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ANOTHER PROOF FOR THE REMOVABLE SINGULARITIES OF THE HEAT EQUATION

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ABSTRACT. We give two different simple proofs for the removable singularities of the heat equation in $(\Omega \setminus \{x_0\}) \times (0, T)$, where $x_0 \in \Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 3$. We also give a necessary and sufficient condition for removable singularities of the heat equation in $(\Omega \setminus \{x_0\}) \times (0, T)$ for the case $n = 2$.

Singularities of solutions of partial differential equations appear in many problems. For example, singularities appear in the study of the solutions of the harmonic map [13] and the harmonic map heat flow [3]. In [14] S. Sato and E. Yanagida studied the solutions for a semilinear parabolic equation with moving singularities. Singularities of solutions also appear in the study of hyperbolic partial differential equations [15] and in the study of the touchdown behavior of the micro-electromechanical systems equation [4], [6], [5].

It is interesting to find a necessary and sufficient condition for the solutions of the equations to have removable singularities. In [8] S.Y. Hsu proved the following theorem.

Theorem 1. *Let $n \geq 3$ and let $0 \in \Omega \subset \mathbb{R}^n$ be a domain. Suppose u is a solution of the heat equation*

$$u_t = \Delta u \tag{1}$$

in $(\Omega \setminus \{0\}) \times (0, T)$. Then u has removable singularities at $\{0\} \times (0, T)$ if and only if for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $\overline{B_{R_0}(0)} \subset \Omega$ depending on t_1, t_2 and δ, such that

$$|u(x, t)| \leq \delta |x|^{2-n} \tag{2}$$

for any $0 < |x| \leq R_0$ and $t_1 \leq t \leq t_2$.

The proof in [8] is based on the Green function estimates of [9] and a careful analysis of the behavior of the solution near the singularities using the Duhamel principle. In this paper we will use the Schauder estimates for the heat equation [2], [12], and the technique of [1] and [7] to give two different simple proofs of the above result. We also obtain the following result for the solution of the heat equation in two dimensions.

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Theorem 2. *Let $0 \in \Omega \subset \mathbb{R}^2$ be a domain. Suppose u is a solution of the heat equation in $(\Omega \setminus \{0\}) \times (0,T)$. Then u has removable singularities at $\{0\} \times (0,T)$ if and only if for any $0 < t_1 < t_2 < T$ and $\delta \in (0,1)$ there exists $\overline{B_{R_0}(0)} \subset \Omega$ depending on t_1, t_2 and δ such that*

$$|u(x,t)| \le \delta(\log(1/|x|))^{-1} \tag{3}$$

for any $0 < |x| \le R_0$ and $t_1 \le t \le t_2$.

Remark 3. Note that the function $\log|x|$ satisfies the heat equation in $(\mathbb{R}^2 \setminus \{0\}) \times (0,\infty)$, but it has non-removable singularities on $\{0\} \times (0,\infty)$ and it does not satisfy (3). Hence (3) is sharp.

We start with some definitions. For any set A we let χ_A be the characteristic function of the set A. Let $0 \in \Omega \subset \mathbb{R}^n$ be a bounded domain. We say that a solution u of the heat equation (1) in $(\Omega \setminus \{0\}) \times (0,T)$ has removable singularities at $\{0\} \times (0,T)$ if there exists a classical solution v of (1) in $\Omega \times (0,T)$ such that $u = v$ in $(\Omega \setminus \{0\}) \times (0,T)$. For any $R > 0$ let $B_R = B_R(0) = \{x : |x| < R\} \subset \mathbb{R}^n$.

Proof of Theorem 1. Suppose u has removable singularities in $\{0\} \times (0,T)$. By the same argument as in the proof in section 3 of [8], for any $0 < t_1 < t_2 < T$ and $\delta \in (0,1)$ there exists $\overline{B}_{R_0} \subset \Omega$ depending on t_1, t_2 and δ such that (2) holds.

Suppose (2) holds. Then for any $0 < t_1 < t_2 < T$ and $\delta \in (0,1)$ there exists $\overline{B}_{R_0} \subset \Omega$ depending on t_1, t_2 and δ such that (2) holds for any $0 < |x| \le R_0$ and $t_1 \le t \le t_2$.

