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A COMBINATORIAL CONSTRUCTION OF HIGH ORDER ALGORITHMS FOR FINDING POLYNOMIAL ROOTS OF KNOWN MULTIPLICITY

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Abstract. We construct a family of high order iteration functions for finding polynomial roots of a known multiplicity s. This family is a generalization of a fundamental family of high order algorithms for simple roots that dates back to Schröder's 1870 paper. It starts with the well known variant of Newton's method $\hat{B}_2(x) = x - s \cdot p(x)/p'(x)$ and the multiple root counterpart of Halley's method derived by Hansen and Patrick. Our approach demonstrates the relevance and power of algebraic combinatorial techniques in studying rational root-finding iteration functions.

1. Introduction

In [4] we introduced symmetric functions to the study of iterative root-finding algorithms, thus revealing the combinatorial nature of rational root-finding iteration functions. In particular, we gave a combinatorial interpretation of the *Basic Family* - a fundamental family of high order algorithms for finding simple roots of a polynomial. The Basic Family first appeared in Schröder's 1870 paper [14] and was later rediscovered many times under different representations. In one representation, it is known as König's family (see [1]). In [7], Kalantari et al. gave a purely algebraic derivation of the Basic Family and revealed many interesting minimality and uniqueness properties of this family. For some related works in this direction, see [8], [9], [10], [11] and [6].

In this article we demonstrate the power of our combinatorial approach by deriving an efficient algorithm to construct a generalized Basic Family for roots with a known multiplicity. Our exposition is organized as follows: In Section 2 we give a brief review of relevant results from [4]. In Section 3 we present a combinatorial construction of the generalized Basic Family and prove its convergence property. In Section 4 we show the connection between the Basic Family and its generalization.

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The main idea of this article was developed in the first author's Ph.D. dissertation [5], supervised by the second author.

2. Preliminaries

First we recall some notation and results from the theory of symmetric functions. For a definitive treatment on this subject, see MacDonald [13].

For each integer $r \geq 1$, the r-th elementary symmetric function e_r is the sum of all products of r distinct variables x_i, the r-th complete symmetric function h_r is the sum of all monomials of total degree (the sum of exponents of all its variables) r in the variables $x_1, x_2, \ldots$, and the r-th power sum is the sum of r-th power of $x_1, x_2, \ldots$:

$$\begin{aligned} e_r &= \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r}, \\ h_r &= \sum_{a_1 + a_2 + \cdots = r} x_1^{a_1} x_2^{a_2} \ldots, \\ q_r &= \sum_{i \geq 1} x_i^r. \end{aligned}$$

Additionally, define $e_0 = h_0 = 1$.

The generating functions for e_r, h_r and q_r are

$$\begin{aligned} E(t) &= \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t), \\ H(t) &= \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}, \\ Q(t) &= \sum_{r \geq 1} q_r t^{r-1} = \sum_{i \geq 1} \frac{x_i}{1 - x_i t}. \end{aligned}$$

Thus, $H(t) = E(-t)^{-1}$ and

$$Q(t) = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)} = \frac{E'(-t)}{E(-t)},$$

where $E'(-t)$ denotes the derivative of E evaluated at $-t$.

The above relations among generating functions imply the following recursions:

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \qquad \text{for} \quad n = 1, 2, \ldots,$$

$$n h_n = \sum_{r=1}^{n} q_r h_{n-r} \qquad \text{for} \quad n = 1, 2, \ldots,$$

$$n e_n = \sum_{r=1}^{n} (-1)^{r-1} q_r e_{n-r} \qquad \text{for} \quad n = 1, 2, \ldots.$$

We now make a connection between symmetric functions and root-finding algorithms.

Let $p(x)$ be a polynomial of degree n with complex coefficients and let $\theta_1, \ldots, \theta_n$ be its complex roots. Then,

$$p(x) = c \prod_{i=1}^{n} (x - \theta_i),$$

where c is the leading coefficient of $p(x)$.

