PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 8, August 2010, Pages 2853–2862
S 0002-9939(10)10327-X
Article electronically published on April 15, 2010

A PALEY-WIENER THEOREM FOR THE ASKEY-WILSON FUNCTION TRANSFORM

LUÍS DANIEL ABREU AND FETHI BOUZEFFOUR

(Communicated by Peter A. Clarkson)

ABSTRACT. We define an analogue of the Paley-Wiener space in the context of the Askey-Wilson function transform, compute explicitly its reproducing kernel and prove that the growth of functions in this space of entire functions is of order two and type $\ln q^{-1}$, providing a Paley-Wiener Theorem for the Askey-Wilson transform. Up to a change of scale, this growth is related to the refined concepts of exponential order and growth proposed by J. P. Ramis. The Paley-Wiener theorem is proved by combining a sampling theorem with a result on interpolation of entire functions due to M. E. H. Ismail and D. Stanton.

1. INTRODUCTION

Let $M(r;f) = \sup\{|f(z)| : |z| \leq r\}$ and consider the space $\mathcal{A}$, constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$, satisfying

$$M(r;f) = O(e^{\pi r}). \tag{1.1}$$

Consider also the space $\mathcal{PW}$ constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$ such that, for some $u \in L^2(-\pi,\pi)$,

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{izt} u(t)\, dt. \tag{1.2}$$

A celebrated classical theorem of Paley and Wiener says that

$$\mathcal{A} = PW.$$

The growth condition (1.1) means that $f : \mathbb{C} \longrightarrow \mathbb{C}$ has *order one* and type π and the space PW is called the Paley-Wiener space of *band-limited* functions; it is the *reproducing kernel Hilbert space* mapped via the Fourier transform into L^2 functions supported on the interval $[-\pi,\pi]$. See [25] for more details.

Received by the editors June 18, 2008, and, in revised form, December 10, 2009.
2010 *Mathematics Subject Classification.* Primary 33D45, 30D15; Secondary 44A20.
Key words and phrases. Askey-Wilson function, Paley-Wiener theorem, reproducing kernels, sampling theorem.
The research of the first author was partially supported by CMUC/FCT and FCT postdoctoral grant SFRH/BPD/26078/2005, POCI 2010 and FSE.

Another famous result, the Whittaker-Shannon-Kotel′nikov sampling theorem, asserts that every function in the space PW admits the following representation:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x-n)}{\pi (x-n)}. \tag{1.3}$$

As a result, research concerning extensions of the sampling theorem has been historically associated with the corresponding extensions of the Paley-Wiener theorem.

The sampling theorem is known to hold for more general transforms, including the Hankel, Dunkl and Jacobi function transforms [14], [7], [26]; and the Paley-Wiener theorem is known to extend to such special function transforms [10].

Many sampling theorems have been recently considered in the q-case [1], [2], [5], [17]. When thinking about these extensions, one should keep in mind that many of the classical q-functions are special cases of a very general basic hypergeometric function known as the *Askey-Wilson function*. This fact is known as the "Askey-Wilson transform scheme" [9].

Recently, one of us has found a sampling theorem for the Askey-Wilson function transform [6]. Thus, it is natural to ask for the associated Paley-Wiener theorem. It is the purpose of this paper to address this question, providing a Paley-Wiener theorem for the Askey-Wilson function transform. This will be done after rephrasing the results in [6] in the convenient reproducing kernel Hilbert space setting.

Recent research concerning q-difference equations [20], interpolation of entire functions [18] and moment problems [4] strongly suggests that in order to deal with basic hypergeometric functions one should use the following concepts. A function f has *logarithmic order* ρ if

$$\limsup_{r\to+\infty} \frac{\ln \ln M(r;f)}{\ln \ln r} = \rho$$

and f with logarithmic order ρ has logarithmic type c if

$$\limsup_{r\to+\infty} \frac{\ln M(r;f)}{(\ln r)^{\rho}} = c.$$

This is because basic hypergeometric functions are of order zero and therefore require a refined concept of order to define their growth. However, we will approach the topic in a slightly different manner in this paper: Instead of considering a function in μ, we will consider a function in $z = q^{\mu}$ and use the classical definitions of order and type in μ. Looking at objects from this point of view, our Askey-Wilson Paley-Wiener space turns out to be constituted by functions of order two with type $\ln(1/q)$. This is equivalent to saying that, in the variable $z = q^{\mu}$, they have logarithmic order two and logarithmic type $\ln(1/q)$.

