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ON THE LENGTH OF CRITICAL ORBITS OF STABLE QUADRATIC POLYNOMIALS

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Abstract. We use the Weil bound of multiplicative character sums, together with some recent results of N. Boston and R. Jones, to show that the *critical orbit* of quadratic polynomials over a finite field of q elements is of length $O(q^{3/4})$, improving upon the trivial bound q.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of q elements. For a polynomial $f \in \mathbb{F}_q[X]$ we define the sequence of iterations:

$$f^{(0)}(X) = X, \qquad f^{(n)}(X) = f\left(f^{(n-1)}(X)\right), \quad n = 1, 2, \ldots .$$

Following [1, 2, 8, 9], we say that f is *stable* if all polynomials $f^{(n)}$ are irreducible over $\mathbb{F}_q$.

We now assume that q is odd.

As in [9], for a quadratic polynomial $f(X) = aX^2 + bX + c \in \mathbb{F}_q[X]$, $a \neq 0$, we define $\gamma = -b/(2a)$ as the unique critical point of f (that is, the zero of the derivative f') and consider the set

$$\mathrm{Orb}(f) = \{f^{(n)}(\gamma) \ : \ n = 2, 3, \ldots\},$$

which is called the *critical orbit* of f. Clearly there is some t such that $f^{(t)}(\gamma) = f^{(s)}(\gamma)$ for some positive integer $s < t$. Then $f^{(n+t)}(\gamma) = f^{(n+s)}(\gamma)$ for any $n \geq 0$. Accordingly, for the smallest value of t_f with the above condition, we have

$$\mathrm{Orb}(f) = \{f^{(n)}(\gamma) \ : \ n = 2, \ldots, t_f\}$$

and $\#\mathrm{Orb}(f) = t_f - 1$ or $\#\mathrm{Orb}(f) = t_f - 2$ (depending on whether $s = 1$ or $s \geq 2$ in the above). It is shown in [7, 8, 9] that critical orbits play a very important role in the dynamics of polynomial iterations.

Trivially we have $t_f \leq q + 1$. In fact, by the *Birthday Paradox* one expects that t_f is of order $q^{1/2}$ (for a sufficiently large q). Indeed, it is natural to expect that the map $x \mapsto f(x)$ behaves like a random map on $\mathbb{F}_q$, for which the trajectory length is of this order; see [4] for a detailed treatment of cycle structure of random maps on finite sets. For example, the Pollard integer factorisation algorithm (where a quadratic

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polynomial $f(X) = X^2 + c$ is iterated in a residue ring; see [3, Section 5.2.1]) is based on this assumption.

Here we obtain a nontrivial upper bound on the orbit length of stable quadratic polynomials:

Theorem 1. *For any odd q and any stable quadratic polynomial $f \in \mathbb{F}_q[X]$ we have*

$$t_f = O\left(q^{3/4}\right).$$

By [9, Proposition 3], a quadratic polynomial $f \in \mathbb{F}_q[X]$ is stable if the *adjusted orbit*

$$\overline{\mathrm{Orb}}(f) = \{-f(\gamma)\} \cup \mathrm{Orb}(f)$$

contains no squares. We also recall that $\alpha \in \mathbb{F}_q$ is a square if either $\alpha = 0$ or $\alpha^{(q-1)/2} = 1$, which can be tested (via repeated squaring) in $O(\log q)$ field operations. Combining these with the bound of Theorem 1, we immediately obtain:

Corollary 2. *For any odd q, a quadratic polynomial $f \in \mathbb{F}_q[X]$ can be tested for stability in time $q^{3/4+o(1)}$.*

Our proof is based on the Weil bound for multiplicative character sums with polynomials; see [6, Theorem 11.23].

Finally, we remark that estimating the size of the set of stable quadratic polynomials $aX^2 + bX + c \in \mathbb{F}_q[X]$ is a very interesting question to which we hope our technique can apply as well.

2. Proof of Theorem 1

Let χ be the quadratic character of $\mathbb{F}_q$.

By [9, Proposition 3], if a quadratic polynomial $f \in \mathbb{F}_q[X]$ is stable, then $\mathrm{Orb}(f)$ contains no squares, that is, $\chi\left(f^{(n)}(\gamma)\right) = -1$, $n = 2, 3, \ldots$.