Define $r_j = 1/(x - \theta_j)$, $j = 1, \ldots, n$. Then we have

Proposition 2.1 (Jin and Kalantari [4], Lemma 3.1)**.**

$$p^{(i)}(x) = i!p(x)e_i(r_1, \ldots, r_n),$$

where e_i is the i-th elementary symmetric function.

For each integer $m \geq 2$, define the m-th order Basic Family iteration function

$$B_m(x) = x - \frac{h_{m-2}(r_1, \ldots, r_n)}{h_{m-1}(r_1, \ldots, r_n)}.$$

This is equivalent to the original definition of Basic Family given by Kalantari et al. [7].

Proposition 2.2 (Jin and Kalantari [4], Corollary 3.4)**.** *$B_m(x)$ has an m-th order of convergence to a simple root and linear convergence to a multiple root.*

Proposition 2.1 and the recurrence relation between h_r and e_r give rise to an efficient algorithm to compute B_m.

Algorithm 1: Evaluation of the m-th order iteration function B_m.

input : $m, p(x)$, and x_0.
output: $B_m(x_0)$.

```
h[0] = 1;
for i from 1 to m − 1 do
    e[i] = p^(i)(x_0)/(i! p(x_0));
end
for i from 1 to m − 1 do
    h[i] = Σ_{r=0}^{i−1} (−1)^{i−r−1} e[i − r] h[r];
end
output B_m(x_0) = x_0 − h[m − 2]/h[m − 1];
```

The first three members of the Basic Family are

$$B_2(x) = x - \frac{p(x)}{p'(x)}, \tag{2.1}$$

which is the well known Newton's method,

$$B_3(x) = x - \frac{2p(x)p'(x)}{2p'(x)^2 - p(x)p''(x)}, \tag{2.2}$$

which is Halley's method, and the 4th order method

$$B_4(x) = x - \frac{6p(x)p'(x)^2 - 3p(x)^2p''(x)}{p(x)^2p'''(x) + 6p'(x)^3 - 6p(x)p'(x)p''(x)}. \tag{2.3}$$

For a root of multiplicity $s \geq 2$, there is a well known generalization of the 2nd order Newton's method:

$$\hat{B}_2(x) = x - s\frac{p(x)}{p'(x)}.$$

A generalization of the 3rd order Halley's method was derived by Hansen and Patrick [3, Equation (8.2)].

In this article, we shall show that such generalization exists for all Basic Family iteration functions, and it can be computed efficiently due to its simple combinatorial characterization.

3. $\{\hat{B}_m\}_{m=2}^{\infty}$ - A FAMILY OF HIGH ORDER METHODS FOR ROOTS OF MULTIPLICITY s

The key ingredient in our combinatorial construction is a generalization of power sums and complete symmetric functions.

Define the generalized power sums

$$\hat{q}_r = \frac{q_r}{s} = \frac{1}{s}\sum_{i\geq 1} x_i^r \qquad \text{for} \quad r = 1, 2, ... \tag{3.1}$$

The generalized complete symmetric functions $\hat{h}_r$ are defined through the following recursion:

$$\begin{aligned} \hat{h}_0 &= 1, \\ \hat{h}_r &= \frac{1}{r}\sum_{i=0}^{r-1} \hat{q}_{r-i}\hat{h}_i \qquad \text{for} \quad r = 1, 2, ... \end{aligned} \tag{3.2}$$

Define the m-th order generalized Basic Family iteration functions

$$\hat{B}_m(x) = x - \frac{\hat{h}_{m-2}(r_1, \ldots, r_n)}{\hat{h}_{m-1}(r_1, \ldots, r_n)} \qquad \text{for} \quad m = 2, 3, ... \tag{3.3}$$

Note that when $s = 1$, $\hat{B}_m$ reduces to B_m, an m-th order method for simple roots. In the following subsection, we shall show that $\hat{B}_m$ is an m-th order method for roots of multiplicity s.

3.1. Proof of convergence property of $\hat{B}_m$.

Recall that the falling factorial is defined as

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x-1)\ldots(x-n+1) \quad \text{for} \ n = 1, 2, ...$$

We first express $\hat{h}_k(r_1, \ldots, r_n)$ as a polynomial in r_1.

