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AN INTEGRAL EQUATION ON HALF SPACE

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Abstract. Let R^n_+ be the n-dimensional upper half Euclidean space, and let α be any real number satisfying $0 < \alpha < n$. In this paper, we consider the integral equation

$$u(x) = \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) u^\tau(y), \quad u(x) > 0, \quad \forall x \in R^n_+, \tag{1}$$

where $\tau = \dfrac{n+\alpha}{n-\alpha}$, and $x^* = (x_1, \cdots, x_{n-1}, -x_n)$ is the reflection of the point x about the hyperplane $x_n = 0$. We use a new type of moving plane method in integral forms introduced by Chen, Li and Ou to establish the regularity and rotational symmetry of the solution of the above integral equation.

1. Introduction

Let R^n be the $n-$dimensional Euclidean space, and let α be a real number satisfying $0 < \alpha < n$. Consider the integral equation

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u^\tau(y) dy \tag{2}$$

with $\tau = \dfrac{n+\alpha}{n-\alpha}$. It arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequalities (see [L]). In [L], Lieb classified the maximizers of the functional and thus obtained the best constant in the Hardy-Littlewood-Sobolev inequalities. He then posed the classification of all the critical points of the functional, i.e. the solutions of the integral equation (2), as an open problem.

In [CLO], Chen, Li, and Ou solved Lieb's open problem by using the method of moving planes. They proved that

Proposition 1. *Every positive regular solution $u(x)$ of (2) is radially symmetric and decreasing about some point x_0 and therefore assumes the form*

$$c\left(\frac{t}{t^2 + |x-x_0|^2}\right)^{(n-\alpha)/2}, \tag{3}$$

with some positive constants c and t.

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They also established the equivalence between the integral equation and the following well-known family of semi-linear partial differential equations:

$$(-\Delta)^{\alpha/2}u = u^{(n+\alpha)/(n-\alpha)}, \quad x \in R^n. \tag{4}$$

In the special case $\alpha = 2$, a series of results has been found concerning the classification of the solutions (cf. [GNN], [CGS], [CL], and [Li]). Recently, Wei and Xu [WX] generalized these results to the cases that α is any even number between 0 and n. Apparently, for any real value of α between 0 and n, equation (4) is also of practical interest and importance.

For more results concerning integral equations and the method of moving planes, please see [BN], [CL1], [CL2], [CL3], [CLO1], [CLO2] [CY], [CJ], [CJ1], [F], [LLim], [LiM] [MC], [MC2], [MZ], [O], [Se], and the references therein.

On the upper half Euclidean space,

$$R^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in R^n \mid x_n > 0\}.$$

The same equation naturally arises with Dirichlet boundary conditions. In particular, when α is an even number, it is interesting to study the following higher order semilinear elliptic equation:

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}}u(x) = u^\tau(x), & \forall x \in R^n_+; \\ (-\triangle)^k u(x) = 0, & \forall x \in \partial R^n_+, \quad k = 0, 1, \cdots, \dfrac{\alpha}{2} - 1. \end{cases} \tag{5}$$

The corresponding integral equation in the upper half space R^n_+ is

$$u(x) = C \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^\tau(y)dy, \tag{6}$$

and we will investigate its qualitative properties in this paper. One of our motivation is

Theorem 1. *Let $u(x)$ be the smooth solution of* (6)*. Then $u(x)$ satisfies* (5)*.*

We believe that the converse is also true. Of course, the integral equation is also interesting in its own right.

We prove regularity, rotational symmetry, and monotonicity of the solutions for the integral equations.

Theorem 2. *Let $u(x)$ be a solution of* (1)*. Assume that $u(x) \in L^{\frac{2n}{n-\alpha}}(R^n_+)$. Then $u(x)$ is in $L^q(R^n_+) \cap L^\infty(R^n_+)$, for any $1 < q < \infty$. Hence u is continuous.*

Theorem 3. *Every positive solution $u(x)$ of* (1) *is rotationally symmetric about some line parallel to the x_n-axis.*

In section 2, we will use the regularity lifting method to prove Theorem 2. In section 3, we will obtain rotational symmetry of the solutions by using the method of moving planes. In section 4, we will show the relation between the integral equations and PDEs.

2. Regularity of solutions

Since we shall use an equivalent form of the Hardy-Littlewood-Sobolev Inequality, we now list it as a lemma.

