PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 6, June 2010, Pages 2135–2140
S 0002-9939(10)10272-X
Article electronically published on February 4, 2010

BENEDICKS' THEOREM FOR THE HEISENBERG GROUP

E. K. NARAYANAN AND P. K. RATNAKUMAR

(Communicated by Michael T. Lacey)

Abstract. If an integrable function f on the Heisenberg group is supported on the set $B \times \mathbb{R}$ where $B \subset \mathbb{C}^n$ is compact and the group Fourier transform $\hat{f}(\lambda)$ is a finite rank operator for all $\lambda \in \mathbb{R} \setminus \{0\}$, then $f \equiv 0$.

1. Introduction

The uncertainty principle says that a nonzero function and its Fourier transform cannot both be sharply localized. There are several manifestations of this principle. We refer the reader to the excellent survey article by Folland and Sitaram [6] and also the monograph by S. Thangavelu [10].

In this paper we are interested in a variant of Benedicks' theorem on the Heisenberg group. Recall that Benedicks' theorem [2] states the following. Let $f \in L^2(\mathbb{R}^n)$, if both the sets $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ and $\{\xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0\}$ have finite Lebesgue measure, then $f \equiv 0$. In the context of noncommutative Lie groups the Fourier transform is an operator valued function. We measure the "smallness" of the Fourier transform in terms of the rank of these operators.

To state our result, we need to recall briefly the representation theory of the Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is topologically $\mathbb{C}^n \times \mathbb{R}$, with the group law

$$(z,t) \cdot (w,s) = (z + w, t + s + \frac{1}{2}\Im(z \cdot \bar{w})).$$

Under this group law, $\mathbb{H}^n$ becomes a two step nilpotent Lie group with center $Z = \{0\} \times \mathbb{R}$. The infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ are parametrized by $\lambda \in \mathbb{R} \setminus \{0\}$. Each such λ defines a representation π_λ, realized on $L^2(\mathbb{R}^n)$ by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}\, e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$. The representation π_λ is clearly unitary and it is well known that they are irreducible on $L^2(\mathbb{R}^n)$. In fact, a famous theorem of Stone and von Neumann says that any irreducible unitary representation of $\mathbb{H}^n$ that is nontrivial at the center is (unitarily) equivalent to π_λ for some λ (see [5]).

Received by the editors April 15, 2009, and, in revised form, October 28, 2009.
2010 *Mathematics Subject Classification.* Primary 42B10; Secondary 22E30, 43A05.
Key words and phrases. Benedicks' theorem, Weyl transform, uncertainty principles.
The first author was supported in part by a grant from UGC via DSA-SAP.

If $f \in L^1(\mathbb{H}^n)$, we can define the group Fourier transform by

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)\, dz\, dt.$$

Since π_λ is an isometry, a simple norm estimate shows that $\hat{f}(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^n)$. Moreover, if $f \in L^2(\mathbb{H}^n)$, then $\hat{f}(\lambda)$ turns out to be a Hilbert-Schmidt operator and the Plancherel theorem for the Heisenberg group reads as

$$\int_{\mathbb{H}^n} |f(z,t)|^2 dz\, dt = (2\pi)^{-n-1} \int \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda.$$

Our main result is the following :

Theorem 1.1. *Let $f \in L^1(\mathbb{H}^n)$ be supported on a set of the form $B \times \mathbb{R}$, where $B \subset \mathbb{C}^n$.*

1. *If B is a compact set and $\hat{f}(\lambda)$ is a finite rank operator for all $\lambda \neq 0$, then $f \equiv 0$.*
2. *If B has finite Lebesgue measure and $\hat{f}(\lambda)$ is a rank one operator for all $\lambda \neq 0$, then $f \equiv 0$.*

Remark 1. Note that our result is in sharp contrast with the situation on other Lie groups. For example, in the Euclidean case, the Fourier transform of any nontrivial $f \in L^1(\mathbb{R}^n)$ gives rise to a rank one operator, via multiplication by $\hat{f}(\xi)$. Next, if G is a noncompact connected semisimple Lie group and K is a maximal compact subgroup of G, then it can be shown that a function in $L^1(G/K)$ has a Fourier transform, which is a rank one operator. More generally, if $f \in L^1(G)$ transforms according to a fixed unitary irreducible representation of the compact group K on the right, then the group Fourier transform of f is a finite rank operator.