We have organized the paper in the following way. The next section reviews the definitions of the Askey-Wilson polynomials and functions and provides a short outline of the L^2 theory of the Askey-Wilson transform. Then, in the third section, we present a detailed study of the reproducing kernel Hilbert space which is naturally associated to the Askey-Wilson function transform (in much the same way PW is associated to the Fourier transform). We compute a basis for this space as well as the explicit formula for the reproducing kernel and recover by this method the sampling theorem of [6]. Finally, in the last section we prove a Paley-Wiener theorem, by describing the growth of functions in the reproducing kernel Hilbert space in terms of their order and type.

2. The Askey-Wilson function transform

2.1. The Askey-Wilson polynomials. Choose a number q such that $0 < q < 1$. The notational conventions from [12],

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}),$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad (a_1,...,a_m;q)_n = \prod_{l=1}^{m}(a_l;q)_n, \quad |q|<1,$$

where $n = 1, 2, \ldots,$ will be used. The symbol ${}_{r+1}\phi_r$ stands for the function

$${}_{r+1}\phi_r\left(\begin{array}{c} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{array}\middle| q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_{r+1};q)_n}{(q,b_1,\ldots,b_r;q)_n}z^n.$$

The Askey-Wilson polynomials $p_n(x;a,b,c,d)$, with $x = \frac{z+z^{-1}}{2}$, are defined by

$$p_n(\frac{z+z^{-1}}{2};a,b,c,d) = \frac{(ab,ac,ad;q)_n}{a^n}{}_4\phi_3\left(\begin{array}{c} q^{-n},q^{n-1}abcd,az,a/z \\ ab,ac,ad \end{array}\middle| q;q\right). \tag{2.1}$$

If $a,b,c,d \in \mathbb{C}$ are four reals or two reals and one pair of conjugates or two pairs of conjugates such that $|ab|, |ac|, |ad|, |bc|, |cd| < 1$, then the Askey-Wilson polynomials are real-valued and their orthogonality can be written as an integral over $x = \frac{z+z^{-1}}{2} \in [-1,1]$ plus a finite sum over a discrete set with mass points outside $[-1,1]$. This finite sum does not occur if $|a|, |b|, |c|, |d| < 1$. When $\max(|a|, |b|, |c|, |d|) < 1$, the Askey-Wilson polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} p_n(x;a,b,c,d)p_m(x;a,b,c,d)w(x)dx = h_n\delta_{m,n},$$

where

$$w(x) = \frac{(x^2,1/x^2;q)_\infty \sin\theta}{(ax,a/x,bx,b/x,cx,c/x,dx,d/x;q)_\infty}$$

and

$$h_n = \frac{2\pi(abcdq^{2n};q)_\infty(abcdq^{n-1};q)_n}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}.$$

The Askey-Wilson function is defined as

$$\phi_\gamma(z) = \frac{1}{(bc,q/ad;q)_\infty}{}_4\phi_3\left(\begin{array}{c} \tilde{a}/\gamma,\tilde{a}\gamma,az,a/z \\ ab,ac,ad \end{array}\middle| q;q\right)$$
$$+\frac{(\tilde{a}/\gamma,\tilde{a}\gamma,qb/d,qc/d,az,a/z;q)_\infty}{(q\gamma/\tilde{d},q/\gamma\tilde{d},ab,ac,bc,ad/q,qz/d,q/zd;q)_\infty}{}_4\phi_3\left(\begin{array}{c} q\gamma/\tilde{d},q/\gamma\tilde{d},qz/d,q/zd \\ qb/d,qc/d,q^2/ad \end{array}\middle| q;q\right),$$

where

$$\begin{aligned} \tilde{a} &= \sqrt{q^{-1}abcd}, \\ \tilde{b} &= ab/\tilde{a} = q\tilde{a}/cd, \\ \tilde{c} &= ac/\tilde{a} = q\tilde{a}bd, \\ \tilde{d} &= ad/\tilde{a} = q\tilde{a}/bc. \end{aligned}$$