Lemma 2.1 (An Equivalent Form of the Hardy-Littlewood-Sobolev Inequality)**.** *Let $g \in L^p(R^n)$ for $\dfrac{n}{n-\alpha} < p < \infty$. Define*

$$Tg(x) = \int_{R^n} |x-y|^{\alpha-n} g(y)dy.$$

Then

$$\|Tg\|_{L^p(R^n)} \le C(n,p,\alpha)\|g\|_{L^{\frac{np}{n+\alpha p}}(R^n)}.$$

Theorem 2.1. *Let $u(x)$ be a solution of* (1)*. Assume that $u(x) \in L^{\frac{2n}{n-\alpha}}(R^n_+)$. Then $u(x)$ is in $L^q(R^n_+) \cap L^\infty(R^n_+)$, for any $1 < q < \infty$. Hence u is continuous.*

Proof of Theorem 2.1. The proof is divided into three steps.

Step 1. We first show that $u(x) \in L^q(R^n_+)$, $\forall x \in R^n_+$. Define

$$a(x) = u^{\tau-1}(x).$$

Then

$$u(x) = \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) a(y)u(y)dy.$$

For a positive number A, define

$$a_A(x) = \begin{cases} a(x), & \text{if } |a(x)| \ge A \text{ or } |x| \ge A; \\ 0, & \text{elsewhere.} \end{cases}$$

Let $a_B(x) = a(x) - a_A(x)$. Obviously, $|a_B(x)| \le A$, and $a_B(x)$ vanishes outside the ball $B_A(0)$.

Define

$$T_A v(x) = \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) a_A(y)v(y)dy,$$

$$F_A(x) = \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) a_B(y)u(y)dy.$$

Then equation (1) can be written as

$$u(x) = T_A u(x) + F_A(x).$$

We will show that, for any $1 < p < \infty$,
(i) $T_A(x)$ is a contracting map from $L^p(R^n_+)$ to $L^p(R^n_+)$ for A large, and
(ii) $F_A(x)$ is in $L^p(R^n_+)$.

(i) Assume $v \in L^p(R^n_+)$. Then

$$|T_A v(x)| \le \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) |a_A(y)v(y)|dy.$$

For any $p > \dfrac{n}{n-\alpha}$, we first apply the Hardy-Littlewood-Sobolev Inequality (Lemma 2.1) to obtain

$$\|T_A v\|_{L^p(R^n_+)} \le C\|a_A v\|_{L^{\frac{np}{n+\alpha p}}(R^n_+)}.$$

Then, we apply the Hölder inequality to the right hand side of the above:

$$\|T_A v\|_{L^p(R^n_+)} \le C\|a_A\|_{L^s(R^n_+)}\|v\|_{L^r(R^n_+)}.$$

Here we have chosen $s = \dfrac{n}{\alpha}$ and $r = p$. Then, it follows that

$$\|T_A v\|_{L^p(R^n_+)} \le C\|a_A\|_{L^{\frac{n}{\alpha}}(R^n_+)}\|v\|_{L^p(R^n_+)}.$$

Since $a(x) \in L^{\frac{n}{\alpha}}(R^n_+)$, by the definition of $a_A(x)$, one can choose a large number A such that

$$C\|a_A\|_{L^{\frac{n}{\alpha}}(R^n_+)} \le \frac{1}{2}$$

and hence arrive at

$$\|T_A v\|_{L^p(R^n_+)} \le \frac{1}{2}\|v\|_{L^p(R^n_+)}.$$

That is, T_A: $L^p(R^n_+) \to L^p(R^n_+)$ is a contracting map.

(ii) Consider

$$F_A(x) = \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) a_B(y)u(y)dy.$$

Obviously,

$$|F_A(x)| \le \int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) |a_B(y)u(y)|dy.$$

For any $p > \dfrac{n}{n-\alpha}$, we first apply the Hardy-Littlewood-Sobolev Inequality to obtain

$$\|F_A\|_{L^p(R^n_+)} \le C\|a_B u\|_{L^{\frac{np}{n+\alpha p}}(R^n_+)}.$$

Then, we apply the Hölder inequality to the right hand side of the above:

$$\|a_B u\|_{L^{\frac{np}{n+\alpha}}(R^n_+)} \le \|a_B\|_{L^s(R^n_+)}\|u\|_{L^r(R^n_+)} = \|a_B\|_{L^s(B_A(0))}\|u\|_{L^r(R^n_+)}.$$

