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INTEGRAL CONDITIONS ON THE UNIFORM ASYMPTOTIC STABILITY FOR TWO-DIMENSIONAL LINEAR SYSTEMS WITH TIME-VARYING COEFFICIENTS

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Abstract. This paper is concerned with the uniform asymptotic stability of the zero solution of the linear system $\mathbf{x}' = A(t)\mathbf{x}$ with $A(t)$ being a 2×2 matrix. Our result can be used without knowledge about a fundamental matrix of the system.

1. Introduction

We consider the linear system

$$\mathbf{x}' = A(t)\mathbf{x} = \begin{pmatrix} -\,e(t) & f(t) \\ -\,g(t) & -\,h(t) \end{pmatrix}\mathbf{x}, \tag{1}$$

where the prime denotes d/dt; the coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are continuous for $t \geq 0$, and they are allowed to change sign. It is clear that system (1) has the zero solution $(x(t), y(t)) \equiv (0,0)$.

In the case where $e(t) \equiv h(t)$ and $f(t) \equiv g(t)$, a fundamental matrix $X(t)$ for system (1) is given by

$$X(t) = \begin{pmatrix} \cos G(t) & \sin G(t) \\ -\sin G(t) & \cos G(t) \end{pmatrix} \exp(-H(t)), \tag{2}$$

where

$$G(t) = \int_0^t g(\tau)d\tau \quad \text{and} \quad H(t) = \int_0^t h(\tau)d\tau.$$

Let $\|\mathbf{x}\|$ be the Euclidean norm of a vector $\mathbf{x}$. Then, we have

$$\|X(t)X^{-1}(s)\| \stackrel{\text{def}}{=} \sup_{\|\mathbf{x}\|=1} \|X(t)X^{-1}(s)\mathbf{x}\| = \exp(-H(t)+H(s))$$

for $0 \leq s \leq t < \infty$. Following Theorem 1 in the book [2, p. 54], in general, the zero solution of (1) is uniformly asymptotically stable (for the definition, see Section 2) if and only if there exist positive constants R and ρ such that

$$\|X(t)X^{-1}(s)\| \leq R\exp(-\rho(t-s)) \quad \text{for } \ 0 \leq s \leq t < \infty. \tag{3}$$

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We therefore conclude that a necessary and sufficient condition for the zero solution of (1) to be uniformly asymptotically stable is that

$$\int_s^t h(\tau)d\tau \geq \rho(t-s) - \sigma \quad \text{for } 0 \leq s \leq t < \infty \tag{4}$$

with $\rho > 0$ and $\sigma > 0$ in the special case where $e(t) \equiv h(t)$ and $f(t) \equiv g(t)$.

For example, consider system (1) with

$$e(t) = h(t) = 0.1 + \sin t \quad \text{and} \quad f(t) = g(t) = \frac{1}{2-\sin t}. \tag{5}$$

Then

$$\int_s^t h(\tau)d\tau = 0.1(t-s) - \cos t + \cos s \geq 0.1(t-s) - 2.$$

Hence, condition (4) is satisfied with $\rho = 0.1$ and $\sigma = 2$, and therefore the zero solution of (1) with (5) is uniformly asymptotically stable.

As another method, Floquet theory is available for this example, because all coefficients are periodic functions with period 2π. Note that $X(2\pi)$ is the monodromy matrix of (1), where $X(t)$ is given in (2). Let λ_1 and λ_2 be the eigenvalues of the monodromy matrix $X(2\pi)$ (λ_1 and λ_2 are often called the Floquet multipliers of (1)). It follows from Floquet theory that the zero solution of (1) is uniformly asymptotically stable if and only if the Floquet multipliers λ_1 and λ_2 have magnitudes strictly less than 1. For example, Floquet theory can be found in the books [1, 3, 4, 6, 9].

