PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 7, July 2010, Pages 2335–2339
S 0002-9939(10)10288-3
Article electronically published on February 4, 2010

THE WEAK LEFSCHETZ PROPERTY AND POWERS OF LINEAR FORMS IN $\mathbb{K}[x, y, z]$

HAL SCHENCK AND ALEXANDRA SECELEANU

(Communicated by Bernd Ulrich)

ABSTRACT. We show that an Artinian quotient of an ideal $I \subseteq \mathbb{K}[x, y, z]$ generated by powers of linear forms has the Weak Lefschetz Property. If the syzygy bundle of I is semistable, the property follows from results of Brenner-Kaid. Our proof works without this hypothesis, which typically does not hold.

1. Introduction

Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_r]$ be an ideal such that $A = S/I$ is Artinian. Then A has the *Weak Lefschetz Property* (WLP) if there is an $\ell \in S_1$ such that for all m, the map μ_ℓ

$$A_m \xrightarrow{\cdot \ell} A_{m+1}$$

is either injective or surjective. We assume $char(\mathbb{K}) = 0$; as shown in [6], WLP behaves in very subtle ways in positive characteristic. In [1], Anick shows that if $r = 3$ and I is generated by generic forms, then A has WLP. In [5], Harima-Migliore-Nagel-Watanabe introduced the syzygy bundle of I to study the WLP, and this bundle also plays a key role in recent work of Brenner-Kaid [3].

Definition 1.1. If $I = \langle f_1, \ldots, f_n \rangle$ is $\langle x_1, \ldots, x_r \rangle$-primary, and $deg(f_i) = d_i$, then the syzygy bundle $\mathcal{S}(I) = \widetilde{Syz(I)}$ is a rank $n - 1$ bundle defined via

$$0 \longrightarrow Syz(I) \longrightarrow \bigoplus_{i=1}^{n} S(-d_i) \xrightarrow{[f_1,\ldots,f_n]} S.$$

The cokernel of the rightmost map is S/I, which vanishes as a sheaf.

Definition 1.2. A vector bundle $\mathcal{E}$ on projective space is said to be semistable if for every coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$

$$\frac{c_1(\mathcal{F})}{rk(\mathcal{F})} \leq \frac{c_1(\mathcal{E})}{rk(\mathcal{E})}, \text{ where } c_1 \text{ denotes the first Chern class.}$$

By a result of Grothendieck, every vector bundle on $\mathbb{P}^1$ splits as a sum of line bundles [7], so for a given line L, if $\mathcal{E}$ has rank k, then

$$\mathcal{E}|_L \simeq \bigoplus \mathcal{O}_L(a_i), \text{ with } a_1 \geq a_2 \geq \cdots \geq a_k.$$

Received by the editors July 9, 2009, and, in revised form, November 6, 2009.
2010 *Mathematics Subject Classification.* Primary 13D02, 14J60, 13C13, 13C40, 14F05.
Key words and phrases. Weak Lefschetz property, Artinian algebra, powers of linear forms.
The first author was supported by NSF grant no. 07–07667 and NSA grant no. 904-03-1-0006.

If $\mathcal{E}$ is semistable, then [7] for a generic line L the tuple $(a_1, a_2, \ldots, a_k)$ does not vary, $(a_1, a_2, \ldots, a_k)$ is the *generic splitting type* of $\mathcal{E}$, and if $\mathcal{E}$ is semistable, then $|a_i - a_{i+1}| \leq 1$.

For the remainder of the paper we focus on the case $r = 3$, so henceforth S denotes $\mathbb{K}[x, y, z]$. In [3], Brenner and Kaid show that if $A = S/I$ is Artinian with $\mathcal{S}(I)$ semistable of generic splitting type $(a_1, \ldots, a_{n-1})$, then A has WLP iff $|a_1 - a_{n-1}| \leq 1$. As a corollary of this, they recover a result of Harima-Migliore-Nagel-Watanabe [5] that every Artinian complete intersection in S has WLP. They also completely characterize WLP for almost complete intersections, showing that in this case if $\mathcal{S}(I)$ is not semistable, then WLP holds.

