ and $\Psi(\pi(\widetilde{C_R})) = \sqrt{\frac{R}{zR'}} S_1$, where π is the canonical quotient map $\mathcal{TC}_R$ to $\mathcal{OC}_R$ and 1 is the constant map in X_R taking constant value 1. Moreover the C^*-algebra $\mathcal{OC}_R$ is simple and purely infinite.

Proof. For any $\xi \in X_{R,h}$ and $a \in A$, let $p(z) = \xi(z, R(z))$, $\rho(a) = \pi(T_a)$, $V_\xi = n^{1/2} \rho(p) \pi(\widetilde{C_R})$. We have

$$\rho(a) V_\xi = n^{1/2} \rho(ap) \pi(\widetilde{C_R}) = V_{a \cdot \xi}.$$

Since, for $b \in A$

$$\pi(\widetilde{C_R}) \rho(b) = \pi(\widetilde{C_R} T_b) = \pi(T_{b \circ R} \widetilde{C_R}) = \rho(b \circ R) \pi(\widetilde{C_R}),$$

we have that

$$\begin{aligned} V_\xi \rho(b) &= n^{1/2} \rho(p) \pi(\widetilde{C_R}) \rho(b) = n^{1/2} \rho(p) \rho(b \circ R) \pi(\widetilde{C_R}) \\ &= n^{1/2} \rho(p(b \circ R)) \pi(\widetilde{C_R}) = V_{\xi \cdot b}. \end{aligned}$$

For any $\eta \in X_{R,h}$, define $q(z) = \eta(z, R(z))$. By Proposition 4,

$$\begin{aligned} V_\xi^* V_\eta &= n \pi(\widetilde{C_R}^*) \rho(\overline{p}) \rho(q) \pi(\widetilde{C_R}) = n \pi(\widetilde{C_R}^* T_{\overline{p} q} \widetilde{C_R}) \\ &= n \pi(T_{\mathcal{L}_R(\overline{p} q)}) = \rho(n \mathcal{L}_R(\overline{p} q)) = \rho((\xi | \eta)_{A,h}). \end{aligned}$$

Set $v_k(z, R(z)) = n^{-1/2} Q_{k-1}(z) R_{k-1}(z)$ for $1 \leq k \leq n$. By Lemma 5,

$$\sum_{k=1}^{n} V_{v_k} V_{v_k}^* = \sum_{k=1}^{n} \pi(T_{Q_{k-1} R_{k-1}} \widetilde{C_R} \widetilde{C_R}^* (T_{Q_{k-1} R_{k-1}})^*) = I.$$

By the universality and the simplicity of $\mathcal{O}_{R,h}(\mathbb{T})$, there exists an isomorphism $\Omega : \mathcal{O}_{R,h}(\mathbb{T}) \to \mathcal{OC}_R$ such that $\Omega(\hat{S}_\xi) = V_\xi$ and $\Omega(a) = \rho(a)$ for $\xi \in X_{R,h}$, $a \in A$. Let Φ be the map in Lemma 1 and put $\Psi = \Phi \circ \Omega^{-1}$. Then Ψ is the desired isomorphism. Since it is proved in [11] that the C^*-algebra $\mathcal{O}_R(J_R)$ is simple and purely infinite, so is the C^*-algebra $\mathcal{OC}_R$. The rest is now clear. $\square$

Remark. It is important to notice that the element $\pi(\widetilde{C_R})$ of the composition operator in the quotient algebra corresponds exactly to the implementing isometry operator in Exel's crossed product $A \rtimes_{\alpha, \mathcal{L}_R} \mathbb{N}$ in [7], which depends on the transfer opearator $\mathcal{L}_R$. It follows directly from the fact that $\mathcal{O}_{R,h}(\mathbb{T})$ is naturally isomorphic to $A \rtimes_{\alpha, \mathcal{L}_R} \mathbb{N}$.