The function ϕ_γ is introduced in [16], and it can also be defined as a single ${}_8\phi_7$ with a very-well-poised ${}_8W_7$ structure [24]. The function ϕ_γ is meromorphic in γ.

Moreover, its poles are simple and can be removed by multiplying it by the factor $(q\gamma/\tilde{d}, q/\gamma\tilde{d}; q)_\infty$.

Now we will define the Askey-Wilson function transform, following the construction in [8]. A new weight function is defined as

$$W(x) = \Delta(x)\,\Theta(x),$$

where, using the notation $\theta(x) = (x, q/x; q)_\infty$ for the renormalized Jacobi theta function, the function Θ is defined as

$$\Theta(x) = \frac{\theta(dx, d/x)}{\theta(dtx, dt/x)}.$$

For generic parameters a, b, c, d such that the weight function W has simple poles, we define a measure v, depending on these parameters, by

$$\begin{aligned} \int f(x)\,dv(x) &= \frac{K}{4i\pi}\int_{\mathbb{T}} f(x)\phi_\gamma(x)\,W(x)\,\frac{dx}{x} \\ &\quad + \frac{K}{2}\sum_{x\in D}\left(f(x) + f\left(x^{-1}\right)\right)\ \mathrm{Res}_{y=x}\left(\frac{W(y)}{y}\right), \end{aligned}$$

where K is a constant (the exact value will not be required), $S = S_- \cup S_+$ is the infinite discrete set given by

$$\begin{aligned} S_- &= \{dtq^k;\ k\in\mathbb{Z},\ dtq^k < -1\}, \\ S_+ &= \{aq^k;\ k\in\mathbb{Z},\ aq^k > 1\}. \end{aligned}$$

In the next sections we will often refer to the measure defined above as being of the form $v = v_c + \nu_d$, where v_c is the continuous measure

$$dv_c(x) = \Theta(x)\Delta(x)\,dx/x$$

and ν_d is the discrete part, supported in the set S.

Now, let $L^2_+(v)$ be the Hilbert space with respect to the measure v constituted by functions f satisfying $f(x) = f\left(x^{-1}\right)$, ν-almost everywhere. The *Askey-Wilson function transform* is defined by

$$(\mathcal{F}f)(\gamma) = \int f(x)\,\phi_\gamma(x)\,dv(x)$$

for compactly supported functions $f \in L^2_+(\nu)$. Let $L^2_+(\widetilde{v})$ be the same space with respect to the same measure, but replacing the parameters a, b, c, d by the dual parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. The main result in [8] states that $\mathcal{F}$ extends to an *isometric isomorphism*

$$\mathcal{F} : L^2_+(v) \to L^2_+(\widetilde{v}).$$

3. The Askey-Wilson Paley-Wiener space

3.1. **Reproducing kernel Hilbert spaces.** We will now introduce some concepts concerning reproducing kernel Hilbert spaces. This exposition is taken from [14], [11] and [21].

Let H_{rep} be a class of complex-valued functions, defined in a set $X \subset \mathbb{C}$, such that H_{rep} is a Hilbert space. We say that $k(\gamma, x)$ is a *reproducing kernel* of H_{rep} if $k(\gamma, x) \in H_{rep}$ for every $\gamma \in X$ and every $f \in H_{rep}$ satisfies the reproducing equation

$$f(\gamma) = \langle f(.), k(., \gamma)\rangle_{H_{rep}}.$$

Now we will use the language in Saitoh [21] and proceed to give a brief account of the required results.

