PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 8, August 2010, Pages 3021–3024
S 0002-9939(10)10363-3
Article electronically published on March 23, 2010

A CORRECTION TO "ADJUGATES IN BANACH ALGEBRAS"

R. M. BRITS

(Communicated by Nigel J. Kalton)

ABSTRACT. Let A be a semisimple unital Banach algebra. We show that $\mathrm{rank}_A(ab) = \mathrm{rank}_A(ba)$ for all $a, b \in A$ if and only if $\mathrm{soc}(A)$ is contained in the center of A, and $ab \in \mathrm{soc}(A)$ implies $ba \in \mathrm{soc}(A)$ for all $a, b \in A$. This corrects an erroneous statement in R.E. Harte and C. Hernández, *Adjugates in Banach algebras*, Proc. Amer. Math. Soc. 134(5) (2005), 1397–1404.

If A is a semisimple Banach algebra with identity element $\mathbf{1}$, then the rank of an element $a \in A$ is defined by

$$\mathrm{rank}_A(a) = \sup_{x \in A} \#\sigma'_A(xa) = \sup_{x \in A} \#\sigma'_A(ax) \leq \infty. \tag{1}$$

Here $\sigma_A(x)$ denotes the spectrum of $x \in A$, $\sigma'_A(x) = \sigma_A(x)\backslash\{0\}$ and $\#\sigma'_A(x)$ is the number of elements in $\sigma'_A(x)$. It can be shown that the set of finite rank elements of A coincides with the socle of A (denoted $\mathrm{soc}(A)$).

Theorem 0.3 of this paper corrects the erroneous claim in [4] (see (2.5)) that $\mathrm{rank}_A(ab) = \mathrm{rank}_A(ba)$ for all $a, b \in A$. The difficulty here stems from the fact that Jacobson's lemma ([1, p. 33]) is generally speaking not valid for permutations of more than two elements. So although $\sigma'_A(ab) = \sigma'_A(ba)$ for all $a, b \in A$, it is not necessarily true that $\mathrm{rank}_A(ab) = \mathrm{rank}_A(ba)$ for all $a, b \in A$. This is already evident from the fact that in the operator case the standard rank, $\mathrm{rank}(T) = \dim \mathcal{R}(T)$, is equivalent to the spectral rank (1). We start with two lemmas. The first, Lemma 0.1, is an extension of Aupetit and Mouton's diagonalization theorem.

Lemma 0.1. *Let A be a semisimple Banach algebra and $0 \neq a \in \mathrm{soc}(A)$. If there exists $y \in A$ commuting with a such that $\mathrm{rank}_A(a) = \#\sigma'_A(ya)$, then there exist $n = \mathrm{rank}_A(a)$ mutually orthogonal minimal projections $p_1, \ldots, p_n$ and non-zero scalars $\alpha_1, \ldots, \alpha_n$ (not necessarily distinct) such that*

$$a = \sum_{j=1}^{n} \alpha_j p_j.$$

Proof. Suppose $\mathrm{rank}_A(a) = \#\sigma'_A(ay) = n$ and that $ya = ay$. We first show that this hypothesis implies that we can actually take y invertible. If a is invertible, then for $b \in A$ arbitrary observe that

$$\#\sigma'_A(b) = \#\sigma'_A(a(a^{-1}b)) \leq \mathrm{rank}_A(a) = n,$$

Received by the editors January 15, 2009 and, in revised form, October 16, 2009.
2010 *Mathematics Subject Classification.* Primary 46H05, 46H10, 47A10.

which shows that every element of A has a finite spectrum. Consequently, if a is invertible, the Hirschfeld–Johnson criterion implies that A is finite dimensional, and so the Wedderburn–Artin theorem forces y to be invertible. On the other hand, if $0 \in \sigma_A(a)$, then $0 \in \sigma_A(ax)$ for all $x \in A$ because a, and hence also ax, is a left topological divisor of zero. Since the function $\lambda \mapsto a(\lambda - y)$ is analytic from $\mathbb{C}$ into A, and $0 \in \sigma_A(a(\lambda - y))$ for all $\lambda \in \mathbb{C}$ the scarcity theorem [1, Theorem 3.4.25] says that $\{\lambda \in \mathbb{C} : \#\sigma'_A(a(\lambda - y)) < n\}$ is discrete in $\mathbb{C}$. Hence we can find λ in the resolvent set of y such that $\#\sigma'_A(a(\lambda - y)) = n = \operatorname{rank}_A(a)$. So without loss of generality we may assume $y \in A^{-1}$. By Aupetit and Mouton's diagonalization theorem [2, Theorem 2.8] there exist mutually orthogonal minimal projections $p_1, \ldots, p_n$ and distinct non-zero scalars $\lambda_1, \ldots, \lambda_n$ such that $ay = \sum_{j=1}^{n} \lambda_j p_j$. Since for each j,