Here,

$$\frac{1}{s} + \frac{1}{r} = \frac{n+\alpha}{np}.$$

By the boundedness of $a_B(x)$, we see that s can be arbitrary. Since $u \in L^{\frac{2n}{n-\alpha}}(R^n_+)$, we take $r = \dfrac{2n}{n-\alpha}$, and hence

$$\frac{1}{p} = \frac{1}{s} + \frac{n-3\alpha}{2n} = \frac{(2+s)n - 3\alpha p}{2ns}.$$

Since s is arbitrary, and as $s \to \infty, p \to \dfrac{2n}{n-3\alpha}$, we conclude that p can be chosen to be any of the following values for the given dimension n, for which $F_A(x) \in L^p(R^n_+)$:

$$\begin{cases} 1 < p < \infty, & \text{when } 3 \le n \le 3\alpha, \\ 1 < p < \dfrac{2n}{n-3\alpha}, & \text{when } n > 3\alpha. \end{cases}$$

If $n \leq 3\alpha$, we are done. If $n \geq 3\alpha$, we repeat the above process. After a few steps, we will arrive at

$$u(x) \in L^p(R^n_+), \quad \forall\, 1 < p < \infty.$$

Step 2. In this step, we are going to show that

$$u(x) \in L^\infty(R^n_+).$$

We split the integral into two parts:

$$\begin{aligned} u(x) &= \int_{B_1(x)} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^\tau(y)dy \\ &+ \int_{R^n_+ \setminus B_1(x)} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^\tau(y)dy \\ &\leq \int_{B_1(x)} \frac{1}{|x-y|^{n-\alpha}} u^\tau(y)dy + \int_{R^n_+ \setminus B_1(x)} \frac{1}{|x-y|^{n-\alpha}} u^\tau(y)dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

First consider I_2. Since $\dfrac{1}{|x-y|^{n-\alpha}} < 1$, and by the result in Step 1, $u \in L^\tau(R^n_+)$, we have $I_2 < C_1$.

Then for I_1, we apply the Hölder inequality,

$$I_1 \leq \left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)p}} dy \right)^{\frac{1}{p}} \left(\int_{B_1(x)} |u^\tau(y)|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}}.$$

Choose an appropriate p, so that $(n-\alpha)p < n$, and hence

$$\left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-\alpha)p}} dy \right)^{\frac{1}{p}} < C_2.$$

Since $u(x) \in L^q(R^n_+)$, $\forall q > \dfrac{n}{n-\alpha}$,

$$\left(\int_{B_1(x)} |u^\tau(y)|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}} < C_3.$$

We conclude that

$$u(x) \in L^\infty(R^n_+).$$

Step 3. Now we prove the continuity of $u(x)$. To this end, we divide the integral into two parts:

$$u(x) - u(z) = I_1 + I_2,$$

where

$$I_1 = \int_{B_R(x)} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) - \left(\frac{1}{|z-y|^{n-\alpha}} - \frac{1}{|z^*-y|^{n-\alpha}} \right) \right] u^\tau(y)dy.$$

$$I_2 = \int_{R^n_+ \setminus B_R(x)} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) - \left(\frac{1}{|z-y|^{n-\alpha}} - \frac{1}{|z^*-y|^{n-\alpha}} \right) \right] u^\tau(y)dy.$$

Since

$$\int_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^\tau(y)dy < \infty,$$

one can choose R sufficiently large so that I_2 is small.

We then estimate I_1. Obviously, we have

$$\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}} \to 0,$$

uniformly for all $y \in B_R(x)$, as $|x-z| \to 0$. Noticing that $|x^* - z^*| \to 0$, as $|x-z| \to 0$, we also have

$$\frac{1}{|x^*-y|^{n-\alpha}} - \frac{1}{|z^*-y|^{n-\alpha}} \to 0,$$

uniformly for all $y \in B_R(x)$, as $|x-z| \to 0$. Hence

$$\begin{aligned} &\lim_{|x-z|\to 0} \int_{B_R(x)} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}} \right) - \left(\frac{1}{|x^*-y|^{n-\alpha}} - \frac{1}{|z^*-y|^{n-\alpha}} \right) \right] u^\tau(y)dy \\ &= \int_{B_R(x)} \lim_{|x-z|\to 0} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}} \right) - \left(\frac{1}{|z^*-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) \right] u^\tau(y)dy \\ &= 0. \end{aligned}$$

Therefore $u(x)$ is continuous.

This completes the proof of the theorem. □

3. Symmetry of solutions

Theorem 3.1. *Every positive solution $u(x)$ of (1) is rotationally symmetric about some line parallel to the x_n-axis.*

To prove the theorem, we need a few lemmas.

