UNIFORM ASYMPTOTIC EXPANSIONS OF THE TRICOMI-CARLITZ POLYNOMIALS

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Abstract. The Tricomi-Carlitz polynomials satisfy the second-order linear difference equation
\[(n + 1)f^{(\alpha)}_{n+1}(x) - (n + \alpha) x f^{(\alpha)}_n(x) + f^{(\alpha)}_{n-1}(x) = 0, \quad n \geq 1,\]
with initial values \(f^{(\alpha)}_0(x) = 1\) and \(f^{(\alpha)}_1(x) = \alpha x\), where \(x\) is a real variable and \(\alpha\) is a positive parameter. An asymptotic expansion is derived for these polynomials by using the turning-point theory for three-term recurrence relations developed by Wang and Wong [Numer. Math. 91 (2002) and 94 (2003)]. The result holds uniformly in regions containing the critical values \(x = \pm 2/\sqrt{\nu}\), where \(\nu = n + 2\alpha - 1/2\).

1. Introduction

The Tricomi polynomials are defined by
\[(1.1) \quad t^{(\alpha)}_n(x) = \sum_{k=0}^{n} \binom{x-\alpha}{k} \frac{x^{n-k}}{(n-k)!}, \quad n = 0, 1, 2, \ldots,\]
which satisfy the recurrence relation
\[(1.2) \quad (n + 1)t^{(\alpha)}_{n+1}(x) - (n + \alpha) t^{(\alpha)}_n(x) + x t^{(\alpha)}_{n-1}(x) = 0, \quad n \geq 1,\]
with initial values \(t^{(\alpha)}_0(x) = 1, \quad t^{(\alpha)}_1(x) = \alpha\). Tricomi [7] observed that \(\{t^{(\alpha)}_n(x)\}\) is not a system of orthogonal polynomials, since the recurrence relation (1.2) fails to have the required form [6, p.135]. However, Carlitz [1] discovered that if one sets
\[(1.3) \quad f^{(\alpha)}_n(x) = x^n t^{(\alpha)}_n(x^{-2}),\]
then \(f^{(\alpha)}_n(x)\) satisfies
\[(1.4) \quad (n + 1)f^{(\alpha)}_{n+1}(x) - (n + \alpha) x f^{(\alpha)}_n(x) + f^{(\alpha)}_{n-1}(x) = 0, \quad n \geq 1,\]
with initial values $f^{(\alpha)}_0(x) = 1$, $f^{(\alpha)}_1(x) = \alpha x$. Furthermore, he gave the orthogonal relation

$$
\int_{-\infty}^{\infty} f^{(\alpha)}_m(x) f^{(\alpha)}_n(x) d\psi^{(\alpha)}(x) = \frac{2e^\alpha}{(n+\alpha)n!} \delta_{mn},
$$

where $\psi^{(\alpha)}(x)$ is the step function whose jumps are

$$
d\psi^{(\alpha)}(x) = \frac{(k+\alpha)^{k-1}e^{-k}}{k!} \quad \text{at} \quad x = x_k = \pm(k+\alpha)^{-1/2}, \quad k = 0, 1, 2, \ldots.
$$

The generating function

$$
\exp \left\{ \frac{w}{x} + \frac{1-\alpha x^2}{x^2} \log(1-wx) \right\} = \sum_{n=0}^{\infty} f^{(\alpha)}_n(x) w^n
$$

can be derived from the recurrence relation (1.4) when $x \neq 0$. The series in (1.7) is convergent for $|wx| < 1$.

Asymptotic behavior of $f^{(\alpha)}_n(x)$ has been investigated by Goh and Wimp [2] [3]. In their first paper, they used an elementary approach to show that

$$
f^{(\alpha)}_n(y/\sqrt{\alpha}) = e^{\alpha/y^2} \frac{x^{n/2}y^{-\alpha(y^2)/y^2-1}}{\Gamma(n+\alpha(y^2)/y^2)}, \quad \text{as} \quad n \to \infty,
$$

