A NON-FINITELY GENERATED ALGEBRA OF FROBENIUS MAPS

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(Communicated by Bernd Ulrich)

ABSTRACT. This paper describes an Artinian module over a ring of prime characteristic whose algebra of Frobenius maps is not finitely generated. This settles a question raised by Lyubeznik and Smith.

1. Introduction

The purpose of this paper is to answer a question raised by Gennady Lyubeznik and Karen Smith in [LS]. This question involves the finite generation of a certain non-commutative algebra which we define below (cf. section 3 in [LS]).

Let $S$ be any commutative algebra of prime characteristic $p$. For any $S$-module $M$ and all $e \geq 0$ we let $\mathcal{F}^e(M)$ denote the set of all additive functions $\phi : M \to M$ with the property that $\phi(sm) = s^e \phi(m)$ for all $s \in S$ and $m \in M$. Note that for all $e_1, e_2 \geq 0$ and $\phi_1 \in \mathcal{F}^{e_1}(M)$, $\phi_2 \in \mathcal{F}^{e_2}(M)$, the composition $\phi_2 \circ \phi_1$ is in $\mathcal{F}^{e_1 + e_2}(M)$. Note also that each $\mathcal{F}^e(M)$ is a module over $\mathcal{F}^0(M) = \text{Hom}_S(M, M)$ via $\phi_0 \circ \phi = \phi_0 \circ \phi$. We now define $\mathcal{F}(M) = \bigoplus_{e \geq 0} \mathcal{F}^e(M)$ and endow it with the structure of a $\text{Hom}_S(M, M)$-algebra with multiplication given by composition.

In section 2 we construct an example of an Artinian module over a complete local ring $S$ for which $\mathcal{F}(M)$ is not a finitely generated $\text{Hom}_S(M, M)$-algebra, thus giving a negative answer to the question raised in section 3 of [LS].

2. The Example

Let $\mathbb{K}$ be a field of characteristic $p > 0$, $R = \mathbb{K}[x, y, z]$, and let $I \subseteq R$ be an ideal. Let $E$ be the injective hull of the residue field of $R$ and let $f$ denote the standard Frobenius map of $E$ (cf. section 4 in [K]). Write $S = R/I$ and let $E_S$ be the injective hull of the residue field of $S$.

Notice that as $S$ is complete, $\mathcal{F}^0(E_S) = \text{Hom}_S(E_S, E_S) \cong S$; the $S$-module $\mathcal{F}^e(E_S)$ of $p^e$th Frobenius maps on $E_S$ is given by $(I[p^e] : I) f^e$ (cf. section 4 in [K]).

For all $e \geq 1$ write $K_e = (I[p^e] : I)$. We define

$$L_e = \sum_{1 \leq \beta_1, \ldots, \beta_s \leq p \atop \beta_1 + \cdots + \beta_s = e} K_{\beta_1} K_{p^{\beta_1}} K_{p^{\beta_2}} \cdots K_{p^{\beta_s}} K_{p^{\beta_1 + \cdots + \beta_s - 1}}.$$

**Proposition 2.1.** Fix any $e \geq 1$, and let $\mathcal{F}_e \subseteq \mathcal{F}(E_S)$ be the $S$-subalgebra of $\mathcal{F}(E_S)$ generated by $\mathcal{F}^0(E_S), \ldots, \mathcal{F}^{e-1}(E_S)$. We have that $\mathcal{F}_e \cap \mathcal{F}(E_S) = L_e f^e$. 

Received by the editors June 5, 2009 and, in revised form, December 16, 2009.

2010 Mathematics Subject Classification. Primary 13A35, 13E10.
Proof. Any element in \( F_{<e} \cap F^e(ES) \) can be written as a sum of elements of the form \( \phi_1 \cdots \phi_s \), where for all \( 1 \leq j \leq s \) we have \( \phi_j \in F_{\beta_j}(ES) \) \((1 \leq \beta_j < e)\) and \( \beta_1 + \cdots + \beta_s = e \). Each such \( \phi_j \) equals \( a_j f^{\beta_j} \) where \( a_j \in K_{\beta_j} \), so
\[
\phi_1 \cdots \phi_s = a_1 f^{\beta_1} a_2 f^{\beta_2} a_3 f^{\beta_3} \cdots a_s f^{\beta_s}
\]
\[
= a_1 a_2 a_3^{\beta_1 + \beta_2} \cdots a_s^{\beta_1 + \cdots + \beta_s - 1} f^{\beta_1 + \cdots + \beta_s} \in L_e f^e,
\]
hence \( F_{<e} \cap F^e(ES) \subseteq L_e f^e \).