For any $0 < |x| \le R_0$, let

$$w(y,s) = u(|x|y, |x|^2 s) \quad \forall 0 < |y| \le R_0/|x|, t_1/|x|^2 \le s \le t_2/|x|^2. \tag{4}$$

Then w is a solution of (1) in $(\overline{B}_1 \setminus \{0\}) \times (|x|^{-2}t_1, |x|^{-2}t_2)$. By (2),

$$|w(y,s)| \le \delta(|x||y|)^{2-n} \quad \forall 0 < |y| \le R_0/|x|, t_1/|x|^2 \le s \le t_2/|x|^2. \tag{5}$$

Let $t_1 < t_3 < t_2$. Then

$$\frac{t_3}{|x|^2} - \frac{t_1}{|x|^2} \ge \frac{t_3 - t_1}{R_0^2}. \tag{6}$$

By the parabolic Schauder estimates [2], [12], (5) and (6), there exists a constant $C_1 > 0$ such that

$$|\nabla w(y,s)| \le C_1 \sup_{\substack{1/2 \le |z| \le 1 \\ |x|^{-2}t_1 \le \tau \le |x|^{-2}t_2}} w(z,\tau) \le C_2 \delta |x|^{2-n} \tag{7}$$

holds for any $2/3 \le |y| \le 3/4$, $t_3/|x|^2 \le s \le t_2/|x|^2$, where $C_2 = 2^{n-2}C_1$. By (4) and (7),

$$\begin{aligned} &|\nabla u(z,t)| \le C_2 \delta |x|^{1-n} \quad \forall |z| = \frac{3}{4}|x|, 0 < |x| \le R_0, t_3 \le t \le t_2 \\ \Rightarrow\ &|\nabla u(z,t)| \le C_2 \delta |z|^{1-n} \quad \forall |z| \le \frac{3}{4}R_0, t_3 \le t \le t_2. \end{aligned} \tag{8}$$

Let $R_1 = 3/(4R_0)$. We will now use a modification of the proof of Lemma 2.3 of [1] and Lemma 2.1 of [7] to complete the argument. We will first show that u satisfies

(1) in $\Omega \times (t_1, t_2)$ in the distribution sense. Since u satisfies (1) in $(\Omega \setminus \{0\}) \times (0, T)$, for any $0 < \varepsilon < R_1$ and $\eta \in C_0^\infty(\Omega \times (0, T))$ we have

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon} u\eta \, dx\Big|_{t_3}^{t_2} = \int_{t_3}^{t_2}\int_{\Omega\setminus B_\varepsilon} u\eta_t \, dxdt - \int_{t_3}^{t_2}\int_{\Omega\setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dxdt \\ - \int_{t_3}^{t_2}\int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt,\end{aligned} \tag{9}$$

where $\partial u/\partial n$ is the derivative of u with respect to the unit outward normal at ∂B_ε. By (8),

$$\limsup_{\varepsilon\to 0}\left|\int_{t_3}^{t_2}\int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt\right| \le C_2\delta(t_2 - t_3)|\partial B_1|\|\eta\|_{L^\infty}.$$

Since $\delta > 0$ is arbitrary, it follows that

$$\lim_{\varepsilon\to 0}\int_{t_3}^{t_2}\int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt \, dxdt = 0. \tag{10}$$

By (8) and the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\int_{t_3}^{t_2}\int_{\Omega\setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dxdt = \int_{t_3}^{t_2}\int_{\Omega} \nabla u \cdot \nabla \eta \, dxdt. \tag{11}$$

Letting $\varepsilon \to 0$ in (9), by (10) and (11) it follows that

$$\int_{\Omega} u\eta \, dx\Big|_{t_3}^{t_2} = \int_{t_3}^{t_2}\int_{\Omega} u\eta_t \, dxdt - \int_{t_3}^{t_2}\int_{\Omega} \nabla u \cdot \nabla \eta \, dxdt \quad \forall t_3 \in (t_1, t_2). \tag{12}$$