For a given real number λ, define

$$\Sigma_\lambda = \left\{ x = (x_1, x_2, \cdots, x_n) \in R^n_+ \mid x_1 < \lambda \right\}$$

and let

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$$

and

$$u_\lambda(x) = u(x^\lambda).$$

Lemma 3.1. *For any solution $u(x)$ of (1), we have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) - \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) \right] [(u^\tau(y) - u_\lambda^\tau(y)]\, dy,$$

where $x^{\lambda *} = (2\lambda - x_1, x_2, \cdots, x_{n-1}, -x_n)$ is the reflection of x^λ about the plane ∂R^n_+.

Proof. Since $|x^\lambda - y| = |x - y^\lambda|$, we have

$$u(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^\tau(y) dy + \int_{\Sigma_\lambda} \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) u_\lambda^\tau(y) dy$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) u^\tau(y) dy + \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u_\lambda^\tau(y) dy.$$

Then

$$\begin{aligned} &u(x) - u_\lambda(x) \\ &= \int_{\Sigma_\lambda} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) - \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) \right] [(u^\tau(y) - u_\lambda^\tau(y)]\, dy. \end{aligned}$$

□

This completes the proof of the lemma.

Lemma 3.2. *For any $x, y \in \Sigma_\lambda$*

$$\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) - \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) > 0.$$

Proof. Write

$$\begin{aligned} \left(|x-y|^2\right)^{\frac{n-\alpha}{2}} &= \left[(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2\right]^{\frac{n-\alpha}{2}} := t^{\frac{n-\alpha}{2}}, \\ \left(|x^*-y|^2\right)^{\frac{n-\alpha}{2}} &= \left[(x_1-y_1))^2 + (x_2-y_2)^2 + \cdots + (-x_n-y_n)^2\right]^{\frac{n-\alpha}{2}} := m^{\frac{n-\alpha}{2}}, \\ \left(|x^\lambda-y|^2\right)^{\frac{n-\alpha}{2}} &= \left[(2\lambda-x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2\right]^{\frac{n-\alpha}{2}} \\ &:= (a+t)^{\frac{n-\alpha}{2}}, \\ \left(|x^{\lambda *}-y|^2\right)^{\frac{n-\alpha}{2}} &= \left[(2\lambda-x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (-x_n-y_n)^2\right]^{\frac{n-\alpha}{2}} \\ &:= (a+m)^{\frac{n-\alpha}{2}}, \end{aligned}$$

where

$$a = 4\lambda^2 - 2\lambda(x_1 + y_2) + 4x_1y_1.$$

Since

$$|x^\lambda - y|^2 > |x - y|^2,$$

i.e. $t + a > t$, we see that $a > 0$. So one only need to consider the monotonicity of the function

$$f(x) = \frac{1}{x^r} - \frac{1}{(x+a)^r}.$$

This can be seen from

$$f'(x) = \frac{-r}{x^{r+1}} + \frac{r}{(x+a)^{r+1}} < 0.$$

From this and the fact that $m > t$,

$$\begin{aligned}
&\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) - \left(\frac{1}{|x^\lambda-y|^{n-\alpha}} - \frac{1}{|x^{\lambda*}-y|^{n-\alpha}}\right)\\
&= \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda-y|^{n-\alpha}}\right) - \left(\frac{1}{|x^*-y|^{n-\alpha}} - \frac{1}{|x^{\lambda*}-y|^{n-\alpha}}\right)\\
&= \left(\frac{1}{|t|^{\frac{n-\alpha}{2}}} - \frac{1}{|t+\alpha|^{\frac{n-\alpha}{2}}}\right) - \left(\frac{1}{|m|^{\frac{n-\alpha}{2}}} - \frac{1}{|m+a|^{\frac{n-\alpha}{2}}}\right) > 0.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

The Proof of Theorem 3.1. We will use the method of moving planes to derive the symmetry and monotonicity of $u(x)$.

Step 1 (prepare to move the plane from near $x_1 = -\infty$).