Lemma 3.1. *Assume that θ_1 is a root of multiplicity s', i.e., $\theta_1 = \theta_2 = \cdots = \theta_{s'}$, and $\theta_i \neq \theta_1$ for $i = s'+1, \ldots, n$. Define $r_j = 1/(x - \theta_j)$, $j = 1, \ldots, n$. Then for every integer $k \geq 0$,*

$$\hat{h}_k(r_1, \ldots, r_n) = \sum_{i=0}^{k} \frac{(k - i + s'/s - 1)_{k-i}}{(k-i)!}\, \hat{h}_i(r_{s'+1}, \ldots, r_n)\, r_1^{k-i}. \tag{3.4}$$

Proof. By induction on k.

For $k = 0$, (3.4) holds trivially. Now assume that (3.4) holds for $k < m$, $m \in \mathbb{Z}^+$. Then for $k = m$, by (3.2) we have the following recursion:

$$\hat{h}_m(r_1, \ldots, r_n) = \frac{1}{m}\sum_{j=0}^{m-1} \hat{q}_{m-j}(r_1, \ldots, r_n)\hat{h}_j(r_1, \ldots, r_n). \tag{3.5}$$

By (3.1) and induction assumption, we have

$$\hat{q}_{m-j}(r_1, \ldots, r_n) = \frac{s'}{s} r_1^{m-j} + \hat{q}_{m-j}(r_{s'+1}, \ldots, r_n) \quad \text{for} \ m - j > 0,$$

$$\hat{h}_j(r_1, \ldots, r_n) = \sum_{i=0}^{j} \frac{(j - i + s'/s - 1)_{j-i}}{(j-i)!}\, \hat{h}_i(r_{s'+1}, \ldots, r_n)\, r_1^{j-i} \quad \text{for} \ j < m.$$

Plugging the above identities into (3.5), multiplying out, and collecting like terms of powers of r_1, we get

$$\hat{h}_m(r_1, \ldots, r_n) = \frac{1}{m} \sum_{i=0}^{m} c_i r_1^{m-i},$$

where

$$\begin{aligned} c_i &= \frac{s'}{s} \hat{h}_i(r_{s'+1}, \ldots, r_n) \sum_{j=i}^{m-1} \frac{(j-i+s'/s-1)_{j-i}}{(j-i)!} \\ &+ \frac{(m-i+s'/s-1)_{m-i}}{(m-i)!} \sum_{j=1}^{i} \hat{h}_{i-j}(r_{s'+1}, \ldots, r_n) \hat{q}_j(r_{s'+1}, \ldots, r_n) \\ &= \frac{(m-i+s'/s-1)_{m-i}}{(m-i-1)!} \hat{h}_i(r_{s'+1}, \ldots, r_n) \\ &+ \frac{(m-i+s'/s-1)_{m-i}}{(m-i)!} i \hat{h}_i(r_{s'+1}, \ldots, r_n) \\ &= m \frac{(m-i+s'/s-1)_{m-i}}{(m-i)!} \hat{h}_i(r_{s'+1}, \ldots, r_n). \end{aligned}$$

Thus,

$$\hat{h}_m(r_1, \ldots, r_n) = \sum_{i=0}^{m} \frac{(m-i+s'/s-1)_{m-i}}{(m-i)!} \hat{h}_i(r_{s'+1}, \ldots, r_n) r_1^{m-i}.$$

So (3.4) holds for $k = m$. Also, by induction, it holds for all $k \geq 0$. □

Using the above structural lemma about $\hat{h}_k$, it is easy to derive the convergence property of $\hat{B}_m$.