Hence u is a distribution solution of (1) in $\Omega \times (t_1, t_2)$. By (2), for any $1 \le p < \frac{n}{n-2}$ there exists a constant $C_p' > 0$ such that

$$\sup_{t_1\le t\le t_2}\int_{B_{R_0}} u(x,t)^p \, dx \le C_p'. \tag{13}$$

By (12) and (13) and an argument similar to the proof of [11] and section 1 of [10], $u \in L^\infty_{loc}(B_{R_0} \times (t_1, t_2))$. We now let v be the solution of

$$\begin{cases} v_t = \Delta v & \text{in } B_{R_1} \times (t_3, t_2), \\ \dfrac{\partial v}{\partial n}(x,t) = \dfrac{\partial u}{\partial n}(x,t) & \text{on } \partial B_{R_1} \times (t_3, t_2), \\ v(x, t_3) = u(x, t_3) & \text{in } B_{R_1}. \end{cases} \tag{14}$$

For any $0 \le h \in C_0^\infty(B_{R_1})$ and $t_3 < t \le t_2$ let η be the solution of

$$\begin{cases} \eta_t + \Delta \eta = 0 & \text{in } B_{R_1} \times (t_3, t), \\ \dfrac{\partial \eta}{\partial n}(x,t) = 0 & \text{on } \partial B_{R_1} \times (t_3, t), \\ \eta(x, t) = h(x) & \text{in } B_{R_1}. \end{cases} \tag{15}$$

By the maximum principle,

$$0 \le \eta \le \|h\|_{L^\infty} \quad \text{in } B_{R_1} \times (t_3, t). \tag{16}$$

Then by (14) and (15),

(17)

$$\begin{aligned}\int_{B_{R_1}\setminus B_\varepsilon}(u-v)\eta\,dx\bigg|_{t_3}^{t} &= \int_{t_3}^{t}\int_{B_{R_1}\setminus B_\varepsilon}[(u-v)\eta_t+(u-v)_t\eta]\,dxdt\\ &= \int_{t_3}^{t}\int_{B_{R_1}\setminus B_\varepsilon}[(u-v)\eta_t+\Delta(u-v)\eta]\,dxdt\\ &= \int_{t_3}^{t}\int_{B_{R_1}\setminus B_\varepsilon}(u-v)(\eta_t+\Delta\eta)\,dxdt\\ &\quad -\int_{t_3}^{t}\int_{\partial B_\varepsilon}\eta\frac{\partial}{\partial n}(u-v)\,d\sigma dt+\int_{t_3}^{t}\int_{\partial B_\varepsilon}(u-v)\frac{\partial\eta}{\partial n}\,d\sigma dt\\ &= -\int_{t_3}^{t}\int_{\partial B_\varepsilon}\eta\frac{\partial}{\partial n}(u-v)\,d\sigma dt+\int_{t_3}^{t}\int_{\partial B_\varepsilon}(u-v)\frac{\partial\eta}{\partial n}\,d\sigma dt.\end{aligned}$$

By (2),

$$\left|\int_{t_3}^{t}\int_{\partial B_\varepsilon}(u-v)\frac{\partial\eta}{\partial n}\,d\sigma dt\right|\le C\varepsilon\to 0 \quad \text{as } \varepsilon\to 0. \tag{18}$$

By (8) and (16),

$$\limsup_{\varepsilon\to 0}\left|\int_{t_3}^{t}\int_{\partial B_\varepsilon}\eta\frac{\partial}{\partial n}(u-v)\,d\sigma dt\right|\le C\delta. \tag{19}$$

Since $\delta>0$ is arbitrary, by (19) it follows that

$$\lim_{\varepsilon\to 0}\left|\int_{t_3}^{t}\int_{\partial B_\varepsilon}\eta\frac{\partial}{\partial n}(u-v)\,d\sigma dt\right|=0. \tag{20}$$

Letting $\varepsilon\to 0$ in (17), by (18) and (20),

$$\int_{B_{R_1}}(u-v)(x,t)h(x)\,dx=\int_{B_{R_1}}(u-v)(x,t_3)\eta(x,t_3)\,dx=0. \tag{21}$$