uniformly for all $y$ in a compact set $K \subseteq \mathbb{C}\setminus[-1, 1]$. (There is a minor error in the statement of their result; the factor $\alpha^{-n/2}$ in the denominator on the left-hand side of (1.8) is missing in their equation (23) in [2].) The validity of (1.8) can be verified by a direct application of Darboux’s method [10] p. 116]. Goh and Wimp also observed that all zeros of $f^{(\alpha)}_n(y/\sqrt{\alpha})$ lie in the interval $[-1, 1]$. In their second paper, they used a saddle-point method to study the asymptotics of $f^{(\alpha)}_n(z/\sqrt{n})$ for $z$ in neighborhoods of $z = \pm i/2$. Note that the scales in their two papers are different; in [2] the scale is $y/\sqrt{\alpha}$, whereas in [3] the scale is $z/\sqrt{n}$. The behavior of $f^{(\alpha)}_n(x)$ as $\alpha \to \infty$ has been studied by López and Temme [4]. Their result is expressed in terms of Hermite polynomials.

The purpose of this paper is to present an asymptotic expansion for $f^{(\alpha)}_n(t/\sqrt{\nu})$, which holds uniformly for $t$ in $[0, \infty)$, where $\nu = n + 2\alpha - 1/2$. In view of the reflection formula $f^{(\alpha)}_n(-x) = (-1)^n f^{(\alpha)}_n(x)$, a corresponding result can be given for $t$ in $(-\infty, 0]$. Note that our result is not covered in the two papers of Goh and Wimp; in fact, it complements those in [2] and [3]. We also point out that the Tricomi-Carlitz polynomials do not satisfy a second-order differential equation; hence the powerful tools developed for differential equations (see, e.g., [5]) are not applicable. Our approach is to use a turning-point theory recently introduced by Wang and Wong [8] [9] for three-term recurrence relations.

2. Difference equation

Returning to (1.4), we write

$$f^{(\alpha)}_{n+1}(x) - \frac{n+\alpha}{n+1} x f^{(\alpha)}_n(x) + \frac{1}{n+1} f^{(\alpha)}_{n-1}(x) = 0$$

and introduce the sequence $\{K_n\}$ defined by

$$n+1 K_{n+1} = K_{n-1}$$
with $K_0 = 1$ and $K_1 = \sqrt{2/\pi}$. Induction shows that
\begin{equation}
K_n = \frac{1}{2^{n/2} \Gamma \left( \frac{1}{2} n + 1 \right)}.
\end{equation}

With the notation
\begin{equation}
F^{(\alpha)}_n(x) := \frac{f^{(\alpha)}_n(x)}{K_n},
\end{equation}
equation (2.1) can be put in the canonical form considered in [9],
\begin{equation}
F^{(\alpha)}_{n+1}(x) - (A_n x + B_n) F^{(\alpha)}_n(x) + F^{(\alpha)}_{n-1}(x) = 0
\end{equation}
with
\begin{equation}
A_n = \frac{n + \alpha}{n + 1} K_{n+1} = \frac{n + \alpha \Gamma \left( \frac{1}{2} (n + 1) \right)}{\sqrt{2} \Gamma \left( \frac{1}{2} n + 1 \right)}
\end{equation}
and $B_n = 0$. To find an asymptotic expansion for $A_n$, we recall the well-known result [10, p. 47]
\begin{equation}
\frac{\Gamma \left( \frac{1}{2} n + \frac{1}{4} \right)}{\Gamma \left( \frac{1}{2} n + 1 \right)} \sim \sqrt{\frac{2}{n}} \left[ 1 - \frac{1}{4} - \frac{1}{32} n + \cdots \right].
\end{equation}
Thus,
\begin{equation}
A_n \sim n^{1/2} \left[ 1 + \left( \alpha - \frac{1}{4} \right) - \frac{1}{32} \frac{1}{4} n^2 + \cdots \right].
\end{equation}