We compare the values of $u_\lambda(x)$ and $u(x)$. For λ sufficiently negative, we are going to show that

$$u_\lambda(x) - u(x) \geq 0, \ \ \forall x \in \Sigma_\lambda. \tag{7}$$

To this end, let

$$w_\lambda(x) = u_\lambda(x) - u(x)$$

and define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \,|\, w_\lambda(x) < 0\},$$

the set where inequality (7) is violated. We will prove that Σ_λ^- is empty.

$$
\begin{aligned}
& u(x) - u_\lambda(x) \\
= & \int_{\Sigma_\lambda} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) \right. \\
& \left. - \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) \right] [(u^\tau(y) - u_\lambda^\tau(y)]\, dy \\
= & \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) \right. \\
& \left. - \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) \right] [u^\tau(y) - u_\lambda^\tau(y)]\, dy \\
& + \int_{\Sigma_\lambda^-} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) \right. \\
& \left. - \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) \right] [u^\tau(y) - u_\lambda^\tau(y)]\, dy \\
\leq & \int_{\Sigma_\lambda^-} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) \right. \\
& \left. - \left(\frac{1}{|x^\lambda - y|^{n-\alpha}} - \frac{1}{|x^{\lambda *} - y|^{n-\alpha}} \right) \right] [u^\tau(y) - u_\lambda^\tau(y)]\, dy \\
\leq & \int_{\Sigma_\lambda^-} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) [u^\tau(y) - u_\lambda^\tau(y)]\, dy \\
\leq & \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} [u^\tau(y) - u_\lambda^\tau(y)]\, dy \\
= & \tau \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} \psi_\lambda^{\tau-1}(y) [(u(y) - u_\lambda(y)]\, dy \\
\leq & \tau \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} u^{\tau-1}(y) [u(y) - u_\lambda(y)]\, dy,
\end{aligned}
$$

where $\psi_\lambda(y)$ is valued between $u_\lambda(y)$ and $u(y)$. Since on Σ_λ^-,

$$u_\lambda(y) < u(y),$$

we have

$$\psi_\lambda(y) < u(y).$$

We apply the equivalent form of the Hardy-Littlewood-Sobolev Inequality to (8) to obtain, for any $q > \dfrac{n}{n-\alpha}$,

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C \|u^{\tau-1} w_\lambda\|_{L^{\frac{nq}{n+\alpha q}}(\Sigma_\lambda^-)}.$$

Then apply the generalized Hölder inequality to the above. Choose

$$s = \frac{nq}{n+\alpha q}, \ r = q.$$

To ensure

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{t},$$

we choose

$$t = \frac{n}{\alpha}.$$

We thus arrive at

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \le C\|u^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)}\|w_\lambda\|_{L^q(\Sigma_\lambda^-)}, \qquad \forall q > \frac{n}{n-\alpha}. \tag{8}$$

Since $u \in L^{\frac{2n}{n-\alpha}}(R_+^n)$, we can choose N sufficiently large, such that for $\lambda < -N$, we have

$$C\|u^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda)} \le \frac{1}{2}.$$

Now inequality (8) implies that

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0,$$

and therefore Σ_λ^- must be measure zero and hence empty due to the continuity of $w_\lambda(x)$. Therefore, for λ sufficiently negative, we must have

$$w_\lambda(x) \ge 0, \quad \forall x \in \Sigma_\lambda.$$

Step 2 (move the plane to the limiting position to derive symmetry).

We continue to move the plane $T_\lambda = \left\{x \in R_+^n \,|\, x_1 = \lambda\right\}$ to the right as long as (7) holds. Define

$$\lambda_o = \sup\left\{\lambda \,|\, w_\mu(x) \ge 0, \mu \le \lambda, \forall x \in \Sigma_\mu\right\}.$$

We will show that

$$w_{\lambda_o}(x) \equiv 0. \tag{9}$$

Otherwise, there exists y_o, such that

$$w_{\lambda_o}(y_o) > 0.$$

By the continuity of $w_\lambda(x)$, there exists a neighborhood $N(y_o)$ of y_o, such that

$$w_{\lambda_o}(y) > 0, \quad \forall y \in N(y_o).$$

Consequently,

$$\begin{aligned}
& w_{\lambda_o}(x) = u_{\lambda_o}(x) - u(x) \\
= \; & \int_{\Sigma_{\lambda_o}} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right)\right. \\
& \qquad \left. - \left(\frac{1}{|x^{\lambda_o}-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_o *}-y|^{n-\alpha}}\right)\right]\left[u_{\lambda_o}^\tau(y) - u^\tau(y)\right] dy \\
\ge \; & \int_{N(y_o)} \left[\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right)\right. \\
& \qquad \left. - \left(\frac{1}{|x^{\lambda_o}-y|^{n-\alpha}} - \frac{1}{|x^{\lambda_o *}-y|^{n-\alpha}}\right)\right]\left[u_{\lambda_o}^\tau(y) - u^\tau(y)\right] dy \\
> \; & 0.
\end{aligned}$$