On the other hand, for all \( 1 \leq \beta_1, \ldots, \beta_s < e \) such that \( \beta_1 + \cdots + \beta_s = e \),
\[
K_{\beta_1} K_{\beta_2} [p^{\beta_1 + \beta_2}] \cdots K_{\beta_s} [p^{\beta_1 + \cdots + \beta_s - 1}] \subseteq (I[p^{\beta_1 + \cdots + \beta_s}] : I) = (I[p^e] : I),
\]
so \( L_e f^e \subseteq (I[p^e] : I)f^e = F^e(ES) \). A similar argument to the one in the previous paragraph shows that we also have
\[
K_{\beta_1} K_{\beta_2} [p^{\beta_1 + \beta_2}] \cdots K_{\beta_s} [p^{\beta_1 + \cdots + \beta_s - 1}] f^e \subseteq F_{<e},
\]
and we deduce that \( L_e f^e \subseteq F_{<e} \cap F^e(ES) \). □

Now fix \( I \) to be the ideal generated by \( xy \) and \( yz \). We show that \( F(M) \) is not a finitely generated \( S \)-algebra.

**Proposition 2.2.** For all \( e \geq 1 \), \( K_e \) is generated by
\[
\left\{ x^{p^e} y^{p^e-1}, x^{p^e-1} y^{p^e-1} z^{p^e-1}, y^{p^e-1} z^{p^e} \right\}.
\]

**Proof.** For any \( q > 1 \),
\[
(x^q y^q, y^q z^q) : (xy, yz) = \left( (x^q y^q, y^q z^q) : xy \right) \cap \left( (x^q y^q, y^q z^q) : yz \right)
\]
\[
= \left( x^{q-1} y^{q-1}, y^{q-1} z^q \right) \cap \left( x^{q-1} y^{q-1}, y^{q-1} z^{-1} \right)
\]
\[
= \left( x^{q-1} y^{q-1}, x^{q-1} y^{q-1} z^{q-1}, x^q y^{q-1} z^q, y^{q-1} z^q \right)
\]
\[
= \left( x^{q-1} y^{q-1}, x^{q-1} y^{q-1} z^{q-1}, y^{q-1} z^q \right).
\]
□

**Theorem 2.3.** The \( S \)-algebra \( F(ES) \) is not finitely generated.

**Proof.** It is enough to show that for all \( e \geq 1 \), \( F(ES) \) is not in \( F_{<e} \), and we establish this by showing that the generator \( x^{p^e} y^{p^e-1} \) of \( K_e \) is not in \( L_e \).

Since \( L_e \) is a sum of monomial ideals, \( x^{p^e} y^{p^e-1} \in L_e \) if and only if \( x^{p^e} y^{p^e-1} \) is in one of the summands. So we now fix \( e \geq 1 \) and \( 1 \leq \beta_1, \ldots, \beta_s < e \) such that \( \beta_1 + \cdots + \beta_s = e \), and we show that the ideal
\[
K_{\beta_1} K_{\beta_2} [p^{\beta_1 + \beta_2}] \cdots K_{\beta_s} [p^{\beta_1 + \cdots + \beta_s - 1}]
\]
does not contain \( x^{p^e} y^{p^e-1} \).

Since \( z \) does not occur in \( x^{p^e} y^{p^e-1} \), it is enough to show that with \( J_e = x^{p^e} y^{p^e-1} R \),
\[
J_{\beta_1} J_{\beta_2} [p^{\beta_1 + \beta_2}] \cdots J_{\beta_s} [p^{\beta_1 + \cdots + \beta_s - 1}]
\]
does not contain \( x^{p^e} y^{p^e-1} \). The exponent of \( x \) in the generator of the product above is
\[
p^{\beta_1 + (\beta_1 + \beta_2) + \cdots + (\beta_1 + \cdots + \beta_s)} > p^{\beta_1 + \cdots + \beta_s} = p^e,
\]
where the inequality follows from the fact that we must have \( s > 1 \). □
3. A conjecture

Although the example in section 2 settles the question raised in [LS], one might still raise the question of whether such examples exist over “nice” rings, e.g., normal domains.

Let \( \mathbb{K} \) be a field of prime characteristic \( p \), let \( R = \mathbb{K}[x, y, z, u, v, w] \) and let \( I \) be the ideal generated by the \( 2 \times 2 \) minors of the matrix \( \begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix} \). The ring \( S = R/I \) is a normal, Cohen-Macaulay domain (cf. Theorem 7.3.1 in [BH]). Let \( E_S \) be the injective hull of the residue field of \( S \) and, as before, for all \( e \geq 1 \) let \( \mathcal{T}_{<e} \) be the \( S \)-subalgebra of \( \mathcal{T}^e(E_S) \) generated by \( \mathcal{T}^1(E_S), \ldots, \mathcal{T}^{e-1}(E_S) \). Note that \( \mathcal{T}^0(E_S) = S \).

**Conjecture 3.1.** For all \( e \geq 1 \), \( \mathcal{T}^e(E_S) \) is not contained in \( \mathcal{T}_{<e} \), and hence \( \mathcal{T}^e(E_S) \) is not a finitely generated \( S \)-algebra.

I have tested this conjecture using the computer system Macaulay 2 ([GS]), and, for example, in characteristic 2, it holds for \( 1 \leq e \leq 6 \).

**References**


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