Since

$$G(t) = \int_0^t \frac{1}{2-\sin\tau}d\tau = \frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2\tan(t/2)-1}{\sqrt{3}}\right) + \frac{2m\pi}{\sqrt{3}} + \frac{\pi}{3\sqrt{3}}$$

for $m\pi < t \leq (m+1)\pi$, $m = 0, 1, 2, \ldots$, and

$$H(t) = \int_0^t (0.1 + \sin\tau)d\tau = 0.1t + 1 - \cos t$$

for $t > 0$, it follows that $G(2\pi) = 2\pi/\sqrt{3}$ and $H(2\pi) = 0.2\pi$. Hence, from (2) it turns out that the monodromy matrix

$$X(2\pi) = \begin{pmatrix} \cos\left(2\pi/\sqrt{3}\right) & \sin\left(2\pi/\sqrt{3}\right) \\ -\sin\left(2\pi/\sqrt{3}\right) & \cos\left(2\pi/\sqrt{3}\right) \end{pmatrix} e^{-0.2\pi},$$

and therefore the Floquet multipliers λ_1 and λ_2 are the roots of the equation

$$\lambda^2 - 2e^{-0.2\pi}\cos\frac{2\pi}{\sqrt{3}}\lambda + e^{-0.4\pi} = 0;$$

that is,

$$\lambda_1 = e^{-0.2\pi}\left(\cos\frac{2\pi}{\sqrt{3}} + i\sin\frac{2\pi}{\sqrt{3}}\right) \quad \text{and} \quad \lambda_2 = e^{-0.2\pi}\left(\cos\frac{2\pi}{\sqrt{3}} - i\sin\frac{2\pi}{\sqrt{3}}\right).$$

Since the Floquet multipliers have modulus smaller than 1, the zero solution of (1) with (5) is uniformly asymptotically stable.

To confirm whether Coppel's criterion (3) is satisfied or not, of course, we need a fundamental matrix for system (1). Unfortunately, however, we cannot get a concrete expression of a fundamental matrix in the general case where $e(t) \not\equiv h(t)$ or $f(t) \not\equiv g(t)$. On the other hand, if the coefficients $e(t)$, $f(t)$, $g(t)$ and $h(t)$ are periodic, then, without knowledge of a fundamental matrix of (1), the Floquet multipliers λ_1 and λ_2 can be calculated by a numerical scheme. For example, consider system (1) with

$$e(t) = 0, \quad f(t) = g(t) = \frac{1}{2 - \sin t} \quad \text{and} \quad h(t) = 0.1 + \sin t. \tag{6}$$

Then, although we cannot find a fundamental matrix of (1), the Floquet multipliers λ_1 and λ_2 can be estimated as follows:

$$\lambda_1 \approx 0.1875612224300 \quad \text{and} \quad \lambda_2 \approx 2.8443410859625 > 1.$$

Hence, the zero solution of (1) with (6) is not uniformly asymptotically stable.

The fault of Floquet theory is being unable to use it when some of the coefficients of (1) are not periodic. In this paper, we give sufficient conditions for the zero solution of (1) to be uniformly asymptotically stable, which are applicable even in cases where a fundamental matrix cannot be found and system (1) has non-periodic coefficients. In Section 2, we present the main result and give its proof. To illustrate our main result, we take some concrete examples.

2. The main result

We denote the solution of (1) through $(t_0, \mathbf{x}_0) \in [0, \infty) \times \mathbb{R}^2$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. The zero solution of (1) is said to be *uniformly stable* if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be *uniformly attractive* if there exists a $\delta_0 > 0$ such that, for every $\eta > 0$, there is a $T(\eta) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta_0$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for $t \geq t_0 + T$. The zero solution is *uniformly asymptotically stable* if it is uniformly stable and is uniformly attractive. The most important point is that δ and T can be chosen independent of t_0 in the definition of uniform asymptotic stability.

The concept of uniform asymptotic stability plays an essential role in perturbation problems. For example, if the zero solution of (1) is uniformly asymptotically stable and if $\mathbf{f}(t, \mathbf{x})$ and $\lambda(t)$ satisfy that $\|\mathbf{f}(t, \mathbf{x})\| \leq \lambda(t)\|\mathbf{x}\|$ for $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^2$, where

$$\int_0^\infty \lambda(s) ds < \infty,$$

then the zero solution of the perturbed system

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t, \mathbf{x})$$

is uniformly asymptotically stable. However, if δ and T depend on t_0, then we cannot derive this conclusion. For the details, see [7] (also [1, pp. 169–170]). For this reason, the present study has a close relationship with perturbation problems.