Consider a second Hilbert space, H. For each t belonging to a domain X, let $K(.,t)$ belong to H. Then,

$$k(\gamma,x)=\langle K(.,\gamma),K(.,x)\rangle_H$$

is defined on $X\times X$. Suppose that we have an isometric transformation

$$(Fg)(\gamma)=\left\langle g,\overline{K(,\gamma)}\right\rangle_H$$

and denote the set of images by $F(H)$. The following theorem can be found in [21]:

Theorem A. *If F is a one-to-one isometric transformation, the kernel $k(\gamma,x)$ determines uniquely a reproducing kernel Hilbert space for which it is the reproducing kernel. This reproducing kernel Hilbert space is precisely $F(H)$, and it can have no other reproducing kernel. If $\{S_n\}$ is a basis of $F(H)$, then*

$$k(\gamma,x)=\sum_n S_n(\gamma)S_n(x).$$

There is a general formulation of the sampling theorem in reproducing kernel Hilbert spaces [15]. We will use the following "orthogonal basis case".

Theorem B. *With the notation established earlier, we have: If there exists $\{t_n\}_{n\in\mathbf{I}\subset\mathbb{Z}}$ such that $\{K(.,t_n)\}_{n\in\mathbf{I}}$ is an orthogonal basis, we then have the sampling expansion*

$$f(t)=\sum_{n\in\mathbf{I}} f(t_n)\frac{k(t,t_n)}{k(t_n,t_n)}$$

in $F(H)$, pointwise over $\mathbf{I}$ and uniformly over any compact subset of X for which $\|K_t\|$ is bounded.

The chief example of a reproducing kernel Hilbert space is PW. In this situation the reproducing kernel is the function $\sin\pi(x-\gamma)/\pi(x-\gamma)$, the sampling points are $t_n=n$ and the uniformly convergent expansion is the Whittaker-Shannon-Kotel′nikov sampling formula.

3.2. **The Askey-Wilson function reproducing kernel.** Let us look at the reproducing kernel Hilbert space associated to the Askey-Wilson function transform.

The first task is to consider a proper analogue of band-limited functions. This is done by defining a finite continuous Askey-Wilson function in much the same way it was done in [6].

We start by removing the poles of the function $\phi_{\tilde{a}q^\mu}$: Consider a function u_μ, analytic in the variable μ, defined as

$$u_\mu(x,a,b,c,d\mid q)=(\tilde{a}q^\mu;\tilde{a}q^{-\mu};q)_\infty\phi_{\tilde{a}q^\mu}\left(e^{i\theta}\right),\ x=\cos\theta.$$

Then we consider what is going to be the analogue of the transform (1.2): if $\max(|a|,|b|,|c|,|d|)<1$, the *finite continuous Askey-Wilson transform* $\mathcal{J}$ is defined by

$$\mathcal{J}(f)(\mu)=\int_{-1}^{1}f(x)u_\mu(x;a,b,c,d\mid q)\,w(x,a,b,c,d\mid q)dx. \tag{3.1}$$

The continuous Askey-Wilson transform relates to the Askey-Wilson transform as follows: If $\breve{f}$ is the analytic function such that $f(\cos\theta) = \breve{f}\left(e^{i\theta}\right)$, then

$$\mathcal{J}(f)(\mu) = \frac{4i\pi}{K}(\tilde{a}q^{\mu};\tilde{a}q^{-\mu};q)_{\infty}\mathcal{F}\left(\frac{\breve{f}}{\Theta}\right)(\tilde{a}q^{\mu}).$$

Definition 1. The *Askey-Wilson Paley-Wiener space*, $\mathcal{PW}_{AW}$, is the space constituted by the analytic functions $f \in L^2_+(v)$ such that, for some $u \in L^2(w(x,a,b,c,d \mid q), dx)$,

$$f = \mathcal{J}(u).$$

Let us look at this particular setting from the point of view of Theorem A.