$$p_j = \frac{1}{2\pi i} \int_{\Gamma_j} (\lambda - ay)^{-1} \, d\lambda,$$

where Γ_j is a small circle surrounding λ_j and separating λ_j from the remaining spectrum of ay, we see that y^{-1} commutes with the integrand and thus with p_j. From the minimality of p_j there exists a corresponding $\beta_j \in \mathbb{C}$ such that

$$a = \sum_{j=1}^{n} \lambda_j p_j y^{-1} = \sum_{j=1}^{n} \lambda_j p_j y^{-1} p_j = \sum_{j=1}^{n} \lambda_j \beta_j p_j.$$

If some $\beta_j = 0$, then $p_j y^{-1} = y^{-1} p_j$ implies that $y^{-1} p_j = p_j y^{-1} p_j = \beta_j p_j = 0$ and consequently $p_j = 0$, which is a contradiction. So, $\beta_j \neq 0$ and the thesis follows with $\alpha_j = \lambda_j \beta_j$. □

Observe that if $y = \mathbf{1}$, then Lemma 0.1 is precisely Aupetit and Mouton's diagonalization theorem, from which one may further conclude that the α_j are distinct.

For Lemma 0.2 recall that if A is a semisimple Banach algebra and p is a projection in A, then pAp is a semisimple Banach algebra with identity element p. Moreover, if $z \in pAp$, then $\sigma'_A(z) = \sigma'_{pAp}(z)$.

Lemma 0.2. *If A is a semisimple Banach algebra and $a \in \operatorname{soc}(A)$ is a linear combination of mutually orthogonal projections, say $\{p_1, \ldots, p_n\}$, then there exists a finite dimensional, unital and semisimple subalgebra B of A such that $a \in B$ and $\operatorname{rank}_A(a) = \operatorname{rank}_B(a)$. In particular, we can take $B = pAp$ where $p = \sum_{j=1}^{n} p_j$ is the identity in B and, moreover, a is then invertible with respect to p in B.*

Proof. If $a \in \operatorname{soc}(A)$ and $a = \lambda_1 p_1 + \cdots + \lambda_n p_n$, then the orthogonality of the p_j implies that each $p_j \in \operatorname{soc}(A)$. Also, it is obvious that each $p_j \in pAp$. From [2, Theorem 2.16] we have that $\operatorname{rank}_A(a) = \sum_{j=1}^{n} \operatorname{rank}_A(p_j)$ and similarly, applying [2, Theorem 2.16] in the algebra pAp, $\operatorname{rank}_{pAp}(a) = \sum_{j=1}^{n} \operatorname{rank}_{pAp}(p_j)$. But, from the remarks preceding Lemma 0.2 (together with Jacobson's lemma), we have, for each j,

$$\begin{aligned} \operatorname{rank}_{pAp}(p_j) &= \sup_{x \in A} \#\sigma'_{pAp}(p_j p x p) = \sup_{x \in A} \#\sigma'_A(p_j p x p) \\ &= \sup_{x \in A} \#\sigma'_A(p_j x) = \operatorname{rank}_A(p_j), \end{aligned}$$

which proves the first part. The inverse of a in pAp is of course $\sum_{j=1}^{n} \frac{1}{\lambda_j} p_j$. □

In the remaining part of this paper, $Z(A)$ denotes the center of A.

Theorem 0.3. *If A is a semisimple Banach algebra, then* $\operatorname{rank}_A(ab) = \operatorname{rank}_A(ba)$ *for all $a, b \in A$ if and only if both of the following conditions are met:*

(i) $\operatorname{soc}(A) \subset Z(A)$*;*
(ii) $ab \in \operatorname{soc}(A) \Rightarrow ba \in \operatorname{soc}(A)$.