In terms of the notation
\begin{equation}
A_n \sim \nu^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{\nu^s} \quad \text{and} \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{\nu^s},
\end{equation}
used in [9], we have
\begin{equation}
\theta = -\frac{1}{2}, \quad \alpha_0 = 1, \quad \alpha_1 = \alpha - \frac{1}{4}, \quad \alpha_2 = \left( \frac{1}{32} - \frac{\alpha}{4} \right), \ldots
\end{equation}
and $\beta_0 = \beta_1 = \beta_2 = \cdots = 0$. If these expansions are recast in the form
\begin{equation}
A_n \sim \nu^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha'_s}{\nu^s} \quad \text{and} \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{\nu^s},
\end{equation}
where $\nu = n + \tau_0$ and $\tau_0$ is some fixed real number to be determined, it is easily found that
\begin{equation}
\alpha'_0 = 1, \quad \alpha'_1 = \left( \alpha - \frac{1}{4} \right) - \frac{\tau_0}{2}, \ldots
\end{equation}
and $\beta'_0 = \beta'_1 = \beta'_2 = \cdots = 0$. To apply the result in [9], we first choose $\tau_0$ so that $\alpha'_1 = 0$. From (2.11), it is obvious that the choice is
\begin{equation}
\tau_0 = 2\alpha - \frac{1}{2}.
\end{equation}

According to equation (2.4) in [9], the characteristic equation is
\begin{equation}
\lambda^2 - t\lambda + 1 = 0,
\end{equation}
where $t$ is the rescaled variable $x = \nu^{-\frac{1}{2}} t$. The two roots of this equation are
\begin{equation}
\lambda_{\pm} = \frac{1}{2} (t \pm \sqrt{t^2 - 4}).
\end{equation}
The points $t_\pm = \pm 2$ where the two roots coincide are called the turning points of equation (2.4). In view of the symmetry relation $F_n^{(\alpha)}(-x) = (-1)^n F_n^{(\alpha)}(x)$, we may restrict ourselves just to the case $0 < t < \infty$.

We now define the function $\zeta(t)$ introduced in [9] (4.10). With $t_+ = 2$, $\theta = -\frac{1}{2}$, $\alpha'_0 = 1$ and $\beta'_0 = 0$, this function is given by

$$\frac{2}{3} [\zeta(t)]^{3/2} := \log \frac{t + \sqrt{t^2 - 4}}{2} - \frac{1}{t^2} \int_2^t \frac{s^2}{\sqrt{s^2 - 4}} ds, \quad t \geq 2,$$

and

$$\frac{2}{3} [-\zeta(t)]^{3/2} := t^{-2} \int_2^t \frac{s^2}{\sqrt{4 - s^2}} ds - \cos^{-1} \frac{t}{2}, \quad 0 < t < 2.$$

By direct calculation, one obtains

$$\frac{2}{3} [\zeta(t)]^{3/2} = \log \frac{t + \sqrt{t^2 - 4}}{2} - \frac{1}{t^2} \left[ \frac{1}{2} t \sqrt{t^2 - 4} + 2 \log |t + \sqrt{t^2 - 4}| - 2 \ln 2 \right],$$

for $t \geq 2$, and

$$\frac{2}{3} [-\zeta(t)]^{3/2} = \frac{1}{t^2} \left[ -2 \sin^{-1} \frac{t}{2} + \pi + \frac{t}{2} \sqrt{4 - t^2} \right] - \cos^{-1} \frac{t}{2},$$

for $0 < t < 2$. We also define the functions $H_0(\zeta)$ and $\Phi(\zeta)$ introduced in [9] (4.19) and (4.28)]. In the present situation, these functions are given by

$$H_0(\zeta) = -\sqrt{\frac{t^2 - 4}{4 \zeta}} \quad \text{and} \quad \Phi(\zeta) = 0,$$

where $\zeta$ is the function defined in (2.15) and (2.16). Note that in our special case, $\alpha'_1 = \beta'_1 = 0$; hence, according to the definition of $\Phi(\zeta)$ given in (4.28) of [9], the second equation in (2.17) holds for $0 < t < \infty$, instead of $t \geq \delta$, $0 < \delta < 2$.

