PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 6, June 2010, Pages 2061–2064
S 0002-9939(10)10306-2
Article electronically published on January 27, 2010

ON THE UNIQUENESS OF CLASSICAL SOLUTIONS OF CAUCHY PROBLEMS

ERHAN BAYRAKTAR AND HAO XING

(Communicated by Edward C. Waymire)

Abstract. Given that the terminal condition is of at most linear growth, it is well known that a Cauchy problem admits a unique classical solution when the coefficient multiplying the second derivative is also a function of at most linear growth. In this paper, we give a condition on the volatility that is necessary and sufficient for a Cauchy problem to admit a unique solution.

1. Main result

Given a terminal-boundary condition $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $g(x) \leq C(1+x)$ for some constant $C > 0$, we consider the following Cauchy problem:

$$
\begin{aligned}
&u_t + \frac{1}{2}\sigma^2(x)\, u_{xx} = 0, \quad (x,t) \in (0,\infty) \times [0,T),\\
&u(0,t) = g(0), \quad t \leq T,\\
&u(x,T) = g(x),
\end{aligned}
\tag{1}
$$

where $\sigma \neq 0$ on $(0,\infty)$, $\sigma^{-2} \in L^1_{loc}(0,\infty)$ (i.e., $\int_a^b \sigma^{-2}(x)dx < \infty$ for any $[a,b] \subset (0,\infty)$), and $\sigma = 0$ on $(-\infty, 0]$.

A solution $u : \mathbb{R}_+ \times [0,T] \mapsto \mathbb{R}$ of (1) is said to be a *classical solution* if $u \in C^{2,1}((0,\infty) \times [0,T))$. A function $f : \mathbb{R}_+ \times [0,T] \mapsto \mathbb{R}$ is said to be of *at most linear growth* if there exists a constant $C > 0$ such that $|f(x,t)| \leq C(1+x)$ for any $(x,t) \in \mathbb{R}_+ \times [0,T]$.

A well-known sufficient condition for (1) to have a unique classical solution among the functions with at most linear growth is that σ itself is of at most linear growth; see, e.g., Chapter 6 of [8] and Theorem 7.6 on page 366 of [10]. On the other hand, consider the SDE

$$
dX_s^{t,x} = \sigma(X_s^{t,x})\, dW_s, \quad X_t^{t,x} = x > 0. \tag{2}
$$

The assumptions on σ we made below (1) ensure that (2) has a unique weak solution which is absorbed at zero. (See [6].) The solution $X^{t,x}_{\cdot}$ is clearly a local martingale.

Received by the editors September 16, 2009.
2010 *Mathematics Subject Classification.* Primary 35K65, 60G44.
Key words and phrases. Cauchy problem, a necessary and sufficient condition for uniqueness, European call-type options, strict local martingales.
This research is supported in part by the National Science Foundation under grant number DMS-0906257.

Delbaen and Shirakawa show in [4] that $X^{t,x}_\cdot$ is a martingale if and only if

$$\int_c^\infty \frac{x}{\sigma^2(x)}\,dx = \infty, \quad \text{for some } c > 0. \tag{3}$$

(See also [2].)

Below, in Theorem 2, we prove that (3) is also necessary and sufficient for the existence of a unique classical solution of (1). First, in the next theorem, we show that (3), which is weaker than the linear growth condition on σ, is a sufficient condition for the uniqueness.

Theorem 1. *The Cauchy problem* (1) *has a unique classical solution (if any) in the class of functions with at most linear growth if* (3) *is satisfied.*

Proof. It suffices to show that (1) with $g \equiv 0$ has a unique solution $u \equiv 0$. Let us define a sequence of stopping times $\tau_n \triangleq \inf\{s \geq t : X^{t,x}_s \geq n \text{ or } X^{t,x}_s \leq 1/n\} \wedge T$ for each $n \in \mathbb{N}$ and $\tau_0 \triangleq \{s \geq t : X^{t,x}_s = 0\} \wedge T$. Then, as in the proof of Theorem 1.6 in [4], we can show that the function defined by

$$\Psi(x) = \begin{cases} x, & x \leq 1, \\ x + \int_1^x \frac{u}{\sigma^2(u)}(x-u)du, & x \geq 1, \end{cases}$$

satisfies $\mathbb{E}[\Psi(X^{t,x}_{\tau_n})] \leq \Psi(x) + xT/2$. Since Ψ is convex, (3) implies that $\lim_{x\to\infty} \Psi(x)/x = \infty$. Then the criterion of de la Vallée Poussin implies that $\{X^{t,x}_{\tau_n} : n \in \mathbb{N}\}$ is a uniformly integrable family.