Theorem 1. *If* $\max(|a|,|b|,|c|,|d|) < 1$, *then the set* $\mathcal{PW}_{AW}$ *is a Hilbert space of entire functions with reproducing kernel* $k(\gamma,\lambda)$. *The functions*

$$S_n^{(\tilde{a})}(\mu;q) = \frac{(-1)^n q^{\frac{n(n+1)}{2}}\left(1-\tilde{a}^2q^{2n}\right)(\tilde{a}q^{\mu},\tilde{a}q^{-\mu};q)_{\infty}}{(q;q)_n(a,\tilde{a}^2q^n;q)_{\infty}(1-\tilde{a}q^{n+\mu})(1-\tilde{a}q^{n-\mu})}$$

constitute an orthogonal basis of PW_{AW}, *and the reproducing kernel is given explicitly by*

$$k(\gamma,\lambda) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})}(\gamma;q)\, S_n^{(\tilde{a})}(\lambda;q).$$

Proof. To fulfill the conditions in Theorem A, we need to show that the finite continuous Askey-Wilson transform is a one-to-one isomorphism between $\mathcal{A}_{AW}$ and $\mathcal{PW}_{AW}$. To see that it is one-to-one, observe that, since, if

$$\int_{-1}^{1} f(x)u_{\mu}(x;a,b,c,d \mid q)\, w(x,a,b,c,d \mid q)dx = 0, \text{for all } \mu \in \mathbb{C},$$

then we have, in particular, that

$$\int_{-1}^{1} f(x)u_n(x;a,b,c,d \mid q)\, w(x,a,b,c,d \mid q)dx, \quad \text{for } n = 0,1,....$$

Since for integer values of μ, u_{μ} is a multiple of the Askey-Wilson polynomials,

$$u_n(x;a,b,c;d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab,ac,bc;q)_n} d^{-n} p_n(x;a,b,c,d), \tag{3.2}$$

we can use the completeness of the system of the Askey-Wilson polynomials to get $f = 0$. Consequently, $\mathcal{J}(f)$ is one-to-one. From the definition,

$$\mathcal{PW}_{AW} = \mathcal{J}\left[L^2(w(x,a,b,c,d \mid q)dx)\right].$$

Therefore, endowing $\mathcal{PW}_{AW}$ with the inner product

$$\langle \mathcal{J}(f), \mathcal{J}(g)\rangle_{\mathcal{PW}_{AW}} = \int_{-1}^{1} f(x)\overline{g(x)}w(x,a,b,c,d \mid q)dx, \tag{3.3}$$

the finite Askey-Wilson transform $\mathcal{J}$ becomes a Hilbert space isometry between $L^2(w(x,a,b,c,d \mid q)dx)$ and $\mathcal{PW}_{AW}$.

It remains to show that the functions $S_n^{(\tilde{a})}(\mu;q)$ provide an orthogonal basis for $\mathcal{PW}_{AW}$. By the definitions (3.1) and (3.2),

$$\begin{aligned}\mathcal{J}(u_n)(\mu) &= \int_{-1}^{1} u_n(x)u_\mu(x)\, w(x,a,b,c,d \mid q)dx \\ &= \frac{(-1)^n q^{-n(n-1)/2}}{d^n(ab,ac,bc;q)_n}\int_{-1}^{1} p_n(x)u_\mu(x)\, w(x,a,b,c,d \mid q)dx.\end{aligned}$$

Now we can use Proposition 6 of [6] to conclude that

$$\mathcal{J}(u_n)(\mu) = S_n^{(\tilde{a})}(\mu;q).$$

By (3.2), $\{u_n\}$ is an orthogonal basis of $L^2(w(x,a,b,c,d \mid q)dx)$. Since $\mathcal{J}$ is isometric onto $\mathcal{PW}_{AW}$, it follows that $S_n^{(\tilde{a})}(\mu;q)$ is a basis of $\mathcal{PW}_{AW}$. □

Remark 1. The functions $S_n^{(\tilde{a})}(\gamma;q)$ play the same role in our setting as do the functions $\sin\pi(x-n)/\pi(x-n)$ in the Paley-Wiener space.