With this preliminary work done, we can now apply the main result in [9] to conclude that there are constants $C_1(x)$ and $C_2(x)$ such that the polynomials $F_n^{(\alpha)}(x)$ in (2.4) can be expressed as

$$F_n^{(\alpha)}(x) = C_1(x) P_n(x) + C_2(x) Q_n(x),$$

where, with $x = \nu^{-\frac{1}{2}} t$, $P_n(x)$ and $Q_n(x)$ have the asymptotic expansions

$$P_n(\nu^{-\frac{1}{2}} t) \sim \left( \frac{4 \zeta}{t^2 - 4} \right)^{\frac{1}{4}} \left[ A_\nu \left( \nu^{\frac{1}{2}} \zeta \right) \sum_{s=0}^\infty \frac{\tilde{A}_s(\zeta)}{\nu^{s+\frac{1}{2}}} + A_\nu' \left( \nu^{\frac{1}{2}} \zeta \right) \sum_{s=0}^\infty \frac{\tilde{B}_s(\zeta)}{\nu^{s+\frac{1}{2}}} \right]$$

and

$$Q_n(\nu^{-\frac{1}{2}} t) \sim \left( \frac{4 \zeta}{t^2 - 4} \right)^{\frac{1}{4}} \left[ B_\nu \left( \nu^{\frac{1}{2}} \zeta \right) \sum_{s=0}^\infty \frac{\tilde{A}_s(\zeta)}{\nu^{s+\frac{1}{2}}} + B_\nu' \left( \nu^{\frac{1}{2}} \zeta \right) \sum_{s=0}^\infty \frac{\tilde{B}_s(\zeta)}{\nu^{s+\frac{1}{2}}} \right].$$

In (2.19) and (2.20), $A_\nu(\cdot)$ and $B_\nu(\cdot)$ are the Airy functions, the leading coefficients are given by

$$\tilde{A}_0(\zeta) = 1 \quad \text{and} \quad \tilde{B}_0(\zeta) = 0,$$

and the expansions hold uniformly for $0 \leq t < \infty$. 

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3. Determination of $C_1(x)$ and $C_2(x)$

First we examine the behavior of $f_n^{(\alpha)}(x)$ as $n \to \infty$. By using the classical steepest decent method, one can show that to leading order, we have

\begin{equation}
\begin{aligned}
\frac{f_n^{(\alpha)}(t)}{\sqrt{
u}} & \sim \frac{\nu^{\frac{-3}{2}} e^{\nu^{\frac{3}{2}} + \left(\frac{t}{4} - 1\right) \log \lambda_+}}{\sqrt{2\pi \nu(t^2 - 4)^{\frac{3}{2}}}} \cos(\pi \alpha - \frac{\nu \pi}{2}) \\
& \quad + \frac{\nu^{\frac{-2}{3}} e^{\nu^{\frac{1}{2}} + \left(\frac{t}{4} - 1\right) \log \lambda_-}}{\sqrt{2\pi \nu(t^2 - 4)^{\frac{3}{2}}}} 2\sin(\pi \alpha - \frac{\nu \pi}{2}),
\end{aligned}
\end{equation}

when $t > 2$;

\begin{equation}
\begin{aligned}
\frac{f_n^{(\alpha)}(t)}{\sqrt{
u}} & \sim \frac{\sqrt{2} \nu^{\frac{-2}{3}} e^{\nu^{\frac{1}{2}}}}{\sqrt{\pi \nu(4 - t^2)^{\frac{1}{2}}}} \sin \left[ \nu \left( \frac{1}{2t} \sqrt{4 - t^2} + \frac{\pi}{t^2} - \frac{2}{t^2} \sin^{-1} \frac{t}{\sqrt{2}} - \cos^{-1} \frac{t}{2} \right) \\
& \quad + \pi \alpha - \frac{\nu \pi}{4} \right],
\end{aligned}
\end{equation}

when $0 < t < 2$;

\begin{equation}
\begin{aligned}
f_n^{(\alpha)}(\frac{2}{\sqrt{\nu}}) & \sim \nu^{\frac{-2}{3}} e^{\frac{3}{2} \nu^{\frac{1}{2}}} \left( \frac{1}{3\Gamma(\frac{4}{3})} \cos(\pi \alpha - \frac{\nu \pi}{4}) + \frac{\sqrt{3}}{3\Gamma(\frac{4}{3})} \sin(\pi \alpha - \frac{\nu \pi}{4}) \right),
\end{aligned}
\end{equation}