Let

$$\phi_+(t) = \max\{0, \phi(t)\} \quad \text{and} \quad \phi_-(t) = \max\{0, -\phi(t)\}$$

for a continuous function $\phi(t)$. Then, it follows that $\phi(t) = \phi_+(t) - \phi_-(t)$ and $|\phi(t)| = \phi_+(t) + \phi_-(t)$. The function $\phi_+(t)$ is said to be *integrally positive* if

$$\int_I \phi_+(t) dt = \infty$$

for every set $I = \bigcup_{n=1}^{\infty}[\tau_n, \sigma_n]$ such that $\tau_n + \omega < \sigma_n < \tau_{n+1}$ for some $\omega > 0$. For example, $\sin^2 t$ is an integrally positive function (see [5, 8]).

Throughout this paper, we assume that $f(t)g(t) > 0$ and $g(t)/f(t)$ is differentiable for $t \geq 0$. Then, we may define

$$\psi(t) = 2h(t) + \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'.$$

Our main result is as follows:

Theorem 1. *Suppose that $f(t)$, $g(t)$ and $h_+(t)$ are bounded for $t \geq 0$. Suppose also that*

(i) $f(t)g(t) > 0$ *for* $t \geq 0$ *and* $\liminf_{t\to\infty} f(t)g(t) > 0$;

(ii) $\int_0^\infty e_-(t)dt < \infty$, $\int_0^\infty h_-(t)dt < \infty$ *and* $\int_0^\infty \psi_-(t)dt < \infty$;

(iii) $\psi_+(t)$ *is integrally positive.*

Then the zero solution of (1) *is uniformly asymptotically stable.*

Remark 1. As a paper related to Theorem 1, we can cite Hatvani [5]. Under the assumptions that $e(t) \geq 0$, $f(t) = g(t) \geq 0$ and $h(t) \geq 0$ for $t \geq 0$, he has given some sufficient conditions guaranteeing only asymptotic stability for system (1).

Before proving the main theorem, we present some values drawn from assumptions in Theorem 1. From assumption (i) and the boundedness of $f(t)$, $g(t)$ and $h_+(t)$, we can choose positive numbers $\overline{f}$, $\underline{g}$, $\overline{h}$, k and K such that

$$|f(t)| \leq \overline{f}, \quad \underline{g} \leq |g(t)|, \quad h_+(t) \leq \overline{h} \quad \text{and} \quad k \leq \frac{f(t)}{g(t)} \leq K$$

for $t \geq 0$. We may assume that $k \leq 1 \leq K$. From assumption (ii), there exist positive constants L and M such that

$$L = \int_0^\infty (2e_-(s) + \psi_-(s))ds \quad \text{and} \quad M = \int_0^\infty h_-(s)ds.$$

It is known that $\psi_+(t)$ is integrally positive if and only if

$$\liminf_{t\to\infty} \int_t^{t+\gamma} \psi_+(s)ds > 0$$

for every $\gamma > 0$. Hence, there exist an $l > 0$ and a $\hat{t} > 0$ such that

$$\int_t^{t+1} \psi_+(s)ds \geq l \quad \text{for} \quad t \geq \hat{t}.$$

The above-mentioned values are used without notice in the proof of Theorem 1.

Proof of Theorem 1. We will prove the theorem by dividing it into seven steps.

Step 1. To prove the uniform stability of the zero solution of (1), for a given $\varepsilon > 0$, we select

$$\delta(\varepsilon) = \sqrt{\frac{k}{Ke^L}}\,\varepsilon. \tag{7}$$

Needless to say, $\delta < \varepsilon$. Let $t_0 \geq 0$ and $\mathbf{x}_0 = (x_0, y_0)$ be given. We will show that $t \geq t_0$ and $\|\mathbf{x}_0\| = \sqrt{x_0^2 + y_0^2} < \delta$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$. For convenience of notation, we write $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and $(x(t), y(t)) = \mathbf{x}(t)$.