Now, Theorem 1 and Theorem B give the following sampling theorem. This has been proved in [6], but the approach with reproducing kernels provides the uniform convergence that will be used in the next section.

Theorem 2. *For $f \in PW_{AW}$ we have*

$$f(\mu) = \sum_{n=0}^{\infty} f(n)\, S_n^{(\tilde{a})}(\mu;q), \tag{3.4}$$

where $S_n^{(\tilde{a})}(\mu;q)$ is given by

$$S_n^{(\tilde{a})}(\mu;q) = \frac{(-1)^n q^{\frac{n(n+1)}{2}}\left(1-\tilde{a}^2q^{2n}\right)\left(\tilde{a}q^{\mu},\tilde{a}q^{-\mu};q\right)_\infty}{(q;q)_n\left(a,\tilde{a}^2q^n;q\right)_\infty\left(1-\tilde{a}q^{n+\mu}\right)\left(1-\tilde{a}q^{n-\mu}\right)}.$$

The convergence is uniform on every compact subset of the real line.

Proof. Observe that from

$$S_n^{(\tilde{a})}(m;q) = \delta_{n,m},$$

we obtain

$$g(\mu,m) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})}(\mu;q)\, S_n^{(\tilde{a})}(m;q) = S_m^{(\tilde{a})}(\mu;q).$$

Moreover,

$$g(m,m) = S_m^{(\tilde{a})}(m;q) = 1,$$

and the result follows from Theorem 1 and Theorem B. □

4. The Askey-Wilson Paley-Wiener theorem

Recall that the entire function f is of order ρ if

$$\lim_{r\longrightarrow\infty}\frac{\ln\ln(M(r;f))}{\ln r} = \rho.$$

A constant has order zero, by convention.

The entire function f of positive order ρ is of type τ if

$$\lim \frac{\ln(M(r;f))}{r^\rho} = \tau.$$

Definition 2. The space $\mathcal{A}_{AW}$, which will be the analogue of $\mathcal{A}$ in the Askey-Wilson setting, is the space constituted of analytic functions f such that $f \in L^2(w(x,a,b,c,d \mid q),dx)$ and

$$M(r;f) = O(e^{\ln(1/q)r^2}),$$

that is, of order 2 and type $\ln(1/q)$.

It is easy to see that the functions in $\mathcal{A}_{AW}$ satisfy the conditions in [18, Theorem 3.1]. We rewrite this statement as:

Theorem C. *Every $f \in \mathcal{A}_{\mathcal{AW}}$ admits the expansion*

$$f(\mu) = \sum_{n=0}^{\infty} f(n)\, S_n^{(\tilde{a})}(\mu;q)\,. \tag{4.1}$$

The next result is the Paley-Wiener theorem for the Askey-Wilson function transform. The cornerstone of its proof is the fact that the entire function expansion (4.1) and the sampling expansion (3.4) are *exactly* the same.

Theorem 3. *If* $\max(|a|,|b|,|c|,|d|) < 1$*, then* $\mathcal{A}_{AW} = \mathcal{PW}_{AW}$.

Proof. Take $f \in PW_{AW}$. By definition we have, for some $u \in L^2(w(x,a,b,c,d \mid q),dx)$,

$$f(\mu) = \mathcal{J}(u)(\mu) = \int_{-1}^{1} u(x)u_\mu(x)\, w(x,a,b,c,d \mid q)dx.$$

We need to study the growth of

$$M(r;u_\mu)\,.$$

From formula (5.4) in [24] and for $0 \le \theta \le \pi$, we have

$$u_r(x) = \frac{\left(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, qe^{i\theta}/d;q\right)_\infty}{(ab,ac,bc,e^{2i\theta};q)_\infty}\left(q^{1-r}/e^{i\theta}d;q\right)_\infty [1+o(1)], \quad \text{as} \quad r \longrightarrow \infty.$$