Theorem 3.2. *Let s' be the multiplicity of root θ_1, i.e., $\theta_1 = \theta_2 = \cdots = \theta_{s'}$, and $\theta_i \neq \theta_1$ for $i = s'+1, \ldots, n$. Define $r_j = 1/(x - \theta_j)$, $j = 1, \ldots, n$.*

When $s' = s$, $\hat{B}_m(x)$ has an order of convergence m for θ_1, and

$$\lim_{x \to \theta_1} \frac{\hat{B}_m(x) - \theta_1}{(x - \theta_1)^m} = \hat{h}_{m-1}((\theta_1 - \theta_{s'+1})^{-1}, \ldots, (\theta_1 - \theta_n)^{-1}).$$

When $s' \neq s$, $\hat{B}_m(x)$ has at most linear convergence for θ_1, and

$$\lim_{x \to \theta_1} \frac{\hat{B}_m(x) - \theta_1}{x - \theta_1} = \frac{s' - s}{s' + s(m-2)}.$$

Proof. When $s' = s$, equation (3.4) becomes

$$\hat{h}_k(r_1, \ldots, r_n) = \sum_{i=0}^{k} \hat{h}_i(r_{s'+1}, \ldots, r_n) r_1^{k-i}.$$

From (3.3), we have

$$\begin{aligned}\hat{B}_m(x)-\theta_1 &= x-\theta_1-(x-\theta_1)\frac{(x-\theta_1)^{m-2}\hat{h}_{m-2}(r_1,\dots,r_n)}{(x-\theta_1)^{m-1}\hat{h}_{m-1}(r_1,\dots,r_n)}\\ &= (x-\theta_1)\left(1-\frac{\sum_{i=0}^{m-2}\hat{h}_i(r_{s'+1},\dots,r_n)(x-\theta_1)^i}{\sum_{i=0}^{m-1}\hat{h}_i(r_{s'+1},\dots,r_n)(x-\theta_1)^i}\right)\\ &= (x-\theta_1)^m\frac{\hat{h}_{m-1}(r_{s'+1},\dots,r_n)}{\sum_{i=0}^{m-1}\hat{h}_i(r_{s'+1},\dots,r_n)(x-\theta_1)^i}.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{x\to\theta_1}\frac{\hat{B}_m(x)-\theta_1}{(x-\theta_1)^m} &= \lim_{x\to\theta_1}\frac{\hat{h}_{m-1}(r_{s'+1},\dots,r_n)}{1+\sum_{i=1}^{m-1}\hat{h}_i(r_{s'+1},\dots,r_n)(x-\theta_1)^i}\\ &= \hat{h}_{m-1}((\theta_1-\theta_{s'+1})^{-1},\dots,(\theta_1-\theta_n)^{-1}).\end{aligned}$$

When $s'\neq s$, by (3.3) and (3.4) we have

$$\begin{aligned}\hat{B}_m(x)-\theta_1 &= x-\theta_1-(x-\theta_1)\frac{(x-\theta_1)^{m-2}\hat{h}_{m-2}(r_1,\dots,r_n)}{(x-\theta_1)^{m-1}\hat{h}_{m-1}(r_1,\dots,r_n)}\\ &= (x-\theta_1)\left(1-\frac{(m-3+s'/s)_{m-2}/(m-2)!+\sum_{i=1}^{m-2}a_{m-2,i}(x-\theta_1)^i}{(m-2+s'/s)_{m-1}/(m-1)!+\sum_{i=1}^{m-1}a_{m-1,i}(x-\theta_1)^i}\right).\end{aligned}$$

Hence,

$$\lim_{x\to\theta_1}\frac{B_m(x)-\theta_1}{x-\theta_1}=1-\frac{(m-3+s'/s)_{m-2}/(m-2)!}{(m-2+s'/s)_{m-1}/(m-1)!}=\frac{s'-s}{s'+s(m-2)}.$$

□

Remark 3.3. When $m=2$ and $s'\le s/2$, $\hat{B}_m(x)$ does not converge to θ_1 since

$$\left|\frac{s'-s}{s'+s(m-2)}\right|\ge 1.$$

3.2. **An efficient algorithm for computing $\hat{B}_m$.** Let $\hat{Q}(t)$ and $\hat{H}(t)$ be the generating functions for $\hat{q}_r$ and $\hat{h}_r$, respectively. That is,

$$\hat{Q}(t)=\sum_{r\ge1}\hat{q}_r t^{r-1}\qquad\text{and}\qquad \hat{H}(t)=\sum_{r\ge0}\hat{h}_r t^r.$$