Let

$$u(t) = \frac{f(t)}{g(t)} y^2(t) \quad \text{and} \quad v(t) = x^2(t) + u(t).$$

Then, $v(t) \geq x^2(t) + ky^2(t) \geq k\|\mathbf{x}(t)\|^2$ for $t \geq t_0$. Since

$$v'(t) = -2e(t)x^2(t) - \psi(t)u(t) \leq (2e_-(t) + \psi_-(t))v(t)$$

for $t \geq t_0$, we have

$$\begin{aligned} v(t) &\leq \exp\left(\int_{t_0}^{t} (2e_-(s) + \psi_-(s))ds\right) v(t_0) \leq e^L v(t_0) \\ &\leq e^L K(x_0^2 + y_0^2) < Ke^L \delta^2(\varepsilon) = k\varepsilon^2 \end{aligned} \tag{8}$$

for $t \geq t_0$. Hence, we obtain

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon \quad \text{for} \quad t \geq t_0,$$

and therefore, the zero solution of (1) is uniformly stable. This completes the proof of Step 1.

Hereafter, we will show that the zero solution of (1) is uniformly attractive.

Step 2. Let $\delta_0 = 1/\sqrt{Ke^L}$. For every $\eta > 0$, a number $T(\eta)$ is decided as follows. To begin with, let

$$\underline{v} = k\delta^2(\eta), \quad \mu = \min\left\{\frac{\underline{v}}{2},\ \frac{k\underline{g}^2\underline{v}}{8\overline{h}^2}\right\} \quad \text{and} \quad \tau = \hat{t} + \left[\frac{2(1+L)}{l\mu}\right] + 2,$$

where $\delta(\cdot)$ is the number given in (7) and $[c]$ means the greatest integer that is less than or equal to a real number c. Note that $\underline{v}$, μ and τ depend only on η. Consider the definite integral

$$\int_t^{t+\mu\sqrt{k}/(8\overline{f})} \psi_+(s)ds.$$

Then, the upper limit of integration depends only on η, and so does the integral. Let

$$\nu = \liminf_{t\to\infty} \frac{1}{4} \int_t^{t+\mu\sqrt{k}/(8\overline{f})} \psi_+(s)ds.$$

Since $\psi_+(t)$ is integrally positive, the number ν is positive and depends only on η. From assumptions (ii) and (iii) it turns out that there exists a positive number σ depending only on η such that

$$\int_t^{\infty} (2e_-(s) + \psi_-(s))ds \leq \min\left\{\frac{\mu}{4},\ \frac{\mu\nu}{4}\right\} \tag{9}$$

and

$$\int_t^{t+\mu\sqrt{k}/(8\overline{f})} \psi_+(s)ds \geq 2\nu \tag{10}$$

for $t \geq \sigma$, respectively. Using numbers μ, ν, σ and τ, we define

$$T = \sigma + \left(\left[\frac{4}{\mu\nu}\right] + 1\right)\left(\frac{3e^M}{\overline{h}} + \tau\right).$$

Step 3. Let $t_0 \geq 0$ and let $\mathbf{x}_0 = (x_0, y_0)$ be a point satisfying $\|\mathbf{x}_0\| = \sqrt{x_0^2 + y_0^2} < \delta_0$. Consider a solution $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ of (1) through $(t_0, \mathbf{x}_0)$. To prove the uniform attractivity of the zero solution of (1), it is enough to show that there exists a $t^* \in [t_0, t_0 + T]$ such that

$$\|\mathbf{x}(t^*)\| < \delta(\eta). \tag{11}$$

In fact, because of Step 1, if (11) holds, then any solution $\mathbf{x}(t; t^*, \mathbf{x}(t^*))$ of (1) through $(t^*, \mathbf{x}(t^*))$ satisfies that

$$\|\mathbf{x}(t; t^*, \mathbf{x}(t^*))\| < \eta \quad \text{for } \ t \geq t^*.$$

Since $t_0 + T \geq t^*$, it follows that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta \quad \text{for } \ t \geq t_0 + T.$$

There are two cases to consider: (a) $\eta \geq 1/\sqrt{k}$ and (b) $0 < \eta \leq 1/\sqrt{k}$. In case (a), by (7), we have

$$\delta_0 = \frac{1}{\sqrt{Ke^L}} \leq \sqrt{\frac{k}{Ke^L}}\eta = \delta(\eta).$$