Let $-1 < \delta < 0$. Then

$$M(n+\delta;u_\mu) = O\left(\left(q^{1-\delta-n}/d;q\right)_n\right).$$

This implies that

$$M(n+\delta;u_\mu) = O\left((q/d)^n q^{-n(n+1+2\delta)/2}\right).$$

Therefore,

$$\limsup_{r\to\infty} \frac{\ln\ln(M(r;u_\mu))}{\ln r} = 2$$

and

$$\limsup_{r\to\infty} \frac{\ln(M(r;u_\mu))}{r^2} = \ln(1/q)\,.$$

This condition implies that u_μ is of order 2 and type at most $\ln(1/q)$. Therefore,

$$u_\mu(x) \in \mathcal{A}_{AW}.$$

This shows that $f \in \mathcal{A}_{AW}$. Conversely, let $f \in \mathcal{A}_{AW}$. By Theorem C,

$$f(\mu) = \sum_{n=0}^{\infty} f(n)\, S_n^{(\tilde{a})}(\mu;q)\,.$$

In the end of the proof of Theorem 1 we have seen that

$$S_n^{(\tilde{a})}(\mu;q)=\mathcal{J}(u_n)(\mu).$$

Then, the sampling formula of Theorem 2 can be written as

$$\begin{aligned}f(\mu)&=\sum_{n=0}^{\infty}f(n)\,\mathcal{J}(u_n)(\mu)\\&=\sum_{n=0}^{\infty}f(n)\int_{-1}^{1}u_n(x)u_\mu(x)\,w(x,a,b,c,d\mid q)dx.\end{aligned}$$

The uniform convergence of the sampling series allows us to interchange the integral with the sum in such a way that

$$f(\mu)=\int_{-1}^{1}\left(\sum_{n=0}^{\infty}f(n)\,u_n(x)\right)u_\mu(x)\,w(x,a,b,c,d\mid q)dx.$$

Then we have written f in the form

$$f(\mu)=\mathcal{J}(u)(\mu),$$

with

$$u(x)=\left(\sum_{n=0}^{\infty}f(n)\,u_n(x)\right)\in L^2(w(x,a,b,c,d\mid q),dx).$$

As a result, $f\in\mathcal{PW}_{AW}$. □

References

[1] L. D. Abreu, *A q-sampling theorem related to the q-Hankel transform*, Proc. Amer. Math. Soc. 133 (2005), 1197-1203. MR2117222 (2006f:33014)

[2] M. H. Annaby, *q-type sampling theorems,* Result. Math. 44 (3-4) (2003), 214-225. MR2028677 (2004k:33035)

[3] R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 54 (1985), no. 319. MR783216 (87a:05023)

[4] C. Berg, H. L. Pedersen, with an appendix by Walter Hayman, *Logarithmic order and type of indeterminate moment problems,* Proceedings of the International Conference : “Difference equations, special functions and orthogonal polynomials”, Munich, July 25-30, 2005. World Scientific Publishing Company, Singapore, 2007. MR2450118 (2009i:44007)

[5] F. Bouzeffour, *A q-sampling theorem and product formula for continuous q-Jacobi functions,* Proc. Amer. Math. Soc 135 (2007), 2131-2139. MR2299491 (2008k:33055)

[6] F. Bouzeffour, *A Whittaker-Shannon-Kotel′nikov sampling theorem related to the Askey-Wilson functions,* J. Nonlinear Math. Phys. 14 (2007), no. 3, 367–380. MR2350096 (2008h:94031)

[7] Ó. Ciaurri and J. L. Varona, *A Whittaker-Shannon-Kotel′nikov sampling theorem related to the Dunkl transform*, Proc. Amer. Math. Soc. 135 (2007), 2939–2947. MR2317972 (2008c:94013)

[8] E. Koelink, J.V. Stokman, *The Askey-Wilson function transform*, Int. Math. Res. Not. IMRN, no. 22, 1203–1227 (2001). MR1862616 (2003a:33010)

[9] E. Koelink, J.V. Stokman, *The Askey-Wilson function transform scheme,* “Special functions 2000: Current perspective and future directions” (Tempe, AZ), 221–241, NATO Sci. Ser. II Math. Phys. Chem., 30, Kluwer Acad. Publ., Dordrecht, 2001. MR2006290 (2005b:33019)