We now choose a sequence of functions $h_i\in C_0^\infty(B_{R_1})$ converging to $\chi_{\{u>v\}}$ a.e. $x\in B_{R_1}$ as $i\to\infty$. Putting $h=h_i$ in (21) and letting $i\to 0$,

$$\int_{B_{R_1}}(u-v)_+(x,t)\,dx=0 \quad \forall t_3<t\le t_2. \tag{22}$$

By interchanging the roles of u and v we get

$$\int_{B_{R_1}}(v-u)_+(x,t)\,dx=0 \quad \forall t_3<t\le t_2. \tag{23}$$

Hence by (22) and (23),

$$\begin{aligned}&\int_{B_{R_1}}|v-u|(x,t)\,dx=0 \quad \forall t_3<t\le t_2\\ \Rightarrow\quad & u(x,t)=v(x,t) \quad \forall 0<|x|\le R_1, t_3<t\le t_2.\end{aligned} \tag{24}$$

Hence u has removable singularities on $\{0\}\times(t_3,t_2)$. Since $0<t_1<t_3<t_2<T$ is arbitrary, u has removable singularities on $\{0\}\times(0,T)$ and the theorem follows. □

Proof of Theorem 2. Theorem 2 follows by an argument very similar to the proof of Theorem 1 but with (3) replacing (2) in the argument. □

An alternate proof of Theorems 1 *and* 2. We will show that when (2) (respectively (3)) holds, then u has removable singularities at $\{0\}\times(0,T)$. Suppose (2) holds if $n\ge 3$ and (3) holds if $n=2$. We first observe that by the previous argument, for any $0<t_1<t_2<T$, u satisfies (12) and $u\in L^\infty_{loc}(\Omega\times(0,T))$. Let $\overline{B}_{R_1}\subset\Omega$ and let w be the solution of

$$\begin{cases} w_t=\Delta w & \text{in } B_{R_1}\times(t_1,t_2),\\ w=u & \text{on } \overline{B}_{R_1}\times\{t_1\}\cup\partial B_{R_1}\times(t_1,t_2).\end{cases}$$

By the maximum principle,

$$\|w\|_{L^\infty}\le\|u\|_{L^\infty(B_{R_1}\times(t_1,t_2))}<\infty. \tag{25}$$

For any $\varepsilon>0$, let

$$w_\varepsilon=\begin{cases} w-u+\varepsilon|x|^{2-n} & \text{if } n\ge 3,\\ w-u+\varepsilon\log(R_1/|x|) & \text{if } n=2.\end{cases}$$

Then w_ε satisfies

$$\begin{cases} w_{\varepsilon,t}=\Delta w_\varepsilon & \text{in } (B_{R_1}\setminus\{0\})\times(t_1,t_2),\\ w_\varepsilon\ge u & \text{on } \partial B_{R_1}\times(t_1,t_2)\cup\overline{B}_{R_1}\times\{t_1\}.\end{cases}$$

By (2), (3), and (25) there exists a constant $0<r_0<R_1$ such that

$$w_\varepsilon\ge 0 \quad \text{on } \partial B_{r_1}\times[t_1,t_2]$$

for all $0<r_1\le r_0$. By the maximum principle in $(B_{R_1}\setminus B_{r_1})\times(t_1,t_2)$,

$$\begin{aligned} & w_\varepsilon\ge 0 \quad \text{in } (B_{R_1}\setminus B_{r_1})\times(t_1,t_2)\\ \Rightarrow\quad & \begin{cases} w-u+\varepsilon|x|^{2-n}\ge 0 & \forall r_1\le|x|\le R_1, t_1\le t\le t_2 \quad \text{if } n\ge 3,\\ w-u+\varepsilon\log(R_0/|x|)\ge 0 & \forall r_1\le|x|\le R_1, t_1\le t\le t_2 \quad \text{if } n=2\end{cases}\\ \Rightarrow\quad & w\ge u \qquad \forall 0<|x|\le R_1, t_1\le t\le t_2 \quad \text{as } r_1\to 0,\varepsilon\to 0. \end{aligned} \tag{26}$$

Similarly, by considering the function

$$v_\varepsilon=\begin{cases} w-u-\varepsilon|x|^{2-n} & \text{if } n\ge 3,\\ w-u-\varepsilon\log(R_1/|x|) & \text{if } n=2\end{cases}$$

and applying the maximum principle and letting $\varepsilon\to 0$, we get

$$w\le u \quad \forall 0<|x|\le R_1, t_1\le t\le t_2. \tag{27}$$

By (26) and (27) we get (24), and Theorem 1 and Theorem 2 follow. □

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