Hence, letting $t^* = t_0$, we obtain

$$\|\mathbf{x}(t^*)\| = \|\mathbf{x}_0\| < \delta_0 \leq \delta(\eta),$$

namely, (11). This completes the proof. Thus, we have only to consider case (b) from now on. By way of contradiction, we will prove that inequality (11) holds. Suppose that

$$\|\mathbf{x}(t)\| \geq \delta(\eta) \quad \text{for } \ t_0 \leq t \leq t_0 + T.$$

Then, we have

$$0 < \underline{v} = k\delta^2(\eta) \leq k\|\mathbf{x}(t)\|^2 \leq v(t) \tag{12}$$

for $t_0 \leq t \leq t_0 + T$. Using (8) again, we get

$$v(t) \leq e^L K(x_0^2 + y_0^2) < Ke^L \delta_0^2 = 1 \quad \text{for } \ t \geq t_0. \tag{13}$$

Step 4. If $u(t) \geq \mu/2$ for any interval $[\alpha_1, \beta_1] \subset [t_0, t_0 + T]$, then $\beta_1 - \alpha_1 < \tau$, where μ and τ are numbers given in Step 2. In fact, taking into account that

$$\begin{aligned} v'(t) &= -2e(t)x^2(t) - \psi(t)u(t) \\ &= -2e(t)x^2(t) + \psi_-(t)u(t) - \psi_+(t)u(t) \end{aligned}$$

for $t \geq t_0$, from (13) we see that

$$\begin{aligned} 0 \leq \psi_+(t)u(t) &= -v'(t) - 2e(t)x^2(t) + \psi_-(t)u(t) \\ &\leq -v'(t) + (2e_-(t) + \psi_-(t))v(t) \leq -v'(t) + 2e_-(t) + \psi_-(t) \end{aligned} \tag{14}$$

for $t \geq t_0$. Integrating this inequality from α_1 to β_1 and using (12) and (13), we obtain

$$\begin{aligned} \frac{\mu}{2}\int_{\alpha_1}^{\beta_1} \psi_+(s)ds &\leq \int_{\alpha_1}^{\beta_1} \psi_+(s)u(s)ds \leq -\int_{\alpha_1}^{\beta_1} v'(s)ds + \int_{\alpha_1}^{\beta_1} (2e_-(s) + \psi_-(s))ds \\ &\leq v(\alpha_1) - v(\beta_1) + L < 1 + L. \end{aligned} \tag{15}$$

Let

$$m = \left[\frac{2(1+L)}{l\mu}\right] + 1.$$

Taking $m \geq 2(1+L)/(l\mu)$ into account, we see that

$$\begin{aligned}\int_t^{t+m}\psi_+(s)ds &= \int_t^{t+1}\psi_+(s)ds + \int_{t+1}^{t+2}\psi_+(s)ds + \cdots + \int_{t+m-1}^{t+m}\psi_+(s)ds \\ &\geq lm \geq \frac{2(1+L)}{\mu}\end{aligned}$$

for $t \geq \hat{t}$. If $\alpha_1 \geq \hat{t}$, then by (15) we have

$$\int_{\alpha_1}^{\beta_1}\psi_+(s)ds \leq \frac{2(1+L)}{\mu} \leq \int_{\alpha_1}^{\alpha_1+m}\psi_+(s)ds,$$

and therefore $\beta_1 - \alpha_1 \leq m < \tau$. If $\alpha_1 < \hat{t}$, then by (15) we have

$$\int_{\alpha_1}^{\beta_1}\psi_+(s)ds \leq \frac{2(1+L)}{\mu} \leq \int_{\hat{t}}^{\hat{t}+m}\psi_+(s)ds \leq \int_{\alpha_1}^{\alpha_1+\hat{t}+m}\psi_+(s)ds.$$

Hence, $\beta_1 - \alpha_1 \leq \hat{t} + m < \tau$. Thus, it turns out that the beginning sentence of Step 4 is true.