Example. Let $R(z) = z^n$ for $n \geq 2$. Then the Hilbert bimodule X_R over $A = C(\mathbb{T})$ is isomorphic to A^n as a right A-module. In fact, let $u_i(z, w) = \frac{1}{\sqrt{n}} z^{i-1}$ for $i = 1, \ldots, n$. Then $(u_i | u_j)_A = \delta_{i,j} I$ and $\{u_1, \ldots, u_n\}$ is a basis of X_R. Hence $S_i := S_{u_i}$ ($i = 1, ..., n$) are generators of the Cuntz algebra $\mathcal{O}_n$. We see that $(z \cdot u_i)(z, R(z)) = u_{i+1}(z, R(z))$ for $i = 1, ..., n-1$ and $(z \cdot u_n)(z, R(z)) = z^n = (u_1 \cdot z)(z, R(z))$

and that the left multiplication by z is a unitary U. Therefore the C^*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system on the Julia set J_R is the universal C^*-algebra generated by a unitary U and n isometries $S_1, ..., S_n$ satisfying $S_1S_1^* + \cdots + S_nS_n^* = I$, $US_i = S_{i+1}$ for $i = 1, ..., n-1$ and $US_n = S_1U$. In this way, the operator S_1 corresponds to the element $\pi(\widetilde{C_R}) \in \mathcal{OC}_R$ of the composition operator $\widetilde{C_R}$ and the unitary U corresponds to the element $\pi(T_z) \in \mathcal{OC}_R$. Moreover we find that the commutation relation $U^nS_1 = S_1U$ in the C^*-algebra $\mathcal{O}_R(J_R)$ and the commutation relation $T_{R(z)}\widetilde{C_R} = \widetilde{C_R}T_z$ in the Toeplitz-composition C^*-algebra $\mathcal{TC}_R$ are essentially the same.

The Toeplitz-composition C^*-algebra depends on the analytic structure of the Hardy space by the construction. The finite Blaschke product R is not conjugate with z^n by any Möbius automorphism unless $R(z) = \lambda z^n$. But we can show that the quotient algebra $\mathcal{OC}_R$ is isomorphic to $\mathcal{OC}_{z^n}$ as a corollary of our main theorem.

Corollary 7. *Let R be a finite Blaschke product of degree $n \geq 2$ with $R(0) = 0$. Then the quotient C^*-algebra $\mathcal{OC}_R$ is isomorphic to the C^*-algebra $\mathcal{OC}_{z^n}$. Moreover $K_0(\mathcal{OC}_R) \simeq \mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(\mathcal{OC}_R) \simeq \mathbb{Z}$.*

Proof. Let ψ be the strictly increasing continously differentiable function defined in the paragraph before Lemma 3. Since

$$\psi'(\theta) = \frac{e^{i\theta}R'(e^{i\theta})}{R(e^{i\theta})} = 1 + \sum_{k=1}^{n-1} \frac{1-|z_k|^2}{|e^{i\theta} - z_k|^2} > 1,$$

$R|_{\mathbb{T}}$ is an expanding map of degree n and $R|_{\mathbb{T}}$ is topologically conjugate to z^n on $\mathbb{T}$ by [30]. See a general condition given in Martin [16]. Therefore the statement follows from the above theorem and the results in [11]. $\square$

References

[1] P. S. Bourdon and J. H. Shapiro, *Adjoints of rationally induced composition operators*, J. Funct. Anal., **255** (2008), no. 8, 1995–2012. MR2462584 (2009m:47056)

[2] C. C. Cowen, *Linear fractional composition operators H^2*, Integral Eqns. Operator Theory, **11** (1988), 151-160. MR928479 (89b:47044)

[3] C. C. Cowen and E. A. Gallardo-Gutiérrez, *A new class of operators and a description of adjoints of composition operators*, J. Funct. Anal., **238** (2006), 447-462. MR2253727 (2007e:47033)

[4] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397026 (97i:47056)

[5] U. Daepp, P. Gorkin and R. Mortini, *Ellipses and finite Blaschke products*, Amer. Math. Monthly, **109** (2002), 785-795. MR1933701 (2003h:30044)

[6] V. Deaconu and P. S. Muhly, *C^*-algebras associated with branched coverings*, Proc. Amer. Math. Soc., **129** (2001), 1077-1086. MR1814145 (2002a:46075)

[7] R. Exel, *A new look at the crossed-product of a C^*-algebra by an endomorphism*, Ergodic Theory Dynam. Systems, **23** (2003), 1733-1750. MR2032486 (2004k:46119)

[8] C. Hammond, J. Moorhouse and M. E. Robbins, *Adjoints of composition operators with rational symbol*, J. Math. Anal. Appl., **341** (2008), 626-639. MR2394110 (2009b:47035)