Remark 2. In [1] the authors study the "qualitative uncertainty principle" for unimodular groups. Let G be such a group and let $\hat{G}$ be its unitary dual. Let m denote the Haar measure on G and let $\hat{m}$ denote the Plancherel measure on $\hat{G}$. One of the results in [1] states that if $\{x \in G : f(x) \neq 0\} < m(G)$ and $\int_{\hat{G}} rank(\pi(f))\, d\hat{m} < \infty$, then $f \equiv 0$. When G is the Heisenberg group, the above conditions will force the Fourier transform to be supported on a set of finite (Plancherel) measure in addition to the finite rank condition. Notice that Theorem 1.1 requires only the finite rank condition. We thank Michael Lacey for pointing out this reference. We also refer the reader to [7] and [8] for a Benedicks' type theorem on the Heisenberg group.

In the rest of this section, we recall the necessary details about the Weyl transform and the Fourier-Wigner transform. For a suitable function g defined on $\mathbb{C}^n$, the $\lambda-$Weyl transform is defined to be the operator

$$W_\lambda(g) = \int_{\mathbb{C}^n} g(z)\, \pi_\lambda(z)\, dz,$$

where $\pi_\lambda(z) = \pi_\lambda(z,0)$. Clearly $W_\lambda(g)$ defines a bounded operator on $L^2(\mathbb{R}^n)$, if $g \in L^1(\mathbb{C}^n)$. For $g \in L^2(\mathbb{C}^n)$, $W_\lambda(g)$ is a Hilbert-Schmidt operator, and we have the Plancherel Theorem [9]:

$$\int_{\mathbb{C}^n} |g(z)|^2 dz = (2\pi\, |\lambda|)^{-n}\, \|W_\lambda(g)\|_{HS}^2.$$

The $\lambda-$twisted convolution of two functions F and G on $\mathbb{C}^n$ is defined to be

$$F \times_\lambda G(z) = \int_{\mathbb{C}^n} F(z-w)G(w)e^{\frac{i\lambda}{2}\Im(z\cdot\bar{w})}dw.$$

It is known that $W_\lambda(F \times_\lambda G) = W_\lambda(F)W_\lambda(G)$. When $\lambda = 1$, we write $F \times G$ instead of $F \times_1 G$ and call it the twisted convolution of F and G. Similarly $W_1(F)$ will be denoted by $W(F)$ and called the Weyl transform of F.

Let ϕ_1 and ϕ_2 belong to $L^2(\mathbb{R}^n)$. The Fourier-Wigner transform of ϕ_1 and ϕ_2 is a function on $\mathbb{C}^n$ and is defined by

$$A(\phi_1, \phi_2)(z) = \langle \pi_1(z)\phi_1, \phi_2\rangle.$$

The Fourier-Wigner transform satisfies the 'orthogonality relation',

$$\int_{\mathbb{C}^n} A(\phi_1, \phi_2)(z)\ \overline{A(\psi_1, \psi_2)(z)}\ dz = (2\pi)^n \langle \phi_1, \psi_1\rangle\ \langle \psi_2, \phi_2\rangle. \tag{1.1}$$

In fact, if $\{\phi_i : i \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, then the collection $\{A(\phi_i, \phi_j) : i, j \in \mathbb{N}\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$, see [9]. In particular, if $F \in L^2(\mathbb{C}^n)$ is orthogonal to $A(\phi, \psi)$ for all $\phi, \psi \in L^2(\mathbb{R}^n)$, then $F \equiv 0$.

We finish this section with the following theorem (see [3] or [4]), which will be used later.

Theorem 1.2. *Let $F(z) = A(\phi_1, \phi_2)(z)$ where $\phi_1, \phi_2 \in L^2(\mathbb{R}^n)$. If the set $\{z : F(z) \neq 0\}$ has finite Lebesgue measure, then $F \equiv 0$.*

2. Proof of the main result

We start with the following lemma:

Lemma 2.1. *Let $h_j \in L^2(\mathbb{R}^n)$, $j = 1, 2, ..., N$, and set, for $y \in \mathbb{R}^n$,*

$$K_y(\zeta) = \sum_{j=1}^{N} \overline{h_j(\zeta)}h_j(\zeta + y).$$

If $K_y(\zeta) = 0$ for almost all $\zeta \in \mathbb{R}^n$ and $|y| \geq R$, then each h_j is compactly supported.

Proof. Since each $h_j \in L^2(\mathbb{R}^n)$, there exists a set A of Lebesgue measure zero such that $|h_j(\zeta)| < \infty$ for every $\zeta \in \mathbb{R}^n \setminus A$, for $j = 1, 2 \ldots, N$.