Proof. $\Rightarrow$ Since the finite rank elements coincide with $\operatorname{soc}(A)$, it follows trivially that (ii) holds. To prove (i) it suffices to show, from [3, Corollary 2.2], that 0 is the only nilpotent element of $\operatorname{soc}(A)$. Suppose there is a nilpotent finite rank element a with $a^n = 0\,(n \geq 2)$ and $a^k \neq 0$ for $k < n$. With our hypothesis we have, for each $b \in A$, that $\operatorname{rank}_A(aba^{n-1}) = \operatorname{rank}_A(a^n b) = 0$. So

$$\operatorname{rank}_A(a^{n-2}(aba^{n-1})) \leq \operatorname{rank}_A(aba^{n-1}) \Rightarrow \operatorname{rank}_A(a^{n-1}ba^{n-1}) = 0,$$

from which it follows that $(a^{n-1}b)^2 = 0$ and hence that $\sigma_A(a^{n-1}b) = \{0\}$. Since A is semisimple we then have $a^{n-1} = 0$, contradicting the nilpotency index of a.
$\Leftarrow$ Let $a, b \in A$ be arbitrary. From the assumption (ii) we may suppose $\operatorname{rank}_A(ab) = n$. Then, again from (ii), $ba \in \operatorname{soc}(A) \subset Z(A)$. Since $ab \in Z(A)$, Lemma 0.1 gives

$$ab = \alpha_1 p_1 + \cdots + \alpha_n p_n,$$

where the p_i are distinct non-zero and mutually orthogonal minimal projections. Writing $p = \sum_{i=1}^n p_i$ it follows that ab is invertible in the semisimple finite dimensional algebra pAp. If we observe that $(ab)^2 = (ba)^2$, then $(ba)^2$ belongs to and is invertible in pAp. Invertible elements all having equal rank, we get, using Lemma 0.2,

$$\operatorname{rank}_A(ab) = \operatorname{rank}_{pAp}(ab) = \operatorname{rank}_{pAp}((ba)^2) = \operatorname{rank}_A((ba)^2) \leq \operatorname{rank}_A(ba).$$

So we have shown, for all $a, b \in A$, that $\operatorname{rank}_A(ab) \leq \operatorname{rank}_A(ba)$ and hence the theorem is proved. □

The following simple example shows that (ii) is not superfluous in Theorem 0.3: Let $C(l^2)$ be the ideal of compact operators on l^2 and denote the Calkin algebra by $\mathcal{C} = B(l^2)/C(l^2)$. Then $\mathcal{C}$ is a B^*–algebra and hence semisimple. Denote by $\Omega : B(l^2) \to \mathcal{C}$, $\Omega(T) = \tilde{T} = T + C(l^2)$ the canonical quotient homomorphism. If $\operatorname{soc}(\mathcal{C}) \neq \{\tilde{0}\}$, then $\Omega^{-1}(\operatorname{soc}(\mathcal{C})) \neq B(l^2)$ would be a two-sided ideal of $B(l^2)$ properly containing $C(l^2)$, which violates the fact that $C(l^2)$ is the unique non-trivial two-sided ideal of $B(l^2)$. So $\operatorname{soc}(\mathcal{C}) = \{\tilde{0}\}$. Let $T \in B(l^2)$ be defined by the standard unilateral shift followed by the projection on l^2 which annihilates odd coordinates; let $S \in l^2$ be the projection which annihilates even coordinates. So if we take the semisimple algebra $A = \mathcal{C} \times \mathbb{C}^n$, then $\operatorname{soc}(A) \subset Z(A)$. However with $a = (\tilde{S}, \mathbf{1})$ and $b = (\tilde{T}, \mathbf{1})$ we get $\operatorname{rank}_A(ab) = n$ but $\operatorname{rank}_A(ba) = \infty$.

References

[1] B. Aupetit, *A primer on spectral theory*, Springer-Verlag, New York, 1991. MR1083349 (92c:46001)
[2] B. Aupetit and H. du T. Mouton, *Trace and determinant in Banach algebras*, Studia Math. 121(2) (1996), 115–136. MR1418394 (97i:46086)

[3] R. Brits, L. Lindeboom and H. Raubenheimer, *Rank and the Drazin inverse in Banach algebras*, Studia Math. 177(3) (2006), 211–223 MR2284455 (2007j:46079)

[4] R.E. Harte and C. Hernández, *Adjugates in Banach algebras*, Proc. Amer. Math. Soc. 134(5) (2005), 1397–1404. MR2199186 (2007g:46071)

Department of Mathematics, University of Johannesburg, P.O. Box 524, Auckland Park, 2006, Johannesburg, South Africa

E-mail address: rbrits@uj.ac.za