when $t = 2$. Next we recall the well-known asymptotic formulas

\begin{equation}
\begin{aligned}
\text{Ai}(\eta) & \sim \frac{\eta^{-\frac{1}{2}}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} \eta^\frac{3}{2}\right),
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\text{Bi}(\eta) & \sim \frac{\eta^{-\frac{1}{2}}}{\sqrt{\pi}} \exp\left(\frac{2}{3} \eta^\frac{3}{2}\right)
\end{aligned}
\end{equation}

as $\eta \to +\infty$; see [5, p. 392]. Furthermore, since $\nu = n + 2\alpha - 1/2$, it is readily seen from (3.3) and (3.4) that

\begin{equation}
\begin{aligned}
P_n^{(\alpha)}(\nu^{\frac{1}{2}} t) & \sim \frac{\nu^{-\frac{3}{2}} \nu^{\frac{3}{2}}}{2^n \pi \Gamma(\frac{4}{3})} e^{\nu^{\frac{1}{2}} + \left(\frac{t}{4} - 1\right) \log \lambda_+},
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
Q_n^{(\alpha)}(\nu^{\frac{1}{2}} t) & \sim \frac{\sqrt{2} \nu^{\frac{-2}{3}} \nu^{\frac{1}{2}}}{\pi \Gamma(\frac{4}{3})} \frac{\sqrt{\nu^{\frac{3}{2}} + \left(\frac{t}{4} - 1\right) \log \lambda_-}}{\nu^{\frac{1}{2}} + (\frac{t}{4} - 1) \log \lambda_-}.
\end{aligned}
\end{equation}

Comparing the two sides of (2.18), one concludes that

\begin{equation}
\begin{aligned}
C_1(x) & = \sqrt{\pi} \cos(\pi \alpha - \frac{\pi}{x^2}), \quad C_2(x) = \sqrt{\pi} \sin(\pi \alpha - \frac{\pi}{x^2}).
\end{aligned}
\end{equation}

Note that in obtaining (3.6), any one of the three formulas (3.1), (3.2) or (3.3) could have been used. In summary, we have from (2.14), (2.18), (2.19) and (2.20)

\begin{equation}
\begin{aligned}
f_n^{(\alpha)}(\frac{t}{\sqrt{\nu}}) & = \frac{\sqrt{\pi} \cos(\pi \alpha - \frac{\nu \pi}{4})}{2^n \pi \Gamma(\frac{4}{3})} \left( \frac{4\zeta}{\nu^{\frac{3}{2}}} \right)^{\frac{1}{2}} \nu^{\frac{1}{2}} \text{Ai} \left( \nu^{\frac{3}{2}} \zeta \right) + O\left( \frac{1}{\nu} \right) \\
& \quad + \frac{\sqrt{\pi} \sin(\pi \alpha - \frac{\nu \pi}{4})}{2^n \pi \Gamma(\frac{4}{3})} \left( \frac{4\zeta}{\nu^{\frac{3}{2}}} \right)^{\frac{1}{2}} \nu^{\frac{1}{2}} \text{Bi} \left( \nu^{\frac{3}{2}} \zeta \right) + O\left( \frac{1}{\nu} \right),
\end{aligned}
\end{equation}

\begin{equation}
\end{equation}
where \( x = \nu - t \) and \( \nu = n + 2\alpha - 1/2 \). This result holds uniformly for \( 0 \leq t < \infty \).

As a check on the validity of (3.7), we note that

\[
(3.8) \quad f^{(\alpha)}(0) = \begin{cases} (-1)^n & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise}. \end{cases}
\]

From (3.7), we obtain

\[
(3.9) \quad f^{(\alpha)}(0) \sim \frac{e^{n/2}}{\sqrt{\pi n}^{n+1/2}} \sin \left( \frac{n\pi}{2} + \frac{\pi}{2} \right).
\]

In view of Stirling’s formula, (3.9) agrees with (3.8).

### 4. Numerical results

The expansion (3.7) is particularly useful near \( t = 2 \), where the two characteristic roots in (2.14) coincide. As an illustration, we take \( \alpha = 1.9 \) and \( n = 100 \). Table 1 provides exact and approximate values of \( 2^{n/2} \Gamma(n/2 + 1) f^{(\alpha)}(t/\sqrt{\nu}) \). The last column of the table shows the percentage error of the approximations.

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