Suppose $\tilde{u}$ is another classical solution of at most linear growth. Applying Itô's lemma, we obtain

$$\tilde{u}(X^{t,x}_{\tau_n}, \tau_n) = \tilde{u}(x,t) + \int_t^{\tau_n} \left[\tilde{u}_s(X^{t,x}_s, s) + \frac{1}{2}\sigma^2(X^{t,x}_s)\tilde{u}_{xx}(X^{t,x}_s, s)\right] ds + \int_t^{\tau_n} \tilde{u}_x(X^{t,x}_s, s)\sigma(X^{t,x}_s)dW_s.$$

Note that $X^{t,x}_\cdot$ is a martingale on $[t, \tau_n]$ and we have $\tilde{u}_x(X^{t,x}_s, s)$ bounded for $s \leq \tau_n$. Hence the stochastic integral in the above identity has expectation zero. Taking the expectation of both sides of the above identity, we get $\tilde{u}(x,t) = \mathbb{E}\left[\tilde{u}(X^{t,x}_{\tau_n}, \tau_n)\right]$ for each $n \in \mathbb{N}$.

On the other hand, since $\tilde{u}$ is of at most linear growth, there exists a constant C such that $|\tilde{u}(x,t)| \leq C(1+x)$. Therefore $\{\tilde{u}(X^{t,x}_{\tau_n}, \tau_n) : n \in \mathbb{N}\}$ is a uniformly integrable family. This is because it is bounded from above by the uniformly integrable family $\{C(1 + X^{t,x}_{\tau_n}) : n \in \mathbb{N}\}$. As a result,

$$\begin{aligned}\tilde{u}(x,t) &= \lim_{n\to\infty} \mathbb{E}\left[\tilde{u}(X^{t,x}_{\tau_n}, \tau_n)\right] = \mathbb{E}\left[\lim_{n\to\infty} \tilde{u}(X^{t,x}_{\tau_n}, \tau_n)\right] = \mathbb{E}[\tilde{u}(X^{t,x}_{\tau_0}, \tau_0)] \\ &= \mathbb{E}\left[g(X^{t,x}_T)1_{\{\tau_0 = T\}}\right] + \mathbb{E}\left[\tilde{u}\left(X^{t,x}_{\tau_0}, \tau_0\right)1_{\{\tau_0 < T\}}\right] \\ &= \mathbb{E}\left[g(X^{t,x}_T)1_{\{\tau_0 = T\}}\right] + \mathbb{E}\left[g(0)1_{\{\tau_0 < T\}}\right] = 0.\end{aligned}$$

Here the third equality holds since $X^{t,x}_\cdot$ does not explode (i.e., $\inf\{s \geq t : X^{t,x}_s = +\infty\} = \infty$; see Problem 5.3 on page 332 of [11]), and the one before the last equality follows since $X^{t,x}_{\tau_0} = 0$ on the set $\{\tau_0 < T\}$. □

Theorem 2. *If we further assume that $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is locally Hölder continuous with exponent 1/2 and g is of linear growth, then the Cauchy problem in* (1) *has a unique classical solution if and only if* (3) *is satisfied.*

Proof. First, let us prove the existence of a solution. Let $u(x,t) \triangleq \mathbb{E}\, g(X_T^{t,x})$ (the value of a call-type European option). Thanks to the Hölder continuity of σ, it follows from Theorem 3.2 in [5] that u is a classical solution of (1). Moreover, it is of at most linear growth due to the assumption that g is of at most linear growth.

Proof of sufficiency. This follows from Theorem 1.