Step 5. If $u(t) \leq \mu$ for any interval $[\alpha_2, \beta_2] \subset [t_0, t_0 + T]$, then $\beta_2 - \alpha_2 \leq 2e^M/\overline{h}$. In fact, since

$$u(t) = \frac{f(t)}{g(t)}y^2(t), \quad v(t) = x^2(t) + u(t) \quad \text{and} \quad \mu = \min\left\{\frac{\underline{v}}{2}, \frac{k\underline{g}^2\underline{v}}{8\overline{h}^2}\right\},$$

we see that

$$|x(t)| = \sqrt{v(t) - u(t)} \geq \sqrt{\underline{v} - \mu} \geq \sqrt{\frac{\underline{v}}{2}} \tag{16}$$

and

$$|y(t)| = \sqrt{\frac{g(t)}{f(t)}u(t)} \leq \sqrt{\frac{\mu}{k}} \leq \frac{\underline{g}}{2\overline{h}}\sqrt{\frac{\underline{v}}{2}} \tag{17}$$

for $\alpha_2 \leq t \leq \beta_2$. Note that

$$y'(t) - h_-(t)y(t) = -g(t)x(t) - h_+(t)y(t)$$

for $t \geq t_0$. Then, using (16) and (17), we obtain

$$\begin{aligned}\left|\left(\exp\left(-\int_{t_0}^{t}h_-(s)ds\right)y(t)\right)'\right| &\geq \exp\left(-\int_{t_0}^{t}h_-(s)ds\right)(|g(t)||x(t)| - h_+(t)|y(t)|) \\ &\geq \frac{\underline{g}e^{-M}}{2}\sqrt{\frac{\underline{v}}{2}} > 0\end{aligned}$$

for $\alpha_2 \le t \le \beta_2$. Hence, combining this with (17), we get

$$\begin{aligned}
\frac{\underline{g}}{\overline{h}}\sqrt{\frac{\underline{v}}{2}} &\ge |y(\beta_2)| + |y(\alpha_2)| \\
&\ge \left| \exp\left(-\int_{t_0}^{\beta_2} h_-(s)ds\right) y(\beta_2) - \exp\left(-\int_{t_0}^{\alpha_2} h_-(s)ds\right) y(\alpha_2) \right| \\
&= \left| \int_{\alpha_2}^{\beta_2} \left(\exp\left(-\int_{t_0}^{t} h_-(s)ds\right) y(t) \right)' dt \right| \\
&= \int_{\alpha_2}^{\beta_2} \left| \left(\exp\left(-\int_{t_0}^{t} h_-(s)ds\right) y(t) \right)' \right| dt \ge \frac{\underline{g}e^{-M}}{2}\sqrt{\frac{\underline{v}}{2}}(\beta_2 - \alpha_2),
\end{aligned}$$

and therefore $\beta_2 - \alpha_2 \le 2e^M/\overline{h}$. Thus, it turns out that the beginning sentence of Step 5 is true.

Step 6. Let

$$J_i = \left[t_0 + \sigma + (i-1)\left(\frac{3e^M}{\overline{h}} + \tau\right),\ t_0 + \sigma + i\left(\frac{3e^M}{\overline{h}} + \tau\right) \right]$$

for any $i \in \mathbb{N}$. Then, for each $i \in \mathbb{N}$, the length of J_i is $3e^M/\overline{h} + \tau$. We can divide the interval $[t_0 + \sigma, t_0 + T]$ as follows:

$$[t_0 + \sigma, t_0 + T] = J_1 \cup J_2 \cup \cdots \cup J_{[4/(\mu\nu)]+1}.$$

Let us examine the motion of $u(t)$ in the interval J_1. It turns out that there exists a $t_1 \in [t_0 + \sigma, t_0 + \sigma + \tau] \subset J_1$ such that $u(t_1) < \mu/2$. In fact, if $u(t) \ge \mu/2$ for $t \in [t_0 + \sigma, t_0 + \sigma + \tau] \subset [t_0, t_0 + T]$, then by the conclusion of Step 4, we have

$$\tau = t_0 + \sigma + \tau - (t_0 + \sigma) = \beta_1 - \alpha_1 < \tau.$$

This is a contradiction. It also turns out that there exists a $t_2 \in [t_0 + \sigma + \tau, t_0 + \sigma + 3e^M/\overline{h} + \tau] \subset J_1$ such that $u(t_2) > \mu$. In fact, if $u(t) \le \mu$ for $t \in [t_0 + \sigma + \tau, t_0 + \sigma + 3e^M/\overline{h} + \tau] \subset [t_0, t_0 + T]$, then by the conclusion of Step 5, we have

$$\frac{3e^M}{\overline{h}} = t_0 + \sigma + \frac{3e^M}{\overline{h}} + \tau - (t_0 + \sigma + \tau) = \beta_2 - \alpha_2 \le \frac{2e^M}{\overline{h}}.$$

