ANOTHER PROOF FOR THE REMOVABLE SINGULARITIES
OF THE HEAT EQUATION

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Abstract. We give two different simple proofs for the removable singularities
of the heat equation in \((\Omega \setminus \{x_0\}) \times (0, T)\), where \(x_0 \in \Omega \subset \mathbb{R}^n\) is a bounded
domain with \(n \geq 3\). We also give a necessary and sufficient condition for
removable singularities of the heat equation in \((\Omega \setminus \{x_0\}) \times (0, T)\) for the case
\(n = 2\).

Singularities of solutions of partial differential equations appear in many prob-
lems. For example, singularities appear in the study of the solutions of the harmonic
map \[13\] and the harmonic map heat flow \[3\]. In \[14\] S. Sato and E. Yanagida studied
the solutions for a semilinear parabolic equation with moving singularities. Sin-
gularities of solutions also appear in the study of hyperbolic partial differential equa-
tions \[15\] and in the study of the touchdown behavior of the micro-electromechanical
systems equation \[4\], \[6\], \[5\].

It is interesting to find a necessary and sufficient condition for the solutions of
the equations to have removable singularities. In \[8\] S.Y. Hsu proved the following
theorem.

Theorem 1. Let \(n \geq 3\) and let \(0 \in \Omega \subset \mathbb{R}^n\) be a domain. Suppose \(u\) is a solution
of the heat equation

\[ u_t = \Delta u \]

in \((\Omega \setminus \{0\}) \times (0, T)\). Then \(u\) has removable singularities at \(0 \times (0, T)\) if and only
if for any \(0 < t_1 < t_2 < T\) and \(\delta \in (0, 1)\) there exists \(B_{R_0}(0) \subset \Omega\) depending on \(t_1, \ t_2\) and \(\delta\), such that

\[ |u(x, t)| \leq \delta |x|^{2-n} \]

for any \(0 < |x| \leq R_0\) and \(t_1 \leq t \leq t_2\).

The proof in \[8\] is based on the Green function estimates of \[9\] and a careful
analysis of the behavior of the solution near the singularities using the Duhamel
principle. In this paper we will use the Schauder estimates for the heat equation \[2\],
\[12\], and the technique of \[1\] and \[7\] to give two different simple proofs of the above
result. We also obtain the following result for the solution of the heat equation in
two dimensions.
Theorem 2. Let \( 0 \in \Omega \subset \mathbb{R}^2 \) be a domain. Suppose \( u \) is a solution of the heat equation in \((\Omega \setminus \{0\}) \times (0, T)\). Then \( u \) has removable singularities at \( \{0\} \times (0, T) \) if and only if for any \( 0 < t_1 < t_2 < T \) and \( \delta \in (0, 1) \) there exists \( \overline{B}_{R_0}(0) \subset \Omega \) depending on \( t_1, t_2 \) and \( \delta \) such that

\[
|u(x, t)| \leq \delta (\log(1/|x|))^{-1}
\]

for any \( 0 < |x| \leq R_0 \) and \( t_1 \leq t \leq t_2 \).

Remark 3. Note that the function \( \log |x| \) satisfies the heat equation in \((\mathbb{R}^2 \setminus \{0\}) \times (0, \infty)\), but it has non-removable singularities on \( \{0\} \times (0, \infty) \) and it does not satisfy (3). Hence (3) is sharp.

We start with some definitions. For any set \( A \) we let \( \chi_A \) be the characteristic function of the set \( A \). Let \( 0 \in \Omega \subset \mathbb{R}^n \) be a bounded domain. We say that a solution \( u \) of the heat equation (1) in \((\Omega \setminus \{0\}) \times (0, T)\) has removable singularities at \( \{0\} \times (0, T) \) if there exists a classical solution \( v \) of (1) in \( \Omega \times (0, T) \) such that \( u = v \) in \((\Omega \setminus \{0\}) \times (0, T)\). For any \( R > 0 \) let \( B_R = B_R(0) = \{ x : |x| < R \} \subset \mathbb{R}^n \).