By (3.1), we have

$$\hat{Q}(t)=\frac{1}{s}Q(t)=\frac{E'(-t)}{sE(-t)},$$

and recursion (3.2) implies

$$\hat{Q}(t)=\frac{\hat{H}'(t)}{\hat{H}(t)}.$$

Thus,

$$sE(-t)\hat{H}'(t)=E'(-t)\hat{H}(t),$$

which gives rise to the following recursion:

$$\hat{h}_i=\frac{1}{s\cdot i}\sum_{r=0}^{i-1}(-1)^{i-r-1}(i-r+s\cdot r)e_{i-r}\hat{h}_r\qquad\text{for}\quad i=1,2,\dots. \tag{3.6}$$

Given an input x_0, an efficient algorithm to compute $\hat{B}_m(x_0)$ is as follows:

Algorithm 2: Evaluation of the m-th order iteration function $\hat{B}_m$ for roots of multiplicity s at x_0.

input : $m, s, p(x)$, and x_0.
output: $\hat{B}_m(x_0)$.

$\hat{h}[0] = 1$;
for i *from* 1 *to* $m-1$ **do**
  $e[i] = p^{(i)}(x_0)/(i!p(x_0))$;
end
for i *from* 1 *to* $m-1$ **do**
  $\hat{h}[i] = \left(\sum_{r=0}^{i-1}(-1)^{i-r-1}(i-r+s\cdot r)e[i-r]\hat{h}[r]\right)/(s\cdot i)$;
end
output $\hat{B}_m(x_0) = x_0 - \hat{h}[m-2]/\hat{h}[m-1]$;

Remark 3.4. For a polynomial $p(x)$ of degree n, the normalized derivatives $p^{(i)}(x_0)/i!$ can be evaluated in $O(n\log^2 n)$ time (see Kung [12]). Furthermore, $\hat{h}[m]$, the m-th term of a sequence defined via a homogeneous linear recurrence relation, can be computed in $O(n\log n\log m)$ arithmetic operations (see Fiduccia [2]). Therefore, the computational complexity of $\hat{B}_m(x_0)$ is $O(n\log n(\log m + \log n))$.

Using the above algorithm, we can also derive the closed form formulas for $\hat{B}_2, \hat{B}_3$ and $\hat{B}_4$:

$$\hat{B}_2(x) = x - s\frac{p(x)}{p'(x)},$$
$$\hat{B}_3(x) = x - 2s\,p(x)\frac{p'(x)}{(1+s)p'(x)^2 - sp(x)p''(x)},$$
$$\hat{B}_4(x) = x - 3s\,p(x)\frac{(1+s)p'(x)^2 - sp(x)p''(x)}{(2s^2+3s+1)p'(x)^3 - 3s(1+s)p(x)p'(x)p''(x) + s^2p(x)^2p'''(x)}.$$

Remark 3.5. The generalized complete symmetric functions $\hat{h}_i$ also admit an upper Hessenberg determinantal form

$$\hat{h}_i = \det\begin{bmatrix} c(i,1)e_1 & c(i,2)e_2 & \dots & c(i,i-1)e_{i-1} & c(i,i)e_i \\ 1 & c(i-1,1)e_1 & \ddots & \ddots & c(i-1,i-1)e_{i-1} \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c(2,2)e_2 \\ 0 & 0 & \dots & 1 & c(1,1)e_1 \end{bmatrix},$$

where coefficients $c(i,j) = (j + s(i-j))/(s\cdot i)$. This follows from the fact that by expanding the determinant along the first row we arrive at recurrence relation (3.6).

When $s = 1$, all the coefficients $c(i,j) = 1$ and the above formula is equivalent to the determinantal representation of Basic Family given by [7].

4. The connection between B_m and $\hat{B}_m$

In this section, we show that $\hat{B}_m$ is actually the application of B_m to $p(x)^{1/s}$. We start with a generalization of Proposition 2.1.