This is a contradiction. Hence, because of the continuity of $u(t)$, there exists an interval $[\alpha, \beta] \subset [t_1, t_2]$ such that $u(\alpha) = \mu/2$, $u(\beta) = \mu$ and

$$\frac{\mu}{2} \le u(t) \le \mu \quad \text{for } \alpha \le t \le \beta. \tag{18}$$

Hence, by (9) and (13) we have

$$\begin{aligned}
\frac{\mu}{2} = u(\beta) - u(\alpha) &= \int_\alpha^\beta u'(s)ds = \int_\alpha^\beta (-\psi(s)u(s) - 2f(s)x(s)y(s))ds \\
&\le \int_\alpha^\beta (\psi_-(s)v(s) + 2|f(s)x(s)y(s)|)ds \le \frac{\mu}{4} + 2\overline{f}\int_\alpha^\beta |x(s)y(s)|ds.
\end{aligned}$$

Consequently,

$$\frac{\mu}{8\overline{f}} \le \int_\alpha^\beta |x(s)y(s)|ds.$$

Using (13) again, we can estimate that

$$|x(t)| = \sqrt{v(t) - u(t)} < 1 \quad \text{and} \quad |y(t)| = \sqrt{\frac{g(t)}{f(t)}u(t)} \le \sqrt{\frac{v(t)}{k}} < \frac{1}{\sqrt{k}}$$

for $t \ge t_0$. We therefore conclude that

$$\frac{\mu\sqrt{k}}{8\overline{f}} < \beta - \alpha.$$

Step 7. From the conclusion of Step 6 with (9), (10), (14) and (18) it turns out that

$$\begin{aligned}
\mu\nu &\le \frac{\mu}{2}\int_\alpha^{\alpha+\mu\sqrt{k}/(8\overline{f})} \psi_+(s)ds \le \frac{\mu}{2}\int_\alpha^\beta \psi_+(s)ds \\
&\le \int_\alpha^\beta \psi_+(s)u(s)ds \le \int_\alpha^\beta (-v'(s) + 2e_-(s) + \psi_-(s))ds \\
&= v(\alpha) - v(\beta) + \int_\alpha^\beta (2e_-(s) + \psi_-(s))ds \le v(\alpha) - v(\beta) + \frac{\mu\nu}{4}.
\end{aligned}$$

Hence, we have

$$v(\beta) - v(\alpha) \le -\frac{3\mu\nu}{4}.$$

Using (9) and (14) again, we get

$$v(\alpha) - v(t_0 + \sigma) = \int_{t_0+\sigma}^\alpha v'(s)ds \le \int_{t_0+\sigma}^\alpha (2e_-(s) + \psi_-(s))ds \le \frac{\mu\nu}{4}$$

and

$$\begin{aligned}
v\left(t_0 + \sigma + \frac{3e^M}{\overline{h}} + \tau\right) - v(\beta) &= \int_\beta^{t_0+\sigma+3e^M/\overline{h}+\tau} v'(s)ds \\
&\le \int_\beta^{t_0+\sigma+3e^M/\overline{h}+\tau} (2e_-(s) + \psi_-(s))ds \le \frac{\mu\nu}{4}.
\end{aligned}$$

We therefore conclude that

$$\begin{aligned}
\int_{J_1} v'(s)ds &= v\left(t_0 + \sigma + \frac{3e^M}{\overline{h}} + \tau\right) - v(\beta) + v(\beta) - v(\alpha) + v(\alpha) - v(t_0 + \sigma) \\
&\le \frac{\mu\nu}{4} - \frac{3\mu\nu}{4} + \frac{\mu\nu}{4} = -\frac{\mu\nu}{4}.
\end{aligned}$$

By means of the same process as in the proof of Steps 6 and 7, we see that

$$\int_{J_i} v'(s)ds \le -\frac{\mu\nu}{4} \quad \text{for } 1 \le i \le [4/(\mu\nu)] + 1,$$

and therefore

$$v(t_0 + T) - v(t_0 + \sigma) = \sum_{i=1}^{[4/(\mu\nu)]+1} \int_{J_i} v'(s)ds \le -\frac{\mu\nu}{4}\left(\left[\frac{4}{\mu\nu}\right] + 1\right) < -1.$$