It follows that

$$w_{\lambda_o}(x) > 0, \quad \forall x \in \Sigma_{\lambda_o}. \tag{10}$$

For any small $\eta > 0$, we can choose R sufficiently large so that

$$\left(\int_{R^n_+ \setminus B_R(0)} u^{\frac{2n}{n-\alpha}}(y)dy\right)^{\frac{\alpha}{n}} < \eta. \tag{11}$$

From (10), for any $\delta > 0$, there exists $C_o > 0$ such that

$$w_{\lambda_o}(x) > C_o, \quad \forall x \in \Sigma_{\lambda_o - \delta} \cap B_R(0).$$

By the continuity of w_λ with respect to λ, one can choose ϵ sufficiently small so that

$$w_{\lambda_o+\epsilon}(x) \geq 0, \quad \forall x \in \Sigma_{\lambda_o-\delta} \cap B_R(0). \tag{12}$$

Now (11) and (12) imply that $\Sigma^-_{\lambda_o+\epsilon} = \{x \in \Sigma_{\lambda_o+\epsilon} \,|\, u(x) > u_{\lambda_o+\epsilon}(x)\}$ has no intersection with

$$\Sigma_{\lambda_o-\delta} \cap B_R(0);$$

hence it is contained in the union of

$$(\Sigma_{\lambda_o+\epsilon} \setminus \Sigma_{\lambda_o-\delta}) \cap B_R(0) \text{ and } R^n_+ \setminus B_R(0).$$

Obviously,

$$\mu((\Sigma_{\lambda_o+\epsilon} \setminus \Sigma_{\lambda_o-\delta}) \cap B_R(0))$$

is small for small ϵ and δ, and by (11), we have

$$c\|u^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_{\lambda_o+\epsilon})} < \frac{1}{2}. \tag{13}$$

Applying (8) with $\lambda = \lambda_o + \epsilon$, we have

$$\|w_{\lambda_o+\epsilon}\|_{L^q(\Sigma^-_{\lambda_o+\epsilon})} \leq C\|u^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma^-_{\lambda_o+\epsilon})}\|w_{\lambda_o+\epsilon}\|_{L^q(\Sigma^-_{\lambda_o+\epsilon})}, \qquad \forall q > \frac{n}{n-\alpha}. \tag{14}$$

By (13) and (14), we deduce

$$\|w_{\lambda_o+\epsilon}\|_{L^q(\Sigma^-_{\lambda_o+\epsilon})} = 0,$$

and therefore $\Sigma^-_{\lambda_o+\epsilon}$ must be empty, that is,

$$w_{\lambda_o+\epsilon}(x) \geq 0, \quad \forall x \in \Sigma_{\lambda_o+\epsilon}.$$

This contradicts the definition of λ_o, and hence (9) holds.

Since we can choose any direction that is perpendicular to the x_n-axis as the x_1 direction, we have actually shown that the solution $u(x)$ is rotationally symmetric about some axis parallel the to x_n-axis. □

4. The relation between integral equations and PDEs

Theorem 4.1. *Let $u(x)$ be the smooth solution of*

$$u(x) = C\int_{R^n_+}\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right)u^\tau(y)dy.$$

Then $u(x)$ satisfies

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}}u(x) = u^\tau(x), & \forall x \in R^n_+, \\ (-\triangle)^k u(x) = 0, & \forall x \in \partial R^n_+, \end{cases}$$

$k = 0, 1, \cdots, \dfrac{\alpha}{2} - 1.$

Proof. Since

$$(-\triangle)^k\left(\frac{1}{|x-y|^{n-\alpha}}\right)=(-\triangle)^k\left(\frac{1}{|x^*-y|^{n-\alpha}}\right),$$

for $x=x^*$, it is easy to verify that

$$(-\triangle)^k u(x)=0,\ \ k=0,1,\cdots,\frac{\alpha}{2}-1,\quad \forall x\in\partial R^n_+.$$

Then

$$\begin{aligned}(-\triangle)^{\frac{\alpha}{2}}u(x)&=C\int_{R^n_+}(-\triangle)^{\frac{\alpha}{2}}\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x^*-y|^{n-\alpha}}\right)u^\tau(y)dy\\&=C\int_{R^n_+}\delta(x-y)u^\tau(y)dy\\&=Cu^\tau(x).\end{aligned}$$

□

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