Proposition 4.1. *Let $p(x)$ be a polynomial of degree n with complex coefficients, and let $\theta_1, \dots, \theta_n$ be its complex roots. Let $f(x) = p(x)^{1/s}$.*

Define $r_j = 1/(x - \theta_j)$, $j = 1, \dots, n$. Define

$$\tilde{e}_i = \sum_{|\lambda|=i} m_\lambda \prod_{j=1}^{l(\lambda)} \frac{(1/s)_{\lambda_j}}{\lambda_j!},$$

where m_λ is the monomial symmetric function corresponding to partition $\lambda = (\lambda_1, \lambda_2, \dots)$.

Then

$$f^{(i)}(x) = i! f(x) \tilde{e}_i(r_1, \dots, r_n),$$

where $f^{(i)}(x)$ is the i-th derivative of $f(x)$.

Proof. Write

$$f(x) = c^{1/s} \prod_{i=1}^{n} (x - \theta_i)^{1/s},$$

and apply the generalized product rule of differentiation to it. □

The generating function for $\tilde{e}_i$ is

$$\tilde{E}(t) = \sum_{r \geq 0} \tilde{e}_r t^r = \prod_{i \geq 1} (1 + x_i t)^{1/s} = E(t)^{1/s},$$

which is easy to verify using the power series expansion

$$(1 + x_i t)^{1/s} = 1 + \sum_{k=1}^{\infty} \frac{(1/s)_k}{k!} x_i^k t^k.$$

Let $\tilde{B}_m$ be the iteration function obtained by applying B_m to $p(x)^{1/s}$. Then by Algorithm 1, we have

$$\tilde{B}_m(x) = x - \frac{\tilde{h}_{m-2}(r_1, \dots, r_n)}{\tilde{h}_{m-1}(r_1, \dots, r_n)} \qquad \text{for} \quad m = 2, 3, \dots,$$

where $\tilde{h}_0 = 1$, and

$$\tilde{h}_i = \sum_{r=0}^{i-1} (-1)^{i-r-1} \tilde{e}_{i-r} \tilde{h}_r \qquad \text{for} \quad i = 1, 2, \dots.$$

Let $\tilde{H}(t)$ be the generating function for $\tilde{h}_r$. Then from the above recurrence relation between $\tilde{h}_r$ and $\tilde{e}_r$ we have $\tilde{H}(t) = \tilde{E}(-t)^{-1}$. Thus,

$$\frac{\tilde{H}'(t)}{\tilde{H}(t)} = \frac{[E(-t)^{-1/s}]'}{E(-t)^{-1/s}} = \frac{E'(-t)}{sE(-t)} = \hat{Q}(t) = \frac{\hat{H}'(t)}{\hat{H}(t)},$$

which implies $\tilde{H}(t) = \hat{H}(t)$, and $\tilde{h}_r = \hat{h}_r$ for $r = 0, 1, 2, \dots$.

Therefore, $\hat{B}_m \equiv \tilde{B}_m$; i.e., $\hat{B}_m$ is the application of B_m to $p(x)^{1/s}$.

Remark 4.2. $\tilde{e}_r$, $r = 0, 1, 2, \ldots$ are generalized elementary symmetric functions, and the recurrence relations among $\tilde{e}_r$, $\hat{h}_r$ and $\hat{q}_r$ are the same as those among the standard e_r, h_r and q_r.

An explicit formula for $\hat{B}_m$ can be obtained by expanding $\hat{h}_{m-2}$ and $\hat{h}_{m-1}$ into functions of $p(x)$ and its derivatives.

Using the relation between generating functions $\hat{H}(t)$ and $E(t)$,

$$\hat{H}(t) = \sum_{r \geq 0} \hat{h}_r t^r = E(-t)^{-1/s},$$

we can express $\hat{h}_r$ as a polynomial in e_i's directly.