Proof of Theorem 1. Suppose \( u \) has removable singularities in \( \{0\} \times (0, T) \). By the same argument as in the proof in section 3 of [8], for any \( 0 < t_1 < t_2 < T \) and \( \delta \in (0, 1) \) there exists \( \overline{B}_{R_0} \subset \Omega \) depending on \( t_1, t_2 \) and \( \delta \) such that (2) holds.

Suppose (2) holds. Then for any \( 0 < t_1 < t_2 < T \) and \( \delta \in (0, 1) \) there exists \( \overline{B}_{R_0} \subset \Omega \) depending on \( t_1, t_2 \) and \( \delta \) such that (2) holds for any \( 0 < |x| \leq R_0 \) and \( t_1 \leq t \leq t_2 \).

For any \( 0 < |x| \leq R_0 \), let

\[
w(y, s) = u(|x|, |y|, |x|^2 s) \text{ } \forall 0 < |y| \leq R_0/|x|, t_1/|x|^2 \leq s \leq t_2/|x|^2.
\]

Then \( w \) is a solution of (1) in \((\overline{B}_1 \setminus \{0\}) \times (|x|^{-2} t_1, |x|^{-2} t_2)\). By (2),

\[
|w(y, s)| \leq \delta (|x||y|)^{2-n} \text{ } \forall 0 < |y| \leq R_0/|x|, t_1/|x|^2 \leq s \leq t_2/|x|^2.
\]

Let \( t_1 < t_3 < t_2 \). Then

\[
t_3/|x|^2 - t_1/|x|^2 \geq t_3 - t_1/R_0^2.
\]

By the parabolic Schauder estimates [2], [12], (5) and (6), there exists a constant \( C_1 > 0 \) such that

\[
|\nabla w(y, s)| \leq C_1 \text{ sup}_{1/2 \leq |z| \leq 1, |x|^{-2} t_1 \leq |z|^{-2} t_2} w(z, \tau) \leq C_2 \delta |x|^{2-n}
\]

holds for any \( 2/3 \leq |y| \leq 3/4, t_3/|x|^2 \leq s \leq t_2/|x|^2 \), where \( C_2 = 2^{n-2} C_1 \). By (4) and (7),

\[
|\nabla u(z, t)\| \leq C_2 \delta |x|^{1-n} \text{ } \forall |z| = \frac{3}{4}|x|, 0 < |x| \leq R_0, t_3 \leq t \leq t_2
\]

holds for any \( 2/3 \leq |y| \leq 3/4, t_3/|x|^2 \leq s \leq t_2/|x|^2 \), where \( C_2 = 2^{n-2} C_1 \). By (4) and (7),

\[
|\nabla u(z, t)| \leq C_2 \delta |x|^{1-n} \text{ } \forall |z| \leq \frac{3}{4} R_0, t_3 \leq t \leq t_2.
\]

Let \( R_1 = 3/(4R_0) \). We will now use a modification of the proof of Lemma 2.3 of [1] and Lemma 2.1 of [7] to complete the argument. We will first show that \( u \) satisfies
(1) in $\Omega \times (t_1, t_2)$ in the distribution sense. Since $u$ satisfies (1) in $(\Omega \setminus \{0\}) \times (0, T)$, for any $0 < \varepsilon < R_1$ and $\eta \in C_0^\infty(\Omega \times (0, T))$ we have

$$
\int_{\Omega \setminus B_\varepsilon} u \eta \, dx \bigg|_{t_3}^{t_2} = \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} u \eta \, dx dt - \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dx dt - \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt,
$$

(9)

where $\frac{\partial u}{\partial n}$ is the derivative of $u$ with respect to the unit outward normal at $\partial B_\varepsilon$.

By (8),

$$
\limsup_{\varepsilon \to 0} \left| \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt \right| \leq C_2 \delta (t_2 - t_3) |\partial B_1| \|\eta\|_{L^\infty}.
$$

Since $\delta > 0$ is arbitrary, it follows that

$$
\lim_{\varepsilon \to 0} \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt dtdx = 0.
$$

By (8) and the Lebesgue dominated convergence theorem,

$$
\lim_{\varepsilon \to 0} \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dx dt = \int_{t_3}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \eta \, dx dt.
$$