[9] M. T. Jury, *The Fredholm index for elements of Toeplitz-composition C^*-algebras*, Integral Equations Operator Theory, **58** (2007), 341-362. MR2320853 (2008d:47049)

[10] M. T. Jury, *C^*-algebras generated by groups of composition operators*, Indiana Univ. Math. J., **56** (2007), 3171-3192. MR2375714

[11] T. Kajiwara and Y. Watatani, *C^*-algebras associated with complex dynamical systems*, Indiana Math. J., **54** (2005), 755-778. MR2151233 (2006b:46074)

[12] T. L. Kriete, B. D. MacCluer and J. L. Moorhouse, *Composition operators within singly generated composition C^*-algebras*, arXiv:math.FA/0610077.
[13] T. L. Kriete, B. D. MacCluer and J. L. Moorhouse, *Toeplitz-composition C^*-algebras*, J. Operator Theory, **58** (2007), 135-156. MR2336048 (2008f:47035)
[14] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc., **23** (1925), 481-519.
[15] R. A. Martínez-Avendaño and P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Space*, Graduate Texts in Mathematics, 237, Springer, New York, 2007. MR2270722 (2007k:47051)
[16] N. F. G. Martin, *On finite Blaschke products whose restrictions to the unit circle are exact endomorphisms*, Bull. London Math. Soc., **15** (1983), 343-348. MR703758 (84h:30045)
[17] M. J. Martín and D. Vukotié, *Adjoints of composition operators on Hilbert spaces of analytic functions*, J. Func. Anal., **238** (2006), 298-312. MR2253017 (2007e:47035)
[18] J. N. McDonald, *Adjoints of a class of composition operators*, Proc. Amer. Math. Soc., **131** (2003), 601-606. MR1933352 (2003h:47046)
[19] J. Milnor, *Dynamics in One Complex Variable, Introductory Lectures*, Friedr. Vieweg & Shon, Braunschweig, 1999. MR1721240 (2002i:37057)
[20] P. S. Muhly and B. Solel, *On the Morita equivalence of tensor algebras*, Proc. London Math. Soc., **81** (2000), 113-168. MR1757049 (2001g:46128)
[21] P. S. Muhly and M. Tomforde, *Topological quivers*, Internat. J. Math., **16** (2005), 693-755. MR2158956 (2006i:46099)
[22] B. Ninness and F. Gustafsson, *A unifying construction of orthonomal bases for system identification*, IEEE Trans. Automat. Control, **42** (1997), 515-521. MR1442586 (98b:93020)
[23] B. Ninness, H. Hjalmarsson and F. Gustafsson, *Generalized Fourier and Toeplitz results for rational orthonormal bases*, SIAM J. Control Optim., **37** (1999), 429-460. MR1655861 (99i:42052)
[24] E. A. Nordgren, *Composition operators*, Canad. J. Math., **20** (1968), 442-449. MR0223914 (36:6961)
[25] E. Park, *Toeplitz algebras and extensions of irrational rotation algebras*, Canad. Math. Bull., **48** (2005), 607-613. MR2176158 (2006g:46116)
[26] M. V. Pimsner, *A class of C^*-algebras generating both Cuntz-Krieger algebras and crossed product by $\mathbb{Z}$*, Free Probability Theory, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, pp. 189-212. MR1426840 (97k:46069)
[27] J. Renault, *A Groupoid Approach to C^*-algebras*, Lecture Note in Mathematics, 793, Springer, Berlin, 1980. MR584266 (82h:46075)
[28] J. V. Ryff, *Subordinate H^p functions*, Duke Math. J., **33** (1966), 347-354. MR0192062 (33:289)
[29] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext, Tracts in Mathematics, Springer-Verlag, New York, 1993. MR1237406 (94k:47049)
[30] M. Shub, *Endomorphisms of compact differentiable manifolds*, Amer. J. Math., **91** (1969), 175-199. MR0240824 (39:2169)

Graduate School of Mathematics, Kyushu University, Hakozaki, Fukuoka, 812-8581, Japan
E-mail address: h-hamada@math.kyushu-u.ac.jp

Department of Mathematical Sciences, Kyushu University, Hakozaki, Fukuoka, 812-8581, Japan
E-mail address: watatani@math.kyushu-u.ac.jp