Define

$$T = \sum_{r=1}^{\infty} e_r (-t)^r;$$

then $E(-t) = 1 + T$, and by Taylor expansion we have

$$\hat{H}(t) = (1+T)^{-1/s} = 1 + \sum_{i=1}^{\infty} \frac{(-1/s)_i}{i!} T^i.$$

Thus,

$$\begin{aligned}
\hat{h}_k &= (-1)^k \sum_{\substack{|\lambda|=k \\ \lambda=(\lambda_1^{a_1}\ldots\lambda_v^{a_v})}} \frac{(-1/s)_{l(\lambda)}}{l(\lambda)!} \frac{l(\lambda)!}{\prod_{j=1}^{v} a_j!} e_\lambda \\
&= (-1)^k \sum_{\substack{|\lambda|=k \\ \lambda=(\lambda_1^{a_1}\ldots\lambda_v^{a_v})}} \frac{(-1/s)_{l(\lambda)}}{\prod_{j=1}^{v} a_j!} e_\lambda. \\
&= (-1)^k \sum_{\substack{|\lambda|=k \\ \lambda=(\lambda_1^{a_1}\ldots\lambda_v^{a_v})}} (-1/s)_{l(\lambda)} \prod_{j=1}^{v} \frac{[p^{(\lambda_j)}(x)]^{a_j}}{a_j![\lambda_j! p(x)]^{a_j}}.
\end{aligned}$$

References

[1] X. Buff and C. Henriksen, On König's root-finding algorithms, Nonlinearity, 16 (2003) 989–1015. MR1975793 (2004c:37086)

[2] C.M. Fiduccia, An efficient formula for linear recurrences, SIAM J. Comput., 14 (1985) 106–112. MR774930 (86h:39001)

[3] E. Hansen and M. Patrick, A family of root finding methods, Numer. Math., 27 (1976/77) 257–269. MR0433858 (55:6829)

[4] Y. Jin and B. Kalantari, Symmetric functions and root-finding algorithms, Adv. Appl. Math., 34 (2005) 156–174. MR2102280 (2006a:05167)

[5] Y. Jin, Combinatorics of polynomial root-finding algorithms, Ph.D. dissertation, Dept. of Computer Science, Rutgers University, New Brunswick, NJ, October 2005.

[6] Y. Jin, On efficient computation and asymptotic sharpness of Kalantari's bounds for zeros of polynomials, Math. Comp., 75 (2006) 1905–1912. MR2240641 (2007e:65046)

[7] B. Kalantari, I. Kalantari, and R. Zaare-Nahandi, A basic family of iteration functions for polynomial root finding and its characterizations, J. Comput. Appl. Math., 80 (1997) 209–226. MR1455244 (98d:65066)

[8] B. Kalantari, Generalization of Taylor's theorem and Newton's method via a new family of determinantal interpolation formulas and its applications, J. Comput. Appl. Math., 126 (2000) 287–318. MR1806762 (2001m:41035)

[9] B. Kalantari and Y. Jin, On extraneous fixed-points of the basic family of iteration functions, BIT, 43 (2003) 453–458. MR2010487 (2004g:65066)

[10] B. Kalantari, On homogeneous linear recurrence relations and approximation of zeros of complex polynomials, Unusual applications of number theory, 125–143, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 64, Amer. Math. Soc., Providence, RI, 2004. MR2063208 (2005d:12001)

[11] B. Kalantari, An infinite family of bounds on zeros of analytic functions and relationship to Smale's bound, Math. Comp., 74 (2005) 841–852. MR2114651 (2005k:65093)

[12] H.T. Kung, A new upper bound on the complexity of derivative evaluation, Inform. Process. Lett., 2 (1973) 146–147. MR0347135 (49:11855)

[13] I.G. MacDonald, Symmetric Functions and Hall Polynomials, 2nd Edition, The Clarendon Press, Oxford University Press, Oxford, 1995. MR1354144 (96h:05207)

[14] E. Schröder, Ueber unendlich viele Algorithmen zur Auflösung der Gleichungen, Math. Ann., 2 (1870) 317–365. (English translation, "On infinitely many algorithms for solving equations", by G.W. Stewart, TR-92-121, Institute for Advanced Computer Studies, University of Maryland, College Park, MD, 1992.) MR1509664

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