Letting $\varepsilon \to 0$ in (9), by (10) and (11) it follows that

$$
\int_{\Omega \setminus B_{R_0}} u \eta \, dx \bigg|_{t_3}^{t_2} = \int_{t_3}^{t_2} \int_{\Omega} u \eta \, dx dt - \int_{t_3}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \eta \, dx dt \quad \forall t_3 \in (t_1, t_2).
$$

Hence $u$ is a distribution solution of (1) in $\Omega \times (t_1, t_2)$. By (2), for any $1 \leq p < \frac{n}{n-2}$ there exists a constant $C_p' > 0$ such that

$$
\sup_{t_1 \leq t \leq t_2} \int_{B_{R_0}} u(x, t)^p \, dx \leq C_p'.
$$

By (12) and (13) and an argument similar to the proof of [11] and section 1 of [10], $u \in L^\infty_{loc}(B_{R_0} \times (t_1, t_2))$. We now let $v$ be the solution of

$$
\begin{cases}
  v_t = \Delta v & \text{in } B_{R_1} \times (t_3, t_2), \\
  \frac{\partial v}{\partial n}(x, t) = \frac{\partial u}{\partial n}(x, t) & \text{on } \partial B_{R_1} \times (t_3, t_2), \\
  v(x, t_3) = u(x, t_3) & \text{in } B_{R_1}.
\end{cases}
$$

(14)

For any $0 \leq h \in C_0^\infty(B_{R_1})$ and $t_3 < t \leq t_2$ let $\eta$ be the solution of

$$
\begin{cases}
  \eta_t + \Delta \eta = 0 & \text{in } B_{R_1} \times (t_3, t), \\
  \frac{\partial \eta}{\partial n}(x, t) = 0 & \text{on } \partial B_{R_1} \times (t_3, t), \\
  \eta(x, t) = h(x) & \text{in } B_{R_1}.
\end{cases}
$$

(15)

By the maximum principle,

$$
0 \leq \eta \leq \|h\|_{L^\infty} \quad \text{in } B_{R_1} \times (t_3, t).
$$

(16)
Then by (14) and (15),
\[
(17) \quad \int_{B_{R_1} \setminus B_x} (u - v) \eta \, dx \bigg|_{t_3}^t = \int_{t_3}^t \int_{B_{R_1} \setminus B_x} [(u - v) \eta_t + (u - v) \eta] \, dx \, dt \\
= \int_{t_3}^t \int_{B_{R_1} \setminus B_x} [(u - v) \eta_t + \Delta(u - v) \eta] \, dx \, dt \\
= \int_{t_3}^t \int_{B_{R_1} \setminus B_x} (u - v) (\eta_t + \Delta \eta) \, dx \, dt \\
- \int_{t_3}^t \int_{\partial B_x} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt + \int_{t_3}^t \int_{\partial B_x} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt \\
= - \int_{t_3}^t \int_{\partial B_x} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt + \int_{t_3}^t \int_{\partial B_x} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt.
\]
By (2),
\[
(18) \quad \left| \int_{t_3}^t \int_{\partial B_x} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt \right| \leq C \varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\]
By (8) and (16),
\[
(19) \quad \limsup_{\varepsilon \rightarrow 0} \left| \int_{t_3}^t \int_{\partial B_x} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt \right| \leq C \delta.
\]
Since \( \delta > 0 \) is arbitrary, by (19) it follows that
\[
(20) \quad \lim_{\varepsilon \rightarrow 0} \left| \int_{t_3}^t \int_{\partial B_x} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt \right| = 0.
\]
Letting \( \varepsilon \rightarrow 0 \) in (17), by (18) and (20),
\[
(21) \quad \int_{B_{R_1}} (u - v)(x,t) h(x) \, dx = \int_{B_{R_1}} (u - v)(x,t_3) \eta(x,t_3) \, dx = 0.
\]
We now choose a sequence of functions \( h_i \in C_0^\infty(B_{R_1}) \) converging to \( \chi_{\{u > v\}} \) a.e. \( x \in B_{R_1} \) as \( i \rightarrow \infty \). Putting \( h = h_i \) in (21) and letting \( i \rightarrow 0 \),
\[
(22) \quad \int_{B_{R_1}} (u - v)_+(x,t) \, dx = 0 \quad \forall t_3 < t \leq t_2.
\]
By interchanging the roles of \( u \) and \( v \) we get
\[
(23) \quad \int_{B_{R_1}} (v - u)_+(x,t) \, dx = 0 \quad \forall t_3 < t \leq t_2.
\]
Hence by (22) and (23),
\[
(24) \quad \Rightarrow \ u(x,t) = v(x,t) \quad \forall 0 < |x| \leq R_1, t_3 < t \leq t_2.
\]
Hence \( u \) has removable singularities on \( \{0\} \times (t_3, t_2) \). Since \( 0 < t_1 < t_3 < t_2 < T \) is arbitrary, \( u \) has removable singularities on \( \{0\} \times (0, T) \) and the theorem follows. \( \square \)