[10] T. H. Koornwinder, *A new proof of a Paley-Wiener type theorem for the Jacobi transform*, Ark. Mat. 13 (1975), 145–159. MR0374832 (51:11028)

[11] A. G. Garcia, *Orthogonal sampling formulas: A unified approach*, SIAM Rev. 42 (2000), no. 3, 499–512. MR1786936 (2001i:94034)

[12] G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge University Press, Cambridge, UK, 1990. MR1052153 (91d:33034)

[13] G. H. Hardy, *Notes on special systems of orthogonal functions. IV*, Proc. Cambridge Phil. Soc. 37 (1941), 331-348. MR0005145 (3:108b)

[14] J. R. Higgins, *An interpolation series associated with the Bessel-Hankel transform* , J. Lond. Math. Soc. 5 (1972), 707-714. MR0320616 (47:9152)

[15] J. R. Higgins, *A sampling principle associated with Saitoh's fundamental theory of linear transformations.* Analytic extension formulas and their applications (Fukuoka, 1999/Kyoto, 2000), 73–86, Int. Soc. Anal. Appl. Comput., 9, Kluwer Acad. Publ., Dordrecht, 2001. MR1830378 (2002c:46053)

[16] M. E. H. Ismail, M. Rahman, *The associated Askey-Wilson polynomials,* Trans. Amer. Math. Soc. 328 (1991), 201-239. MR1013333 (92c:33019)

[17] M. E. H. Ismail, A. I. Zayed, *A q-analogue of the Whittaker-Shannon-Kotel′nikov sampling theorem*, Proc. Amer. Math. Soc. 131 (2003), 3711-3719. MR1998178 (2004e:33013)

[18] M. E. H. Ismail, D. Stanton, *q-Taylor theorems, polynomial expansions, and interpolation of entire functions.* J. Approx. Theory 123 (2003), no. 1, 125–146. MR1985020 (2004g:30040)

[19] M. E. H. Ismail, D. Stanton, *Applications of q-Taylor theorems,* J. Comput. Appl. Math. 153 (2003), no. 1-2, 259–272. MR1985698 (2004f:33035)

[20] J. P. Ramis, *About the growth of entire functions solutions of linear algebraic q-difference equations*, Ann. Fac. Sci. Toulouse Math. (6) 1 (1992), no. 1, 53–94. MR1191729 (94g:39003)

[21] S. Saitoh, *Integral transforms, reproducing kernels and their applications* (Harlow: Longman), Pitman Research Notes in Mathematics Series, 369, Longman, Harlow, 1997. MR1478165 (98k:46041)

[22] J. V. Stokman, *An expansion formula for the Askey-Wilson function.* J. Approx. Theory 114 (2002), no. 2, 308–342. MR1883410 (2003k:33025)

[23] S. K. Suslov, *Some orthogonal very-well-poised* ${}_8\varphi_7$*-functions*, J. Phys. A 30 (1997), 5877-5885. MR1478393 (98m:33047)

[24] S. K. Suslov, *Some orthogonal very-well-poised* ${}_8\varphi_7$*-functions that generalize the Askey-Wilson polynomials,* Ramanujan J. 5 (2001), 183-218. MR1857183 (2002m:33018)

[25] R. M. Young, *An introduction to nonharmonic Fourier series*, revised first edition, Academic Press, New York, 2001. MR1836633 (2002b:42001)

[26] A. I. Zayed, *Advances in Shannon's sampling theory*, CRC Press, Boca Raton, FL, 1993. MR1270907 (95f:94008)

Department of Mathematics, Centre for Mathematics, School of Science and Technology (FCTUC), University of Coimbra, 3001-454 Coimbra, Portugal

E-mail address: daniel@mat.uc.pt

Faculté des Sciences, Institut Préparatoire aux Études D'Ingénieur de Bizerte, 7021 Jarzouna, Bizerte, Tunisie

E-mail address: Fethi.Bouzeffour@ipeib.rnu.tn