Proof of necessity. If (3) is violated, then $t \to X_t$ is a strict local martingale on each time interval $(0,T)$, $T \in (0,\infty)$; see [4], [2] and especially the remark after Corollary 4.3 in [12]. Using Theorem 3.2 in [5], it can be seen that $u^*(x,t) \triangleq x - \mathbb{E}[X_T^{x,t}] > 0$ is a classical solution of (1) with zero boundary and terminal conditions. (Note that the Hölder continuity assumption on σ is used in this step as well.) This function clearly has at most linear growth. Therefore $u + \lambda u^*$, for any $\lambda \in \mathbb{R}$, is also a classical solution of (1) which is of at most linear growth. □

Remark 1. Observe that Theorems 1 and 2 are analogous to Theorem 7.6 and Remark 7.8 on pages 366 and 368 of [10], respectively. Our contribution is the weakening of the linear growth assumption on σ.

A related result is given by Theorem 4.3 of [5] on put-type European options: When g is of strictly sublinear growth (i.e., $\lim_{x\to\infty} g(x)/x = 0$), then (1) has a unique solution among the functions with strictly sublinear growth (without assuming (3)).

Our result in Theorem 2 complements Theorem 3.2 of [5], which shows that the call-type European option price is a classical solution of (1) of at most linear growth. We prove that Delbaen and Shirakawa's condition (3) is necessary and sufficient to guarantee that the European option price is the only classical solution (of at most linear growth) to this Cauchy problem. [3] and [9] have already observed that the Cauchy problem corresponding to European call options have multiple solutions. (Also see [7] and [1], which consider super hedging prices of call-type options when there are no equivalent local martingale measures.) However, a necessary and sufficient condition under which there is uniqueness/nonuniqueness remains unknown.

Acknowledgment

The authors would like to thank the referee for insightful comments which helped improve their paper.

References

[1] E. Bayraktar, K. Kardaras, and H. Xing. Strict local martingale deflators and American call-type options. Technical report, Boston University and University of Michigan, 2009. Available at http://arxiv.org/pdf/0908.1082.

[2] P. Carr, A. S. Cherny, and M. Urusov. On the martingale property of time-homogeneous diffusions. Technical report, Berlin University of Technology, 2009. Available at http://www.math.tu-berlin.de/~urusov/papers/cev.pdf.

[3] A. Cox and D. Hobson. Local martingales, bubbles and option prices. *Finance & Stochastics*, 9:477–492, 2005. MR2213778 (2006j:91146)

[4] F. Delbaen and H. Shirakawa. No arbitrage condition for positive diffusion price processes. *Asia-Pacific Financial Markets*, 9(3-4):159–168, 2002.

[5] E. Ekström and J. Tysk. Bubbles, convexity and the Black-Scholes equation. *Annals of Applied Probability*, 19 (4):1369–1384, 2009. MR2538074

[6] H. J. Engelbert and W. Schmidt. On one-dimensional stochastic differential equations with generalized drift. In *Stochastic differential systems (Marseille-Luminy, 1984)*, volume 69 of Lecture Notes in Control and Inform. Sci., pages 143–155. Springer, Berlin, 1985. MR798317 (86m:60144)

[7] D. Fernholz and I. Karatzas. On optimal arbitrage. Technical report, Columbia University, 2008. Available at http://www.e-publications.org/ims/submissions/index.php/AAP/user/submissionFile/3403?confirm=lac329b5. To appear in *Annals of Applied Probability*.

[8] A. Friedman. *Stochastic differential equations and applications. Vol. 1.* Probability and Mathematical Statistics, Vol. 28. Academic Press [Harcourt Brace Jovanovich, Publishers], New York, 1975. MR0494490 (58:13350a)

[9] S. L. Heston, M. Loewenstein, and G. A. Willard. Options and bubbles. *Review of Financial Studies*, 20(2):359–390, 2007.

[10] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991. MR1121940 (92h:60127)

[11] I. Karatzas and S. E. Shreve. *Methods of mathematical finance*. Springer-Verlag, New York, 1998. MR1640352 (2000e:91076)

[12] A. Mijatovic and M. Urusov. On the martingale property of certain local martingales. Technical report, Imperial College London, 2009. Available at http://arxiv.org/abs/0905.3701v2.

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, Michigan 48104

E-mail address: erhan@umich.edu

Department of Mathematics and Statistics, Boston University, Boston, Massachusetts 02215

E-mail address: haoxing@bu.edu