**Proof of Theorem 2.** Theorem 2 follows by an argument very similar to the proof of Theorem 1 but with (3) replacing (2) in the argument. \( \square \)
An alternate proof of Theorems 1 and 2. We will show that when (2) (respectively (3)) holds, then \( u \) has removable singularities at \( \{0\} \times (0, T) \). Suppose (2) holds if \( n \geq 3 \) and (3) holds if \( n = 2 \). We first observe that by the previous argument, for any \( 0 < t_1 < t_2 < T \), \( u \) satisfies (12) and \( u \in L^\infty_{\text{loc}}(\Omega \times (0, T)) \). Let \( \overline{B}_{R_1} \subset \Omega \) and let \( w \) be the solution of

\[
\begin{cases}
  w_t = \Delta w & \text{in } B_{R_1} \times (t_1, t_2), \\
  w = u & \text{on } \overline{B}_{R_1} \times \{t_1\} \cup \partial B_{R_1} \times (t_1, t_2).
\end{cases}
\]

By the maximum principle,

\[
\|w\|_{L^\infty} \leq \|u\|_{L^\infty(B_{R_1} \times (t_1, t_2))} < \infty.
\]

For any \( \varepsilon > 0 \), let

\[
w_\varepsilon = \begin{cases}
  w - u + \varepsilon |x|^{2-n} & \text{if } n \geq 3, \\
  w - u + \varepsilon (R_1/|x|) & \text{if } n = 2.
\end{cases}
\]

Then \( w_\varepsilon \) satisfies

\[
\begin{cases}
  w_{\varepsilon,t} = \Delta w_\varepsilon & \text{in } (B_{R_1} \setminus \{0\}) \times (t_1, t_2), \\
  w_\varepsilon \geq u & \text{on } \partial B_{R_1} \times (t_1, t_2) \cup \overline{B}_{R_1} \times \{t_1\}.
\end{cases}
\]

By (2), (3), and (25) there exists a constant \( 0 < r_0 < R_1 \) such that

\[
w_\varepsilon \geq 0 \quad \text{on } \partial B_{r_1} \times [t_1, t_2]
\]

for all \( 0 < r_1 \leq r_0 \). By the maximum principle in \( (B_{R_1} \setminus B_{r_1}) \times (t_1, t_2) \),

\[
w_\varepsilon \geq 0 \quad \text{in } (B_{R_1} \setminus B_{r_1}) \times (t_1, t_2)
\]

\[
\implies \begin{cases}
  w - u + \varepsilon |x|^{2-n} \geq 0 & \forall \varepsilon \leq |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{if } n \geq 3, \\
  w - u + \varepsilon \log(R_0/|x|) \geq 0 & \forall \varepsilon \leq |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{if } n = 2
\end{cases}
\]

(26) \quad \implies w \geq u \quad \forall 0 < |x| \leq R_1, t_1 \leq t \leq t_2 \text{ as } r_1 \to 0, \varepsilon \to 0.

Similarly, by considering the function

\[
v_\varepsilon = \begin{cases}
  w - u - \varepsilon |x|^{2-n} & \text{if } n \geq 3, \\
  w - u - \varepsilon \log(R_1/|x|) & \text{if } n = 2
\end{cases}
\]

and applying the maximum principle and letting \( \varepsilon \to 0 \), we get

\[
w \leq u \quad \forall 0 < |x| \leq R_1, t_1 \leq t \leq t_2.
\]

By (26) and (27) we get (24), and Theorem 1 and Theorem 2 follow. \( \square \)

References


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