It is clear from the definition that semistability can be a difficult property to show. In this paper, we examine a special class of ideals in S which falls outside the classes considered by Anick, Brenner-Kaid, and Harima-Migliore-Nagel-Watanabe. Our main result is

Theorem. *An Artinian quotient of* $\mathbb{K}[x, y, z]$ *by powers of linear forms has WLP.*

2. Proof of the Theorem

We begin by recalling the setup of [3]. Let ℓ be a generic form in S_1 with $L = V(\ell)$, and let I be an ideal such that $A = S/I$ is Artinian. Taking cohomology of the defining sequence for $\mathcal{S}(I)$,

$$0 \longrightarrow \mathcal{S}(I)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^2}(m - d_i) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow 0,$$

we see that

$$A = \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{S}(I)(m)).$$

On the other hand, since $\mathcal{S}(I)$ is a bundle, tensoring the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m+1) \longrightarrow \mathcal{O}_L(m+1) \longrightarrow 0$$

with $\mathcal{S}(I)$ gives the exact sequence

$$0 \longrightarrow \mathcal{S}(I)(m) \longrightarrow \mathcal{S}(I)(m+1) \longrightarrow \mathcal{S}(I)|_L(m+1) \longrightarrow 0.$$

The long exact sequence in cohomology yields a sequence

$$\begin{array}{l} 0 \longrightarrow H^0(\mathcal{S}(I)(m)) \longrightarrow H^0(\mathcal{S}(I)(m+1)) \xrightarrow{\phi_m} H^0(\mathcal{S}(I)|_L(m+1)) \longrightarrow \\ H^1(\mathcal{S}(I)(m)) \xrightarrow{\cdot \ell} H^1(\mathcal{S}(I)(m+1)) \longrightarrow H^1(\mathcal{S}(I)|_L(m+1)) \xrightarrow{\psi_m} \\ H^2(\mathcal{S}(I)(m)) \longrightarrow H^2(\mathcal{S}(I)(m+1)) \longrightarrow H^2(\mathcal{S}(I)|_L(m+1)) = 0. \end{array}$$

Therefore injectivity of μ_ℓ follows from surjectivity of ϕ_m, and surjectivity of μ_ℓ from injectivity of ψ_m. Our next step is to analyze $\mathcal{S}(I)|_L$. To do this, we tensor the defining sequence

$$0 \longrightarrow Syz(I) \longrightarrow \bigoplus_{i=1}^{n} S(-d_i) \longrightarrow I \longrightarrow 0$$

with S/ℓ, yielding the sequence

$$0 \longrightarrow Tor_1^S(I, S/\ell) \longrightarrow Syz(I) \otimes S/\ell \longrightarrow \bigoplus_{i=1}^{n} S/\ell(-d_i) \longrightarrow I \otimes S/\ell \longrightarrow 0.$$

Now $Tor_1^S(I, S/\ell) = 0$, since it is the kernel of

$$I \xrightarrow{\cdot \ell} I(1).$$

After a change of coordinates, $\ell = x$ is generic. Reducing the defining equations of I mod x, we see that $Syz(I) \otimes S/\ell$ is the module of syzygies on $I \otimes S/\ell$, an ideal generated by powers of linear forms in two variables. We make use of the following pair of lemmas from [4] on ideals

$$J = \langle l_1^{\alpha_1}, \dots, l_t^{\alpha_t} \rangle \subseteq \mathbb{K}[y, z] = R,$$

generated by powers of pairwise linearly independent forms.

Lemma 2.1. *Let $0 < \alpha_1 \leq \alpha_2 \cdots \leq \alpha_t$. Then for $m \geq 2$:*

$$l_{m+1}^{\alpha_{m+1}} \notin \langle l_1^{\alpha_1}, \dots, l_m^{\alpha_m} \rangle \Leftrightarrow \alpha_{m+1} \leq \frac{\sum_{i=1}^{m} \alpha_i - m}{m-1}.$$

Lemma 2.2. *If J is minimally generated by $\{l_1^{\alpha_1}, \dots, l_t^{\alpha_t}\}$ and $t \geq 2$, then the socle degree of $\mathbb{K}[y,z]/J$ is $\omega = \left\lfloor \frac{\sum_{i=1}^{t} \alpha_i - t}{t-1} \right\rfloor$, and J has minimal free resolution*

$$0 \longrightarrow R(-\omega-2)^a \oplus R(-\omega-1)^{t-1-a} \longrightarrow \bigoplus_{i=1}^{t} R(-\alpha_i) \longrightarrow J \longrightarrow 0,$$

where $a = \sum_{i=1}^{t} \alpha_i - (t-1)(\omega+1)$.