Hence, from (13) it follows that

$$v(t_0 + T) < v(t_0 + \sigma) - 1 < 0.$$

This contradicts the fact that $v(t) \ge 0$ for $t \ge t_0$. Thus, in case (b) as well as in case (a), inequality (11) holds. The proof of Theorem 1 is thus complete. $\square$

3. Examples

We illustrate our main result with simple examples to which Coppel's criterion and Floquet theory cannot be applied. To present examples, we define a function as follows: let r be a number satisfying $0 < r < 1$ and let

$$p(t) = \begin{cases} \dfrac{t}{2-r^n} + 2(n-1)\left(1 - \dfrac{1}{2-r^n}\right) & \text{if } 2(n-1) \le t < 2n - r^n, \\ \dfrac{t}{r^n} + 2n\left(1 - \dfrac{1}{r^n}\right) & \text{if } 2n - r^n \le t < 2n \end{cases}$$

for any $n \in \mathbb{N}$. It is clear that the graph of $p(t)$ is a broken line (see Figure 1(a) below). As shown in Figure 1(b) below, the composite function $\sin(p(t)\pi)$ changes sign, but it is not periodic and not even almost periodic. It is easy to check that $\max\{0, \sin(p(t)\pi)\}$ is an integrally positive function and $\max\{0, -\sin(p(t)\pi)\}$ is an integrable function.

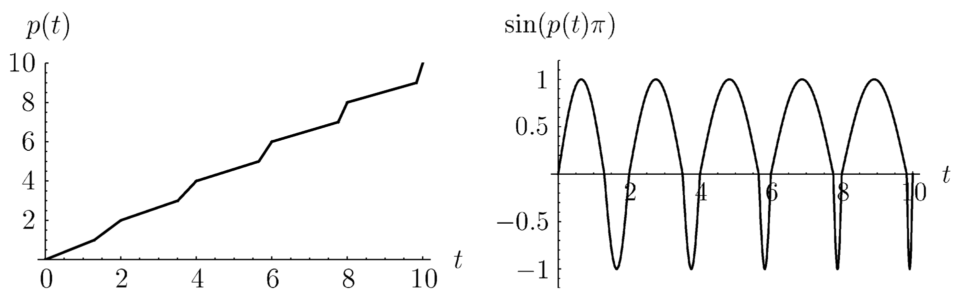

Figure 1. (a) The graph of $p(t)$ with $r = 0.7$; (b) the graph of $\sin(p(t)\pi)$ with $r = 0.7$

Example 1. Consider system (1) with

$$e(t) = 0, \quad f(t) = g(t) = 1 \quad \text{and} \quad h(t) = \sin(p(t)\pi). \tag{19}$$

Then the zero solution is uniformly asymptotically stable.

It is clear that $f(t)$, $g(t)$ and $h_+(t)$ are bounded and $g(t)/f(t)$ is differentiable for $t \ge 0$, and assumption (i) is satisfied. Assumptions (ii) and (iii) are also satisfied. In fact, taking $\psi(t) = 2h(t)$ into account, we see that

$$\int_0^\infty e_-(t)dt = 0, \quad \int_0^\infty h_-(t)dt < \sum_{i=1}^\infty r^i = \frac{r}{1-r}, \quad \int_0^\infty \psi_-(t)dt < \frac{2r}{1-r}$$

and $\psi_+(t)$ is integrally positive. Thus, by virtue of Theorem 1, we conclude that the zero solution is uniformly asymptotically stable.

Example 2. Consider system (1) with

$$e(t) = h(t) = \sin(p(t)\pi), \quad f(t) = 1 \quad \text{and} \quad g(t) = \frac{1+t}{2+t}. \tag{20}$$

Then the zero solution is uniformly asymptotically stable.

Since $\psi(t) = 2h(t) + 1/((1+t)(2+t))$, it turns out that

$$\psi_+(t) > 2h_+(t) \quad \text{and} \quad \psi_-(t) < 2h_-(t)$$

for $t \geq 0$. Hence, it is easy to confirm that all of the assumptions in Theorem 1 are satisfied. We omit the details.

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