We now fix an integer parameter K and note that for any $n \geq 1$, we have simultaneously

$$\chi\left(f^{(k+n)}(\gamma)\right) = -1, \qquad k = 1, \ldots, K,$$

which we rewrite as

$$\chi\left(f^{(k)}\left(f^{(n)}(\gamma)\right)\right) = -1, \qquad k = 1, \ldots, K. \tag{1}$$

Since by the definition of t_f the values $f^{(n)}(\gamma)$, $n = 1, \ldots, t_f - 1$, are pairwise distinct elements of $\mathbb{F}_q$, we derive from (1) that

$$t_f - 1 \leq \#\mathcal{T}_q(K), \tag{2}$$

where

$$\mathcal{T}_q(K) = \left\{x \in \mathbb{F}_q \ : \ \chi\left(f^{(k)}(x)\right) = -1, \ k = 1, \ldots, K\right\}.$$

We have

$$\#\mathcal{T}_q(K) = \frac{1}{2^K} \sum_{x \in \mathbb{F}_q} \prod_{k=1}^{K} \left(1 - \chi\left(f^{(k)}(x)\right)\right) \tag{3}$$

since for every $x \in \mathcal{T}_q(K)$ the product on the right hand side of (3) is 2^K; otherwise it is 0 when $\chi(f^{(k)}(x)) = 1$ for at least one $k = 1, \ldots, K$ (note that since by our assumption $f^{(k)}(X)$ is irreducible over $\mathbb{F}_q$ we have $f^{(k)}(x) \neq 0$ for $x \in \mathbb{F}_q$).

Just expanding the product in (3), we obtain $2^k - 1$ character sums of the shape

$$(-1)^\nu \sum_{x \in \mathbb{F}_q} \chi\left(\prod_{j=1}^{\nu} f^{(k_j)}(x)\right), \qquad 1 \le k_1 < \ldots < k_\nu \le K, \tag{4}$$

with $\nu \ge 1$ and one trivial sum that equals q (corresponding to the terms 1 in the product in (3)).

Clearly $f^{(k)}(X)$ is a polynomial of degree 2^k. Furthermore, by our assumption, each polynomial $f^{(k)}(X)$ is irreducible; therefore none of the polynomials

$$\prod_{j=1}^{\nu} f^{(k_j)}(X) \in \mathbb{F}_q[X], \qquad 1 \le k_1 < \ldots < k_\nu \le K,$$

are a perfect square in the algebraic closure of $\mathbb{F}_q$. Therefore the Weil bound (see [6, Theorem 11.23]) applies to every sum (4) and implies that each of them is $O(2^K q^{1/2})$. Therefore

$$\#\mathcal{T}_q(K) = \frac{1}{2^K} q + O(2^K q^{1/2}). \tag{5}$$

Choosing K to satisfy

$$2^K \le q^{1/4} < 2^{K+1}$$

and combining (2) and (5), we conclude the proof.

3. Comments

It is certainly interesting to obtain nontrivial estimates on the size S_q of the set of triples $(a, b, c) \in \mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q$ which correspond to stable quadratic polynomials $f(X) = aX^2 + bX + c$. Denoting by $F_k(a, b, c)$ the kth element of the critical orbit of f, we see that for any integer parameter K we have

$$S_q \le \#\mathcal{W}_q(K), \tag{6}$$

where

$$\mathcal{W}_q(K) = \left\{(a, b, c) \in \mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q \ : \ \chi\left(F_k(a, b, c)\right) = -1, \ k = 1, \ldots, K\right\},$$

and as before χ denotes the quadratic character of $\mathbb{F}_q$. As in the proof of Theorem 1, we have

$$\#\mathcal{W}_q(K) \le \frac{1}{2^K} \sum_{(a,b,c) \in \mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q} \prod_{k=1}^{K} \left(1 - \chi\left(F_k(a, b, c)\right)\right) \tag{7}$$

since for every triple $(a, b, c) \in \mathcal{W}_q(K)$ the product on the right hand side of (7) is 2^K; otherwise it is either 0 (when $\chi(F_k(a, b, c)) = 1$ for at least one $k = 1, \ldots, K$) or 1 (when $F_1(a, b, c) = \ldots = F_K(a, b, c) = 0$).

Clearly $F_k(a, b, c)$ is a rational function in a, b, c of degree at most $O(2^k)$. Thus expanding the product in (7), we obtain $2^K - 1$ character sums of the shape

$$(-1)^\nu \sum_{(a,b,c) \in \mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q} \chi\left(\prod_{j=1}^{\nu} F_{k_j}(a, b, c)\right), \quad 1 \le k_1 < \ldots < k_\nu \le K, \tag{8}$$

with $\nu \ge 1$ and one trivial sum corresponding to 1 in (7). Assuming that one can prove that the Weil-type bound $O(2^K q^{5/2})$ applies to all of them, we obtain from (6) that $S_q = O(q^3/2^K + 2^K q^{5/2})$ and optimising the choice of K we derive

$S_q = O(q^{11/4})$. In fact, for a nontrivial estimate of S_q it is enough to show that almost all sums (8) admit a nontrivial estimate, which has actually been recently done in [5], where the bound $S_q = O(q^{14/5})$ is obtained.

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