Proposition 2.3. *If $I = \langle l_1^{d_1}, \dots, l_n^{d_n} \rangle \subseteq S$ satisfies*

$$d_{t+1} \leq \frac{\sum_{i=1}^{t} d_i - t}{t-1},$$

for all $t > 1$, then S/I has WLP.

Proof. By Lemma 2.1, the restriction $I \otimes S/\ell$ has the same number of minimal generators and degrees as I, and so it follows from Lemma 2.2 that

$$\mathcal{S}(I)|_L \simeq \mathcal{O}_L(-\omega-2)^a \oplus \mathcal{O}_L(-\omega-1)^{n-1-a}, \tag{1}$$

with

$$\omega = \left\lfloor \frac{\sum_{i=1}^{n} d_i - n}{n-1} \right\rfloor \text{ and } a = \sum_{i=1}^{n} d_i - (n-1)(\omega+1).$$

Suppose $m < \omega$. Then

$$H^0(\mathcal{S}(I)|_L(m+1)) \simeq H^0(\mathcal{O}_L(m-1-\omega))^a \oplus H^0(\mathcal{O}_L(m-\omega))^{n-1-a} = 0,$$

so μ_ℓ is injective. If instead $m \geq \omega$, by Serre duality

$$H^1(\mathcal{S}(I)|_L(m+1) \simeq H^0(\mathcal{O}_L(-m-1+\omega))^a \oplus H^0(\mathcal{O}_L(-m-2+\omega))^{n-1-a} = 0,$$

and thus μ_ℓ is surjective. □

Theorem 2.4. *If $I = \langle l_1^{d_1}, \dots, l_n^{d_n} \rangle \subseteq S$, then S/I has WLP.*

Proof. If

$$d_{t+1} \le \frac{\sum_{i=1}^{t} d_i - t}{t-1},$$

for all $t > 1$, then the theorem follows from Proposition 2.3, so let $d_1 \le d_2 \le \cdots \le d_n$ and suppose that $t+1$ is the first index where

$$d_{t+1} > \frac{\sum_{i=1}^{t} d_i - t}{t-1} \ge \left\lfloor \frac{\sum_{i=1}^{t} d_i - t}{t-1} \right\rfloor = \omega.$$

Thus, $d_i \ge \omega+1$ when $i \ge t+1$. If $d_i \in \{\omega+1, \omega+2\}$ for all $i \ge t+1$, then the shifts appearing in $\mathcal{S}(I)|_L$ are as in equation (1), so the argument of Proposition 2.3 works. Suppose $k \ge t+1$ is the first index such that $d_k \ge \omega+3$. Then

$$\mathcal{S}(I)|_L \simeq \mathcal{O}_L(-\omega-1)^a \oplus \mathcal{O}_L(-\omega-2)^b \bigoplus_{i=k}^{n} \mathcal{O}_L(-d_i),$$

with $a+b = k-2$. If $m < \omega$, the argument of Proposition 2.3 shows that μ_ℓ is injective, so suppose $m \ge \omega$. We show ψ_m is injective by a dimension computation. From the defining sequence for $\mathcal{S}(I)$ we obtain

$$0 \longrightarrow H^2(\mathcal{S}(I)(m)) \longrightarrow H^2(\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^2}(m-d_i)) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^2}(m)) \longrightarrow 0.$$

By Serre duality, $h^2(\mathcal{O}_{\mathbb{P}^2}(m)) = h^0(\mathcal{O}_{\mathbb{P}^2}(-m-3)) = 0$ since $m \ge \omega > 0$, and

$$h^2(\mathcal{S}(I)(m)) = \sum_{i=1}^{n} \binom{d_i-m-1}{2} \text{ and } h^2(\mathcal{S}(I)(m+1)) = \sum_{i=1}^{n} \binom{d_i-m-2}{2}.$$