We work with a fixed representative h_j for each of the class $[h_j] \in L^2(\mathbb{R}^n)$ for which pointwise evaluation makes sense. Hence, for $\zeta \in \mathbb{R}^n \setminus A$,

$$H(\zeta) = (h_1(\zeta), h_2(\zeta), \ldots h_N(\zeta)) \in \mathbb{C}^N.$$

If h_j are nonzero, choose $\zeta_1 \in \mathbb{R}^n \setminus A$ so that $H(\zeta_1)$ is a nonzero vector. Let $B_R(\zeta_1)$ be the open ball of radius R centered at ζ_1. If there is no $\zeta \in \mathbb{R}^n \setminus (B_R(\zeta_1) \cup A)$ such that $H(\zeta)$ is a nonzero vector, we are done. Otherwise, choose $\zeta_2 \in \mathbb{R}^n \setminus (B_R(\zeta_1) \cup A)$ so that $H(\zeta_2)$ is nonzero. By the hypothesis, $H(\zeta_1)$ and $H(\zeta_2)$ are orthogonal vectors in $\mathbb{C}^N$. We repeat this process. That is, if $j \leq N$, choose $\zeta_j \in \mathbb{R}^n \setminus (\bigcup_{l=1}^{j-1} B_R(\zeta_l) \cup A)$ such that $H(\zeta_j)$ is a nonzero vector in $\mathbb{C}^N$ (if there is no such ζ_j we are done). By the hypothesis $H(\zeta_j)$ are orthogonal to each other for $j = 1, 2, \cdots N$. Now, if $\zeta \in \mathbb{R}^n \setminus (\bigcup_{j=1}^{N} B_R(\zeta_j) \cup A)$, then $H(\zeta)$ is orthogonal to $H(\zeta_j)$ for all $j = 1, 2, \cdots N$. It follows that $H(\zeta)$ is zero for $\zeta \in \mathbb{R}^n \setminus (\bigcup_{j=1}^{N} B_R(\zeta_j) \cup A)$, which finishes the proof. $\square$

Our next result is a Benedicks' type theorem for the Weyl transform and is a crucial step in the proof of the main theorem.

Theorem 2.2. *Let $F \in L^1(\mathbb{C}^n)$ be compactly supported. If the Weyl transform $W(F)$ of F is a finite rank operator, then $F \equiv 0$.*

Proof. Let $\overline{G(z)} = F^* \times F(z)$, where $F^*(z) = \overline{F(-z)}$. Then $\bar{G}$ is compactly supported and $\bar{G} = 0$ if and only if $F \equiv 0$, by the Plancherel theorem for the Weyl transform. Now $W(\bar{G}) = W(F)^* W(F)$ is a finite rank, positive, Hilbert-Schmidt operator, and hence by the spectral theorem, we have

$$W(\bar{G})\phi = \sum_{j=1}^{N} \lambda_j \langle \phi, \phi_j \rangle \phi_j, \tag{2.1}$$

where $\{\phi_1, ..., \phi_N\}$ is an orthonormal basis for the range of $W(\bar{G})$, with $W(\bar{G})\phi_j = \lambda_j \phi_j$ and $\lambda_j \geq 0$. Hence

$$\langle W(\bar{G})\phi, \psi \rangle = \sum_{j=1}^{N} \lambda_j \langle \phi, \phi_j \rangle \langle \phi_j, \psi \rangle. \tag{2.2}$$

By (1.1), the above equals

$$(2\pi)^{-n} \sum_{j=1}^{N} \lambda_j \int_{\mathbb{C}^n} A(\phi, \psi)(z)\, \overline{A(\phi_j, \phi_j)(z)}\, dz. \tag{2.3}$$

Also by the definition of the Weyl transform,

$$\langle W(\bar{G})\phi, \psi \rangle = \int_{\mathbb{C}^n} \bar{G}(z)\, A(\phi, \psi)(z)\, dz. \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$G(z) = \sum_{j=1}^{N} A(h_j, h_j)(z),$$

where $h_j(z) = (2\pi)^{-\frac{n}{2}} \sqrt{\lambda_j}\, \phi_j(z)$. Writing $G_y(x) = G(z)$ for $z = (x + iy)$, the above identity reads as

$$G_y(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\zeta + \frac{1}{2}x\cdot y)} \left(\sum_{j=1}^{N} h_j(\zeta + y)\overline{h_j(\zeta)} \right) d\zeta. \tag{2.5}$$

Since G is compactly supported, there exists $R > 0$ such that $G_y \equiv 0$ if $|y| \geq R$. Then (2.5) implies that $\sum_{j=1}^{N} h_j(\zeta + y)\overline{h_j(\zeta)} = 0$ for almost every $\zeta \in \mathbb{R}^n$, provided $|y| \geq R$.

Lemma 2.1 now implies that each h_j is compactly supported and hence that $\sum_{j=1}^{N} h_j(\zeta + y)\overline{h_j(\zeta)}$ is also compactly supported in ζ for each $y \in \mathbb{R}^n$. In view of (2.5), we conclude that $G_y \equiv 0$ for each $y \in \mathbb{R}^n$, hence the proof. □

Now we are in a position to complete the proof of the main theorem.

Proof of Theorem 1.1. Let $f^\lambda(z)$ denote the partial Fourier transform of f in the $t-$variable. That is,

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t)\, e^{i\lambda t}\, dt.$$

Then a simple computation shows that $\hat{f}(\lambda) = W_\lambda(f^\lambda)$.

We start with the proof of (II) in Theorem 1.1. By the hypothesis we have that $f^\lambda(z)$ is supported in the set B (which has finite Lebesgue measure) and $\hat{f}(\lambda) = W_\lambda(f^\lambda)$ is a rank one operator for all λ. We will assume that $\lambda = 1$ and prove that $f^\lambda \equiv 0$. The general case is no different.

It suffices to show that if $F \in L^1(\mathbb{C}^n)$ is supported on a set of finite measure and $W(F)$ is a rank one operator, then $F \equiv 0$. This immediately follows from Theorem 1.2 once we show that F is the Fourier-Wigner transform of two functions in $L^2(\mathbb{R}^n)$. For this, let $\bar{G} = F$. Since $W(\bar{G})$ is a rank one operator, we have $\psi_1, \psi_2 \in L^2(\mathbb{R}^n)$ such that

$$W(\bar{G})\varphi = \langle \varphi, \psi_1 \rangle\ \psi_2,\ \forall \varphi \in L^2(\mathbb{R}^n).$$

Hence, if $\psi \in L^2(\mathbb{R}^n)$ we have

$$\begin{aligned} \langle W(\bar{G})\varphi, \psi \rangle &= \int_{\mathbb{C}^n} \bar{G}(z)\, \langle \pi_1(z)\varphi, \psi \rangle\, dz \\ &= \langle \varphi, \psi_1 \rangle\, \langle \psi_2, \psi \rangle \\ &= (2\pi)^{-n} \int_{\mathbb{C}^n} A(\varphi,\psi)(z)\ \overline{A(\psi_1,\psi_2)(z)}\ dz, \end{aligned}$$

where the last step follows from (1.1). It follows that $G(z) = A(\psi_1, \psi_2)(z)$.

To prove (1), we proceed as above. Taking the Fourier transform in the $t-$variable reduces the problem to $\mathbb{C}^n$. As above we assume that $\lambda = 1$. It suffices to show that if $F \in L^1(\mathbb{C}^n)$ is compactly supported and $W(F)$ is of finite rank, then $F \equiv 0$. But this is precisely the content of Theorem 2.2, hence finishing the proof. □

References

[1] D. Arnal and J. Ludwig, Q.U.P. and Paley-Wiener properties of unimodular, especially nilpotent, Lie groups, *Proc. Amer. Math. Soc.* 125 (1997), no. 4, 1071-1080. MR1353372 (97f:43004)

[2] M. Benedicks, On Fourier transforms of functions supported on sets of finite Lebesgue measure, *J. Math. Anal. Appl.* 106 (1985), 180-183. MR780328 (86f:43006)

[3] P. Jaming, Principe d'incertitude qualitatif et reconstruction de phase pour la transformée de Wigner, *C. R. Acad. Sci. Paris Sér. I Math.* 327 (1998), no. 3, 249-254. MR1650249 (99h:42022)

[4] A. J. E. M. Janssen, Proof of a conjecture on the supports of Wigner distributions, *J. Fourier Anal. Appl.* 4 (1998), no. 6, 723-726. MR1666005 (99j:42014)

[5] G. B. Folland, *Harmonic Analysis in Phase Space*, Annals of Math Studies, 122, Princeton Univ. Press, Princeton, NJ (1989). MR983366 (92k:22017)

[6] G. B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey, *Journal of Fourier Anal. Appl.* 3 (1997), no. 3, 207-238. MR1448337 (98f:42006)

[7] J. F. Price and A. Sitaram, Functions and their Fourier transforms with supports of finite measure for certain locally compact groups, *J. Funct. Anal.* 79 (1988), no. 1, 166-182. MR950089 (90e:43003)

[8] A. Sitaram, M. Sundari and S. Thangavelu, Uncertainty principles on certain Lie groups, *Proc. Indian Acad. Sci. Math. Sci.* 105 (1995), no. 2, 135-151. MR1350473 (96h:43002)

[9] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions,* Mathematical Notes, 42, Princeton Univ. Press, Princeton, NJ (1993). MR1215939 (94i:42001)

[10] S. Thangavelu, *An Introduction to the Uncertainty Principle*, Progress in Math., 217, Birkhäuser Boston, Inc., Boston, MA (2004). MR2008480 (2004j:43007)

Department of Mathematics, Indian Institute of Science, Bangalore 12, India
E-mail address: naru@math.iisc.ernet.in

Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211 019, India
E-mail address: ratnapk@hri.res.in