Thus,

$$\dim \operatorname{im}(\psi_m) = \sum_{i=1}^{n} \max(d_i-m-2, 0).$$

The contributions come from those $d_i \ge m+3 \ge \omega+3$. Our assumption is that

$$\mathcal{S}(I)|_L \simeq \mathcal{O}_L(-\omega-1)^a \oplus \mathcal{O}_L(-\omega-2)^b \bigoplus_{i=k}^{n} \mathcal{O}_L(-d_i),$$

with $a+b = k-2$. Thus for $m \ge \omega$,

$$\begin{aligned} h^1(\mathcal{S}(I)|_L(m+1)) &= \sum_{i=k}^{n} h^1(\mathcal{O}_L(-d_i+m+1)) + h^1(\mathcal{O}_L(m-\omega)^a) + h^1(\mathcal{O}_L(m-\omega-1)^b) \\ &= \sum_{i=k}^{n} h^1(\mathcal{O}_L(-d_i+m+1)) \\ &= \sum_{i=k}^{n} h^0(\mathcal{O}_L(d_i-m-3)) \\ &= \sum_{i=k}^{n} \max(d_i-m-2, 0). \end{aligned}$$

Since this is equal to $\dim \operatorname{im}(\psi_m)$, ψ_m is an inclusion, so that μ_ℓ is surjective. □

It follows from Theorem 2.4 that ideals generated by powers of linear forms in $\mathbb{K}[x,y,z]$ which have unstable syzygy bundles always have WLP. As noted earlier Brenner and Kaid show that almost complete intersections with unstable syzygy bundles have WLP. Thus, it seems reasonable to ask if every ideal in $\mathbb{K}[x,y,z]$ with unstable syzygy bundle has WLP.

Example 2.5. For the ideal $I = \langle x^5, y^5, z^5, x^2yz, xy^2z \rangle \subseteq \mathbb{K}[x, y, z]$, $\mathcal{S}(I)$ is not semistable, by Proposition 2.2 of [2]. The Hilbert function of A is $(1, 3, 6, 10, 13, 13, 10, 6, 3)$, and a computation shows the map from $A_4 \to A_5$ is not full rank, so A does not have WLP.

As noted, Theorem 2.4 need not hold for more than three variables:

Example 2.6. The ring $A = \mathbb{K}[x, y, z, w]/\langle x^3, y^3, z^3, w^3, (x + y + z + w)^3 \rangle$ appears in Example 8.1 of [6] and does not have WLP. The Hilbert function of A is $(1, 4, 10, 15, 15, 6)$, and a computation shows the map from $A_3 \to A_4$ is not full rank. So WLP need not hold for powers of linear forms in more than three variables.

Concluding remarks

The proof of Theorem 2.4 works for any ideal which has the same splitting type as an ideal generated by powers of linear forms, so it would be interesting to find families of such ideals. In light of Example 2.6, we ask: are there reasonable additional hypotheses so that a version of Theorem 2.4 does hold in more than three variables? A second question is if ideals generated by powers of linear forms possess the Strong Lefschetz Property. As pointed out by the referee, the answer is no: SLP fails for the ideal generated by cubes of four general linear forms and for multiplication by a cube of a linear form. However, multiplication by a general form of degree three *does* have maximal rank, so we ask: does multiplication by a general form of any degree induce a multiplication having maximal rank?

Acknowledgements

Computations were performed using Macaulay2, by Grayson and Stillman, available at: `http://www.math.uiuc.edu/Macaulay2/`. Scripts to analyze WLP are available at: `http://www.math.uiuc.edu/∼asecele2`. We thank an anonymous referee for thoughtful comments.

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Department of Mathematics, University of Illinois, Urbana, Illinois 61801
E-mail address: `schenck@math.uiuc.edu`

Department of Mathematics, University of Illinois, Urbana, Illinois 61801
E-mail address: `asecele2@math.uiuc.edu`